Variational Mechanics of Dissipative Systems

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Abstract

Starting from a modified Hamilton's principle conceived for a variational approach to dissipative systems, the various formalisms of classical mechanics(Lagrangian, Hamiltonian, canonical transformations, Poisson's brackets, Hamilton-Jacobi) are covered in a unified and pedagogical way. In particular, we will show that, in the context of the action principle, the time evolution of the system is not canonical. By explicitly solving the modified Hamilton-Jacobi's equation, we will obtain the dynamical solution for the linearly-damped harmonic oscilator. We will also establish Noether's theorem form in our formulation and will later use it to find a conserved quantity associated to the damped harmonic oscilator. Finally, we will briefly show how to carry out the generalization to classical field theories.

RESUMEN

Partiendo de un Principio de Hamilton modificado, concebido para el tratamiento variacional de sistemas disipativos, en este trabajo se desarrollan los diversos formalismos de la mecánica clásica (Lagrangiano, Hamiltoniano, transformaciones canónicas, Hamilton-Jacobi) de una manera unificada y pedagógica. En particular demostramos que en el contexto del principio de acción, la evolución temporal del sistema no es canónica. Resolviendo explícitamente la ecuación modificada de Hamilton-Jacobi obtenemos la solución dinámica para el oscilador armónico linealmente amortiguado. Establecemos la forma que el teorema de Noether asume en esta formulación y se usa para conseguir una cantidad conservada asociada con el oscilador amortiguado. Finalmente mostramos cómo llevar a cabo la generalización a teorías clásicas de campo.

1 INTRODUCTION

It is well known that our models of fundamental interactions are time-reversible.. This is to say that the basic equations describing nature are invariant under the transformation $t \rightarrow t$. This is in sharp contrast with the description of phenomena on a ordinary scale, where dissipation and irreversibility are the norm;

the transition from reversibility to irreversibility is not clear at all[1]. From the perspective of newtonian mechanics, some systems can be phenomenologically simulated by effective forces which provide a reasonable description of dissipation. Linear friction $\sim \vec{v}$, Coulombian friction, $\sim \vec{v}/v$, viscosity $\sim \nabla^2 v$,Dirac's damping $\sim \ddot{v}$ are just some cases of physical relevance.

Within the framework of variational mechanics the situation is less comfortable since, in order to derive the equations of motion from a hamilton's principle, bonding forces doing virtual work equal to zero are needed, and this is not the case with frictional forces. This leads to the introduction of elements of a foreign nature to the variational analysis, such as Raleigh's function[2]. Despite this, a great many authors have entered upon the subject ever since Bateman proposed the Lagrangian in 1931,

$$L(x, \dot{x}, t) = e^{\lambda t}(\frac{1}{2}m\dot{x}^2 - \frac{1}{2}m\omega^2 x^2), \tag{1}$$

where lambda is the damping constant, for the treatment of the linearly-damped oscillator. Bateman's Lagrangian was later "rediscovered" by Caldirola[3] and Kanai[4](see[5] for a detailed historic recount).

It is a known fact that the Lagrangian (1) and its associated Hamiltonian lead to difficulties in interpretation even within the classical domain[6]; i.e., the corresponding energy does not decay in time as it is to be expected from a physical point of view. In the quantum-mechanical case the Bateman lagrangian leads to the violation of the uncertainty relations. Ray's analysis[6] reveals that the Lagrangian (1) corresponds to a frictionless particle of variable mass $m = m_0 e^{\lambda t}$, and the violation of the uncertainty principle is due to the fact that an exponentially-growing mass quickly abandons the quantum realm. The distinction between mathematical Lagrangians (the sub-integral quantity in the action's definition) and physical ones (associated with the dynamics of the system) formulated by Ray[6], is of relevance here, and it is incorporated, up to a point, in our proposed Hamilton's principle for dissipative systems.

