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MASSIVE VECTOR CHERN-SIMONS GRAVITY

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ABSTRACT

We present a second order gravity action which consists of ordinary Einstein action augmented by a first-order, vector like, Chern-Simons quasi topological term. This theory is ghost-free and propagates a pure spin-2 mode. It is diffeomorphism invariant, although its local Lorentz invariance has been spontaneuosly broken.

In three dimensions, it has been pointed out by Deser, Jakiw and Templeton[1] that addition of the tensorial topological Chern-Simons term $S_{TCS} \sim < \omega\partial\omega + \omega^3 >$ to the Einstein action S_E yields a gauge invariant, ghost free, pure spin-2 massive theory. In this paper we present a softer possibility, which also gives rise to a massive spin-2 theory. Instead of the tensorial CS term we introduce the vectorial CS term constructed out of the dreibein variables $e^a = dx^r e^a_r$

$$S_{VCS} \overrightarrow{=} \mu(2\kappa^2)^{-1} < e^a_p \epsilon^{prs} \partial_r e_{sa} > . \tag{1}$$

Here μ is the topological mass of the system, κ is the 3-d gravitational costant, $e_{sa} = e_s{}^b \eta_{ba}$, η_{ba} is the flat Lorentz metric $(-++)$ and ϵ^{prs} is the Levi-Civita density, $\epsilon^{012} = +1$.

This action is diffeomorphism invariant and it is not local Lorentz invariant. It is topological in its world indices. It can not be regarded as fully topological because it needs the flat metric η^{ab} to be a good invariant. Neither it is locally conformally invariant. As it happens with the other two actions we mentioned before, S_{VCS} alone does not contain local excitations. The full action we postulate here is

$$S \overrightarrow{=} (2\kappa^2)^{-1} < e_{pa} \epsilon^{pmn} R_{mn}{}^a(\omega) > + S_{VCS} \overleftarrow{=} S_E + S_{VCS} \tag{2}$$

where $R_{mn}{}^a \overrightarrow{=} \partial_m \omega_n{}^a - \partial_n \omega_m{}^a - \epsilon^a{}_{bc} \omega_m{}^b \omega_n{}^c$ is the planar Riemann tensor, $\omega^a \overrightarrow{=} dx^r \omega_r{}^a$ and $e^a \overrightarrow{=} dx^r e_r{}^a$ being respectively the affinity and the dreibein one-forms.

In spite that Einstein action S_E is both local-Lorentz and diffeomorphism invariant, and the vector-CS term is only diffeomorphism invariant, complexive action S is just diffeomorphism invariant too.

The situation is similar with massive tensor CS-gravity, which is the sum of $S_{TCS} - S_E$. This system is not locally conformal invariant due to the non conformal invariance of the Einstein action. There is a hierarchy of the local symmetries, starting with the tensorial CS term

$$S_{TCS} \overrightarrow{=} (2\mu\kappa^2)^{-1} < \omega_{pa} \epsilon^{pmn} \partial_m \omega_n{}^a - 3^{-1} \epsilon^{pmn} \epsilon_{abc} \omega^a_p \omega^b_m \omega^c_n > \tag{3}$$

which is locally conformal, Lorentz, and diffeomorphism invariant. Then it comes ordinary Einstein action (2) locally Lorentz and diffeomorphism invariant and finally one has S_{VCS} which is only diffeomorphism invariant.

Each of these actions alone has non local excitations. However massive tensorial CS gravity has a pure spin-2 content. Massive vectorial CS gravity, eq. (2), will be shown to have a pure spin-2 content too. One might even go a step further and lose all gauge invariances.

In that case, one ends up with self-dual gravity[2], a first order action on flat three dimensional Minkowski space having no gauge invariance, and a ghost-free, pure spin-2 content.

Independent variations of $\omega_p{}^a$, $e_p{}^a$ in S yield the standard torsionless value of $\omega_p{}^a$ in terms of the dreibein variables.

$$_3 e \omega_p a = e_{pb} e_q{}^a \epsilon^{qrs} \partial_r e_s{}^b - 2^{-1} e_p{}^a e_{qb} \epsilon^{qrs} \partial_r e_s{}^b \tag{4}$$

and the (second order in $e_p{}^a$) field equations

$$E^{pa} \overrightarrow{=} \epsilon^{pmn} R_{mn}{}^a(\omega) + 2\mu \epsilon^{pmn} \partial_m e_n{}^a = 0. \tag{5}$$

Their associated Bianchi identities read

$$\partial_p E^{pa} - \epsilon_{bc}{}^a \omega_p{}^b E^{pc} + 2\mu\epsilon^{nrs}\omega_n{}^a\omega_r{}^b e_{sb} = 0 \tag{6}$$

Insertion of $\omega_p{}^a$ as given by eq. (4) into eq. (5) leads to the second order field equations which determine the dynamics of the system. The Riemann tensor has now a source $\sim \mu\epsilon^{pmn}\partial_m e_n{}^a$ which makes it locally non trivial.

Consequently now we have the possibility of local excitations, as we will show below. Physical variations of the dreibein variables under small diffeomorphisms are $\delta e_r{}^a \sim D_r\xi^a$.

In order to understand the physical content of this theory it is convenient to analyze the associated linearized system, which can be obtained in straighfoward manner by intoducing $e_{pa} = \eta_{pa} + \kappa h_{pa}$, $\omega_p{}^a = \kappa\omega_p{}^a$ in action (2). S then becomes

$$2S^{Lin} =< 2\omega_p{}^a\epsilon^{pmn}\partial_m h_{na} - \eta_{pa}\epsilon^{pmn}\epsilon^a{}_{bc}\omega_m{}^b\omega_n{}^c > +\mu < h_p{}^a\epsilon^{prs}\partial_r h_{sa} > \tag{7}$$

We perform a 2+1 decomposition introducing $q_j \equiv h_{j0}$. S^{Lin} can then be written as

$$\begin{aligned} 2S^{Lin} =&<2h_{00}\{\epsilon^{ij}\partial_i\omega_j{}^0 - \mu\epsilon^{ij}\partial_i q_j\} + 2h_{0l}\{\epsilon^{ij}\partial_i\omega_{jl} + \mu\epsilon^{ij}\partial_i h_{jl}\}+ \\ &+ \mu q_i\epsilon^{ij}\partial_0 q_j - 2\omega_i{}^0\epsilon^{ij}\partial_0 q_j - [2\omega_{ij} + \mu h_{ij}]\epsilon^{il}\partial_0 h_{lj} > + \\ &+ < \omega_{jj}\omega_{ll} - \omega_{ij}\omega_{ji} > + \\ &+ < 2\omega_0{}^0[\epsilon^{ij}\partial_i q_j + \omega_{jj}] + 2\omega_0{}^l[\epsilon^{ij}\partial_i h_{jl} - \omega_l{}^0] > . \end{aligned} \tag{8}$$

$\omega_0{}^0$, $\omega_0{}^l$ constitute multipliers associated with the algebraic constraints

$$\mathcal{D}_0 \equiv \epsilon^{ij}\partial_i q_j + \omega_{jj} = 0, \tag{9a}$$
$$\mathcal{D}_l \equiv \epsilon^{ij}\partial_i h_{jl} - \omega_l{}^0 = 0. \tag{9b}$$

They provide the respective values of ω_{jj}, $\omega_l{}^0$ in terms of the coordinates h_{jl}, q_i. Then we observe that h_{00}, h_{0l} are Lagrange multipliers too. They are asocciated with the diferential constraints

$$\mathcal{C}^0 \equiv \epsilon_{ij}\partial_i\omega_j{}^0 - \mu\epsilon_{ij}\partial_i q_j = 0, \tag{10a}$$
$$\mathcal{C}^l \equiv \epsilon_{ij}\partial_i\omega_{jl} + \mu\epsilon^{ij}\partial_i h_{jl} = 0. \tag{10b}$$

To achieve the unconstrained formulation we introduce the $T + L$ 2-dimensional decompositions

$$q_j =(i\widehat{\partial})_j q^T + \widehat{\partial}_j q^L, \tag{11a}$$
$$\begin{aligned} h_{ij} =&(i\widehat{\partial})_i h_j^T + \widehat{\partial}_i h_j^L \\ \equiv&(i\widehat{\partial})_i(i\widehat{\partial})_j h^{TT} + (i\widehat{\partial})_i\widehat{\partial}_j h^{TL}+ \\ &+\widehat{\partial}_i(i\widehat{\partial})_j h^{LT} + \widehat{\partial}_i\widehat{\partial}_j h^{LL} \end{aligned} \tag{11b}$$

and similarly for ω_{ij}, where $\widehat{\partial}_i$ is the unit gradient, $\widehat{\partial}_j \equiv \rho^{-1}\partial_j$, $\rho \equiv (-\Delta_2)^{\frac{1}{2}}$, $(i\widehat{\partial})_j \equiv -\epsilon_{jl}\widehat{\partial}_l$.