¿From the standpoint of quantum mechanics, the problem of dissipation has been the subject of considerable attention over the last years[5]$^-$[11], not only due to its intrinsic academic importance, but also because of the range of its potential practical applications, which include heavy ions inelastic collision processes, frictional processes related to nuclei fission , and laser theory. In the quantum context, the question of dissipation can be regarded as the *ad hoc* modification of Schrödinger's equation introducing non-linear terms which account for friction[9]$^-$[12], or else, as the application of a quantization prescription starting from a classical formalism, i.e., starting from a Lagrangian and quantizing *á lá* Feynman[15]; or else as a canonical quantization starting from a classical Hamiltonian or starting from Hamilton-Jacobi's equation and quantizing *a la* Schrödinger [12].

In 1986, in reference[13] an alternate way was proposed in order to work out a consistent, physically plausible formalism, leading, after quantization, to Kostin-Langevin's equation[17], which describes a Brownian particle in a thermal bath. Since then, the proposed hamilton's principle has been used as the basis for the

formulation of the damped oscillator's path integrals[14], as well as for deducing the telegraphist's equation which describes the attenuation of charge waves in transmission lines[19].

In this work, we revisit and extend the variational principle proposed in[13]. We have opted for a manifestly pedagogical approach, starting from the action principle and running through the diverse formalisms of variational mechanics. For the sake of simplicity we have considered a system with a single degree of freedom and the simplest of frictional scenarios, the linear one. Nothing conceptually new is added with the extension to more general cases. On the other hand, we have not restricted ourselves to any particular physical system; however, the examples considered all refer to the oscillator since this is the most cited example in the literature. This article is organized as follows. In section 2 we define the action corresponding to the dissipative system as well as establish the Hamilton's modified principle governing its dynamics. In section 3 we develop the canonical formalism, which includes Poisson's brackets and the canonical transformations. Hamilton-Jacobi's theory, along with an example on how to solve the dynamical problem for the harmonic oscillator using Hamilton-Jacobi's equation, is presented in section 4. In section 5 we establish the Noether's theorem corresponding to our formulation. The proof is given in appendix 2. This theorem is then used to obtain a constant of motion associated with the damped oscillator. In appendix 1 we offer a brief introduction on how to extend the principle to the classical field theory.

2 THE DYNAMICAL PRINCIPLE FOR DISSIPATIVE SYSTEMS.

Let us consider a mechanical system in one dimension, under a conservative potential represented by $V(x)$ and under a dissipative force proportional to the speed (as usual, a dot on top of a quantity denotes the time derivative of that quantity). We postulate that the evolution of the system between instants t_1 and t_2 is such that the action given by

$$S = \int_{t_1}^{t_2} e^{\lambda t} L(x, \dot{x})\, dt \tag{2}$$

takes on a stationary value for the real trajectory of the system. It is important to note that $L(x, \dot{x})$ represents the system's physical Lagrangian, i.e. the kinetic energy minus the potential energy,

$$L(x, \dot{x}) = \frac{1}{2} m\dot{x}^2 - V(x), \tag{3}$$

so that the canonical momentum $p = \frac{\partial L}{\partial \dot{x}}$coincides with the kinetic momentum $m\dot{x}$. However, for the purposes of variational mechanics in order to deduce the Euler-Lagrange's equations, the whole sub-integral quantity is to be used as

Lagrangian.. From the action principle defined by equation (3), it is straightforward to obtain the following equation of motion:

$$\ddot{x} + \lambda \dot{x} = -\frac{1}{m}\frac{\partial V(x)}{\partial x}. \tag{4}$$

If there are no external forces acting upon the particle, $V = 0$, although this does not mean we are dealing with a free particle (because of the dissipative force). For a damped harmonic oscillator, we must choose, naturally, $V = \frac{1}{2}kx^2$.

It is worth noting that the action integral defined by equation (3) contains the time parameter t explicitly and in a non-symmetrical way, in other words, the action we are starting with (and therefore all the formalisms and equations of motion thereby deduced), is not invariant under the transformation $t \to -t$, so that an *ab initio* time arrow characteristic of irreversible phenomena is introduced.

3 THE CANONICAL FORMALISM.

3.1 THE HAMILTON'S EQUATIONS

The hamilton's equations can be obtained constructing the Hamiltonian the usual way,

$$H(x,p) \equiv p\dot{x} - L \tag{5}$$

where it is understood that the speed is expressed in terms of the momentum p as usual. Since $L = T - V$ is the physical Lagrangian of the system, it follows that H is the physical Hamiltonian which represents the system's energy $T+V$. Thus, the resulting action is

$$S = \int_{t_1}^{t_2} e^{\lambda t}\,(p\dot{x} - H)\,dt \tag{6}$$

and the Euler-Lagrange equations for the independent variables x and p turn out to be

$$\dot{x} = \frac{\partial H}{\partial p}, \tag{7a}$$

$$\dot{p} = -\frac{\partial H}{\partial x} - \lambda p. \tag{7b}$$

These equation are successful in capturing a fundamental property of dissipative systems which other variational treatments fail to show, i.e., that the Hamiltonian decreases with time, as it should be with one representing the particle's energy. In effect, taking the time derivative and substituting for $\dot{x}$ and $\dot{p}$ by using equations (7a)and (7b), yields

$$\dot{H} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial p}\dot{p} = \frac{\partial H}{\partial x}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial p}(\frac{\partial H}{\partial x} + \lambda p).$$

After simplifying, the result is

$$\dot{H} = -\lambda p \frac{\partial H}{\partial p} < 0. \tag{8}$$

Equation (8) explicitly shows that the energy dissipates, at least for those systems whose Hamiltonian are a homogeneous function of second degree. In the case of the damped harmonic oscillator, $\dot{H} = -\lambda p^2/m$, is obtained which is always negative.

3.2 CANONICAL TRANSFORMATIONS.

As in the standard formulation, we can consider transformations of the canonical variables (x, p) into new ones, function of the latter, which we will denote by (X, P).Since we are to demand that these new variables be canonical, they must satisfy the new Hamilton's equations of the type

$$\dot{X} = \frac{\partial K}{\partial P} \tag{9a}$$

$$\dot{P} = -\frac{\partial K}{\partial X} - \lambda P, \tag{9b}$$

where $K(X, P, t)$ plays the role of the new Hamiltonian. These equations stem from an action similar to (6) and therefore, the sub-integral quantities must differ by the time derivative of a *generating function* or *generator*,which we will denote by F

$$p\dot{x} - H = P\dot{X} - K + e^{-\lambda t}\frac{dF}{dt}. \tag{10}$$

Let's consider in detail the case in which the generator depends on both the old coordinates x, the new coordinates X, and on time, $F = F_1(x, X, t)$. In this case, after multiplying by dt, equation (10) turns out to be

$$p\,dx - H\,dt = P\,dX - K\,dt + e^{-\lambda t}\,\frac{\partial F_1}{\partial x}\,dx + e^{-\lambda t}\,\frac{\partial F_1}{\partial X}\,dX + e^{-\lambda t}\,\frac{\partial F_1}{\partial t}\,dt.$$

Identifying the differential's coefficients on both sides of the latter equation, we obtain the equation for the canonical transformation generated by F_1,

$$\begin{aligned} p &= e^{-\lambda t}\,\frac{\partial F_1}{\partial x} \\ P &= -e^{-\lambda t}\,\frac{\partial F_1}{\partial X} \\ K &= H + e^{-\lambda t}\frac{\partial F_1}{\partial t}. \end{aligned} \tag{11a}$$

We can likewise obtain the equations for the canonical transformations corresponding to a generator of the type $F = F_2(x, P, t)$. the resulting equation

are

$$\begin{aligned} p &= e^{-\lambda t}\frac{\partial F_2}{\partial x} \\ X &= e^{-\lambda t}\frac{\partial F_2}{\partial P} \\ K &= H - \lambda P X + e^{-\lambda t}\frac{\partial F_2}{\partial t}. \end{aligned} \tag{12a}$$

It is similarly easy to obtain the equations corresponding to the generators $F_3(p, X, t)$ and $F_4(p, P, t)$. The corresponding details can be found in reference[13]. It is useful to consider some examples. Let's pick $F_2 = e^{\lambda t} P\, x$. Using equations (12a) we obtain

$$P = p, \qquad X = x, \qquad K = H,$$

so that $e^{\lambda t} P\, x$ is the generator of the identity transformation. Another important is the generator $F_1 = e^{\lambda t} X\, x$. Using equations (11a) yields

$$X = p, \qquad P = -x, \qquad K = H - \lambda P X,$$

which represent the exchange between the coordinate and the momentum. It is trivial to verify that the canonical equations transform into themselves under this transformation.