Eq. (9b) gives the value of ${\omega_l}^0$,

$$ {\omega_l}^0 = -\rho {h^T}_l. \tag{12} $$

Inserting this expression for ${\omega_l}^0$ into $\mathcal{C}^0$, eq. (10a) provides q^T

$$ q^T = -\mu^{-1}\rho h^{TT}. \tag{13} $$

The vectorial differential constraint $\mathcal{C}^l$ determines ω_l^T

$$ {\omega^T}_l = -\mu {h^T}_l. \tag{14} $$

Finally, getting back to $\mathcal{D}_0$ and taking into account eqs. (13) (14) we obtain ω^{LL},

$$ \omega^{LL} = (1+\mu^{-2}\rho^2)\mu h^{TT}. \tag{15} $$

Introducing all this information into S^{Lin} one is led to its (almost canonical) unconstrained expression

$$ 2S^{Lin} = < (\mu h^{TL} - \omega^{LT}) 2\dot{h}^{TT} - 2h^{TT}(\mu^2+\rho^2)h^{TT} + 2\mu h^{TL}\omega^{LT} > . \tag{16} $$

Redefining $2h^{TT} \to h^{TT}$ and introducing the canonical momenta $p \equiv \mu h^{TL} - \omega^{LT}$ eq. (16) transforms into

$$ 2S^{Lin} = < p\dot{h}^{TT} - 2^{-1}h^{TT}(\mu^2+\rho^2)h^{TT} + 2\omega^{LT}(p+\omega^{LT}) > . \tag{17} $$

which, after making independent variations of ω^{LT} attains the standard canonical structure for the unique massive excitation carried out by h^{TT}.

Note that $p \sim \mu h^{TL}$. The self-dual character of the action due to the presence of the vectorial CS term constrains the tranverse part of h_{ij}, $h_j^T = (i\widehat{\partial}_j)h^{TT} + \widehat{\partial}_j h^{TL}$, to carry on both the local physical excitation h^{TT} and its canonical momenta. Moreover action (17) does not contain either of the longitudinal gauge sensitive variables h^{LL}, h^{LT}, q^L (as it must happen) because of the gauge invariance of S^{Lin} with respect to $\delta h_{ij} = \partial_i \xi_j$.

Now we focus on the curved action S. It is convenient to introduce the 2+1 variables $e_{i\bar{j}}$, $q_i \equiv e_{i\bar{0}}$, ${e^0}_{\bar{0}} = n^{-1}$, $e_{0\bar{j}} = -\nu_{\bar{j}}$ and the two dimensional inverse of $e_{i\bar{j}}$, ${e^j}_{\bar{l}}$: ${e^j}_{\bar{l}}{e_j}^{\bar{k}} = \delta_{\bar{l}}^{\bar{k}}$.

It is inmediate to realize that ${}_3e = \epsilon^{rst}{e_r}^{\bar{0}}{e_s}^{\bar{1}}{e_t}^{\bar{2}} = {}_2en$, ${e_0}^{\bar{0}} = n + \nu_{\bar{j}}e^{i\bar{j}}p_i$. In terms of these variables S has the form

$$ \begin{aligned} 2\kappa^2 S =& <2{\omega_0}^{\bar{0}}\{\epsilon^{ij}\partial_i q_j + {}_2ee^{i\bar{j}}\omega_{i\bar{j}}\} \\ &+ 2{\omega_0}^{\bar{l}}\{\epsilon^{ij}\partial_i e_{j\bar{l}} + {}_2eq_j(e^{j\bar{l}}\omega_2 - \omega^{\bar{l}j}) - {}_2e{e^h}_{\bar{l}}{\omega_h}^{\bar{0}}\} \\ &- 2{e_0}^{\bar{0}}\{\epsilon^{ij}\partial_i{\omega_j}^{\bar{0}} - \mu\epsilon^{ij}\partial_i q_j + 2^{-1}{}_2e\omega^{\bar{l}j}\omega_{j\bar{l}} - 2^{-1}{}_2e{\omega^{\bar{l}}}_{\bar{l}}{\omega^{\bar{j}}}_{\bar{j}}\} \\ &+ 2{e_0}^{\bar{l}}\{\epsilon^{ij}\partial_i\omega_{j\bar{l}} + \mu\epsilon^{ij}\partial_i e_{j\bar{l}} + {}_2e(\omega^{\bar{l}j} - e^{j\bar{l}}\omega){\omega_j}^{\bar{0}}\} \\ &+ [\mu q_i - 2{\omega_i}^{\bar{0}}]\epsilon^{ij}\partial_i q_j - [2\omega_{i\bar{l}} + \mu e_{i\bar{l}}]\epsilon^{ih}\partial_0 e_{h\bar{l}} > \\ =& <2{\omega_0}^{\bar{0}}\mathcal{D}_{\bar{0}} + 2{\omega_0}^{\bar{l}}\mathcal{D}_{\bar{l}} + 2e_{0\bar{0}}\mathcal{C}^{\bar{0}} + 2e_{0\bar{l}}\mathcal{C}_{\bar{l}} + \\ &+ [\mu q_i - 2{\omega_i}^{\bar{0}}]\epsilon^{ij}\partial_0 q_j - [2\omega_{i\bar{l}} + \mu e_{i\bar{l}}]\epsilon^{ij}\partial_0 e_{j\bar{l}} > . \end{aligned} \tag{18} $$