3.3 POISSON'S BRACKETS AND CANONICAL INVARIANTS.

The formalism allows an elegant formulation in terms of Poisson's brackets. Let F and G be two functions of the canonical variables x and p. We define Poisson's brackets of F and G in the usual manner,

$$[F, G]_{xp} = \frac{\partial F}{\partial x}\frac{\partial G}{\partial p} - \frac{\partial F}{\partial p}\frac{\partial G}{\partial x}. \tag{13}$$

With this definition, the bracket already preserves its conventional properties, namely, antisymmetry, bilinearity, Leibniz rule and Jacobi's identity properties. In terms of Poisson's bracket, the equations of motion can be rewritten as

$$\dot{x} = [x, H] \tag{14a}$$

$$\dot{p} = [p, H] - \lambda p. \tag{14b}$$

It is easy to corroborate that, if a transformation $(x, p) \to (X, P)$ is canonical, then the corresponding Poisson's bracket is an invariant, this is,

$$[F, G]_{xp} = [F, G]_{XP}$$

and particularly

$$[x, p]_{XP} = 1.$$

Let us now demonstrate the following result. The "volume" of a region in the phase space is a canonical invariant, this is

$$\iint dx\, dp = \iint dX\, dP. \tag{15}$$

To demonstrate this, let's remember that in general, for an arbitrary transformation $(x, p) \rightarrow (X, P)$, it follows that $dX\, dP = J\, dxdp$, where J is the transformation's Jacobian. So it is that the demonstration of (15) is equivalent to demonstrating that the jacobian of a canonical transformation is unity,[20]. Effectively, from its definition and using some known properties of the jacobian, we obtain

$$J \equiv \frac{\partial(X,P)}{\partial(x,p)} = \frac{\partial(X,P)}{\partial(x,P)} \Big/ \frac{\partial(x,p)}{\partial(x,P)} = \frac{\partial(X)}{\partial(x)} \Big/ \frac{\partial(p)}{\partial(P)}. \tag{16}$$

We can make use of a type 2 generator, and using equations (12a) we have

$$\frac{\partial(X)}{\partial(x)} = e^{-\lambda t}\, \frac{\partial^2 F_2}{\partial x\, \partial P} \qquad \text{and} \qquad \frac{\partial(p)}{\partial(P)} = e^{-\lambda t}\, \frac{\partial^2 F_2}{\partial P\, \partial x}$$

so that equation (16) shows that $J = 1$.

It is convenient to consider in this context the motion of the "phase" fluid in time. If we associate the tangent vector $\vec{V} = (\dot{x}, \dot{p})$ to the trajectory of a point $(x(t), p(t))$, then the divergence in the phase space of this vector field is

$$div\, \vec{V} = \frac{\partial\, \dot{x}}{\partial\, x} + \frac{\partial\, \dot{p}}{\partial\, p},$$

and resorting to the canonical equations (7a) and (7b) for substituting $\dot{x}$ and $\dot{p}$, we obtain

$$div\, \vec{V} = -\lambda. \tag{17}$$

This result deserves two comments. In first place it is noticeable that, in our description of the system's evolution, the phase fluid behaves as an compressible fluid, which is to say that Liouville's theorem (which, in the usual formulation of mechanics, establishes the constancy in time for the volume of a region of the phase space), no longer holds. An equivalent way to look at this is to see equation (17) as indicating that a set of different initial conditions is joining toward the same final condition. This lack of symmetry between the initial and final conditions is due, essentially, as we pointed out in the introduction, to the presence of the term $e^{\lambda t}$ in the action. This term breaks the invariance $t \rightarrow -t$ in hamilton's modified principle.

In second place we note that, since the phase volume is a canonical invariant as previously shown, we conclude that the evolution in time *cannot* be considered as a canonical transformation. Otherwise said, the Hamiltonian is not the generator of infinitesimal transformations in time if the Hamiltonian is to represent the dissipative system's energy, as is the case in our formulation.

4 HAMILTON-JACOBI'S THEORY

The essence of the Hamilton-Jacobi's method is to take advantage of the freedom provided by the canonical transformations in order to build a new, jacobian identically zero, so that with that Hamiltonian the integration of Hamilton's equations and therefore, the solving of the dynamical problem is elementary.

Since under a canonical transformation the new Hamiltonian satisfies

$$K(X,P) = H(x,p) + e^{-\lambda t}\frac{\partial F}{\partial t},$$

the demand $K \equiv 0$ results in

$$H(x,p) + e^{-\lambda t}\frac{\partial F}{\partial t} = 0 \tag{18}$$

and the new Hamilton's equation are

$$\begin{aligned} \dot{Q} &= 0 \\ \dot{P} &= -\lambda P. \end{aligned}$$

The integration shows that $Q = Q_0 = const.$, whereas $P = Pe^{-\lambda t}$. Let's pick a generator $F(x,P,t)$. Note that, since P is a known function of t, F is only a function of x and t. Using equations (11a) we obtain Hamilton-Jacobi's modified equation, which must satisfy generator F:

$$e^{-\lambda t}\frac{\partial F}{\partial t} + H(x, e^{-\lambda t}\frac{\partial F}{\partial x}) = 0. \tag{19}$$

It is convenient to define a new generator $\bar{S} \equiv e^{-\lambda t}F$ with respect to which equation (19)is re-written as

$$\frac{\partial \bar{S}}{\partial t} + \lambda \bar{S} + H(x, \frac{\partial \bar{S}}{\partial x}) = 0. \tag{20}$$

Next we will that the solution to equation (20) is precisely the *reduced action*, defined by

$$\bar{S} = e^{-\lambda t}\int e^{\lambda t}\, L\, dt \tag{21}$$

considered as a function of the coordinate and time. The demonstration consists in taking the time derivative on both sides of (21), obtaining

$$\frac{\partial \bar{S}}{\partial t} + \frac{\partial \bar{S}}{\partial x}\,\dot{x} = -\lambda \bar{S} + L,$$

but since $\frac{\partial \bar{S}}{\partial x} = p$, and recalling that $H = p\dot{x} - L$, we see that $\bar{S}$ satisfies equation (20). It is important to point out that it is the reduced action $\bar{S}$ which is directly linked to the physics of the system. In particular, its gradients is the kinetic momentum. Let us remark that the reduced action is not additive, i.e., the action between two instants is not the sum of the actions corresponding to

intermediate instants owing to the exponential outside the integral in definition (21). That is the reason for the appearance of the non-linear term $\lambda\ \bar{S}$ in equation (20), which has important consequences for the quantization of the system and bears responsibility for the appearance of the non-linear term in Kostin-Langevin's equation[17], since the identification of wave amplitudes $\psi \sim e^{-\frac{i}{\hbar}\bar{S}}$ leads, *via* $\lambda\ \bar{S}$ to terms of the form $\lambda \log(\psi/\psi^*)$.

As a way to illustrate , we will solve the dynamic problem for an linearly-damped harmonic oscillator using Hamilton-Jacobi's modified equation (20). For convenience we will take $m = 1$, so that the Hamiltonian of the system is

$$H(x,p) = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\, x^2 \tag{22}$$

so that Hamilton-Jacobi's modified equation is,

$$\frac{\partial \bar{S}}{\partial t} + \lambda \bar{S} + \frac{1}{2}\left[\left(\frac{\partial \bar{S}}{\partial x}\right)^2 + \omega^2\, x^2\right] = 0. \tag{23}$$

Let us now propose the following *ansatz*:

$$\bar{S} = x^2 f(t).$$

Substituting in equation (23) and multiplying by f yields the following differential equation for $f(t)$

$$\frac{df}{dt} + 2f^2 + \lambda f + \frac{\omega^2}{2} = 0$$

whose solution is

$$f(t) = -\frac{1}{2}\omega'\ \tan[\omega'\,(t-c)] - \frac{\lambda}{4}, \tag{24}$$

where c is a constant of integration and $\omega' \equiv \sqrt{\omega^2 - \frac{\lambda^2}{4}}$. Having determined the function $f(t)$, and therefore $\bar{S}$, we can use equations

$$P = \frac{\partial\,(x^2 f)}{\partial c} = P_0\, e^{-\lambda t}, \tag{25}$$

but

$$\frac{\partial\,(x^2 f)}{\partial c} = x^2 \frac{\partial\, f}{\partial c} = x^2\, \omega'\,[1 + \tan^2(\omega' t - c)]. \tag{26}$$

Solving for x and recalling the identity $1 + \tan^2\alpha = \cos^2\alpha$, the result is

$$x = A\, e^{-\lambda t/2}\ \cos\omega' t \tag{27}$$

where $A = P_0^{1/2}/\omega'$. We have set the phase $c = 0$ at the end of the computation. This is one the usual expressions for the solution of the damped oscillator.

5 NOETHER'S THEOREM

It is a well known fact that, within the formulation of variational mechanics (and in field theories), Noether's theorem establishes a profound connection between symmetries properties of the action and the conserved quantities[18]. In this section we will determine the form this theorem takes for our variational principle as well as present an example whereby a constant of motion for the harmonic oscillator is obtained.

Theorem (Noether)

Let us suppose that the action integral of a damped physical system, as given by equation (2) is invariant under the infinitesimal transformation

$$t \rightarrow t' = t + \epsilon\Phi(x,t) \tag{28a}$$

$$x(t) \rightarrow x'(t') = x(t) + \epsilon\Psi(x,t), \tag{28b}$$

then the quantity

$$C \equiv e^{\lambda t}\left[\frac{\partial L}{\partial \dot{x}}(\dot{x}-\Psi) - L\Phi\right] \tag{29}$$

is a constant of motion, this is, $\dot{C}=0$ on the trajectory of the system. We believed it convenient to postpone the demonstration of this theorem for appendix 2, and consider next an application as way of example.

Let us consider the following transformation

$$t \rightarrow t' = t + a \tag{30a}$$

$$x(t) \rightarrow x'(t') = e^{-\frac{\lambda a}{2}}x(t) \tag{30b}$$

which corresponds to a time translation and to a time dilation of the variables linked to damped harmonic oscillator (with mass $m=1$), whose physical Lagrangian is

$$L = \frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2.$$

It is easy to see that action $S = \int e^{\lambda t}(\frac{1}{2}\dot{x}^2 - \frac{\omega^2}{2}x^2)\,dt$ is invariant under the transformation (30a-30b). In effect,

$$S' = \int_{t'_1}^{t'_2} e^{\lambda t'}\,[\frac{1}{2}(\frac{dx'}{dt'})^2 - \frac{\omega^2}{2}x'^2]\,dt' = \int_{t_1+a}^{t_2+a} e^{\lambda t}\,e^{\lambda a}\,[e^{-\lambda a}\frac{\dot{x}^2}{2} - \frac{\omega^2}{2}e^{-\lambda a}x^2]\,dt = S.$$

The infinitesimal form of the transformation (30a-30b) is

$$t' = t + \epsilon, \qquad\qquad x' = x - \frac{1}{2}\epsilon\,\lambda\,x, \tag{31}$$

so that by comparing with (28a,(28b)), we conclude that $\Phi = 1$ whereas $\Psi = -\frac{1}{2}\lambda\,x$. Finally, equation (29) allows us to obtain the constant of motion

$$C = e^{\lambda t}\,(\dot{x}^2 + \lambda\,\dot{x}x + \omega^2\,x^2) \tag{32}$$

associated to that symmetry. This first integral for the damped oscillator has been studied in other contexts and by many authors; see[21] and references therein cited.

5.1 APPENDIX 1: EXTENSION TO FIELD THEORIES

The proposed variational principle may be appropriately extended to continuum systems and fields. We will present here the simplest of generalizations. Let $\phi(\vec{x},t)$ be a physical field whose dynamics is determined by the Lagrangian density $\mathcal{L}(\phi,\partial\phi)$, where ∂ condenses the different time and space derivatives. We will postulate that the equations of motion of the corresponding damped system are the Euler-Lagrange's equation for the action

$$S=\int e^{\lambda t}\,\mathcal{L}(\phi,\partial\phi)\,d^4x. \tag{33}$$

A brief calculation shows the Euler-Lagrange's equations to be

$$\frac{\partial}{\partial t}\left(\frac{\partial\mathcal{L}}{\partial(\frac{\partial\phi}{\partial t})}\right)+\vec{\nabla}.\left(\frac{\partial\mathcal{L}}{\partial(\vec{\nabla}\phi)}\right)+\lambda\frac{\partial\mathcal{L}}{\partial(\frac{\partial\phi}{\partial t})}-\frac{\partial\mathcal{L}}{\partial\phi}=0. \tag{34}$$

For example, if we choose the well known Klein-Gordon's Lagrangian,

$$\mathcal{L}(\phi,\partial\phi)=\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2-\frac{1}{2}\left(\vec{\nabla}\phi\right)^2+m^2\phi^2,$$

equation (34) leads to

$$\frac{\partial^2\phi}{\partial t^2}-\nabla^2\phi+\lambda\frac{\partial\phi}{\partial t}=m^2\phi. \tag{35}$$

The telegraphist's equation, which describes the attenuation of electrical charge waves in transmission lines[19], can be obtained in a similar way.

5.2 APPENDIX 2. DEMONSTRATION OF NOETHER'S THEOREM

In order to demonstrate Noether's theorem, let us note that, from the infinitesimal transformation (31), we can write, neglecting quadratic and superior terms in ϵ,

$$\begin{aligned}\frac{dt'}{dt} &= 1+\epsilon\dot{\Phi} \qquad\qquad \frac{dt}{dt'}=(1+\epsilon\dot{\Phi})^{-1}=1-\epsilon\dot{\Phi}\\ \frac{dx'(t')}{dt'} &= \frac{dt}{dt'}\frac{d\,x'(t)}{dt}=(1-\epsilon\dot{\Phi})\,(\dot{x}+\epsilon\dot{\Psi})=\dot{x}+\epsilon\xi\end{aligned}$$

where $\xi=\dot{\Psi}-\dot{x}\dot{\Phi}$. Using these equations, the variation of the action is

$$\delta S=\int_{t_1}^{t_2} e^{\lambda(t+\epsilon\Phi)}\,L(x+\epsilon\Psi,\,\dot{x}+\epsilon\xi)\,(1+\epsilon\dot{\Phi})\,dt-\int_{t_1}^{t_2} e^{\lambda t}\,L(x,\,\dot{x})\,dt.$$

Expanding in power series, and again neglecting terms of order ϵ^2 and taking into account that the interval of integration is arbitrary, yields

$$\Psi\frac{\partial L}{\partial \dot{x}} + (\dot{\Psi} - \dot{x}\dot{\Phi})\frac{\partial L}{\partial \dot{x}} + \dot{\Phi}L + \lambda\Phi L = 0. \tag{36}$$

We can simplify this expression noting that

$$\frac{d\,(e^{\lambda t}\,L)}{dt} = e^{\lambda t}\,\dot{x}\frac{\partial L}{\partial x} + e^{\lambda t}\,\ddot{x}\frac{\partial L}{\partial \dot{x}} + \lambda\, e^{\lambda t}\,L$$

and using the form of Euler-Lagrange's equations

$$\frac{d}{dt}\left(\frac{\partial\,(e^{\lambda t}\,L)}{\partial \dot{x}}\right) = \frac{\partial(e^{\lambda t}L)}{\partial x},$$

yields

$$e^{-\lambda t}\frac{d}{dt}\left(e^{\lambda t}\frac{\partial\, L}{\partial \dot{x}}\dot{x} - e^{\lambda t}\,L\right) = -\lambda L. \tag{37}$$

Finally, substituting (37) in equation (36), we obtain

$$\frac{d}{dt}\left[e^{\lambda t}\left(\frac{\partial\, L}{\partial \dot{x}}(\dot{x} - \Psi) - L\Phi\right)\right], \tag{38}$$

which concludes the demonstration of the theorem.

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