The interacting structure makes the constraints to become highly non linear, especially the differential ones $\mathcal{C}^{\bar{0}}$, $\mathcal{C}_{\bar{l}}$. The dynamical germ however remains stable, keeping its quadratic self-dual structure identical to the corresponding terms shown in eq. (7). The curved algebraic constraints are

$$\mathcal{D}_{\bar{0}} \equiv \epsilon^{ij}\partial_i q_j + {}_2 e e^{i\bar{j}}\omega_{i\bar{j}} = 0, \tag{19}$$

$$\mathcal{D}_{\bar{l}} \equiv \epsilon^{ij}\partial_i e_{j\bar{l}} + {}_2 e q_j (e^{j\bar{l}}\omega_2 - \omega^{\bar{l}j}) - {}_2 e e^h{}_{\bar{l}}\omega_h{}^{\bar{0}} = 0; \tag{20}$$

while the differential ones (stemming in the Bianchi identities eqs. (6)) take the aspect

$$E^{0\bar{0}} = \mathcal{C}^{\bar{0}} \equiv \epsilon^{ij}\partial_i \omega_j{}^{\bar{0}} - \mu\epsilon^{ij}\partial_i q_j + 2^{-1}{}_2 e \omega^{\bar{l}j}\omega_{j\bar{l}} - 2^{-1}{}_2 e \omega_2^2 = 0, \tag{21}$$

$$E^{0\bar{l}} = \mathcal{C}_{\bar{l}} \equiv \epsilon^{ij}\partial_i \omega_{j\bar{l}} + \mu\epsilon^{ij}\partial_i e_{j\bar{l}} + {}_2 e(\omega^{\bar{l}j} - e^{j\bar{l}}\omega_2)\omega_j{}^{\bar{0}} = 0. \tag{22}$$

To understand the dynamics in the curved case we choose the transverse gauge for both e_{ij} and q_i. This means taking $q^L = h^{LT} = h^{LL} = 0$ when we decompose q_i, e_{ij} acoording to eqs. (11), i.e.

$$q_j = (i\widehat{\partial})_j q^T, \tag{23a}$$

$$e_{i\bar{j}} \equiv (i\widehat{\partial})_i h^T{}_{\bar{j}} \equiv (i\widehat{\partial})_i (i\widehat{\partial})_j h^{TT} + (i\widehat{\partial})_i \widehat{\partial}_j h^{TL}. \tag{23b}$$

Moreover:

$$\omega_i{}^{\bar{0}} = (i\widehat{\partial})_i \mathrm{w}^T + \widehat{\partial}_i \mathrm{w}^L. \tag{24}$$

The dynamical germ becomes

$$\begin{aligned} &< [\mu q_i - 2\omega_i{}^{\bar{0}}]\epsilon^{ij}\partial_0 q_j - [2\omega_{i\bar{l}} + \mu e_{i\bar{l}}]\epsilon^{ij}\partial_0 e_{j\bar{l}} >= \\ &\qquad = < -2\mathrm{w}^L \dot{q}^T - 2\omega^{LL}\dot{h}^{TL} - 2\omega^{LT}\dot{h}^{TT} >, \end{aligned} \tag{25}$$

while the constraints (18)(19)(20)(21) acquire the form:

$$\mathcal{D}_{\bar{0}} \sim \rho q^T = {}_2 e \omega_2, \tag{26}$$

$$\mathcal{D}_j \sim \omega_j{}^{\bar{0}} = -e^{-1}(\rho h^T{}_{\bar{l}}) e_{j\bar{l}} + (i\widehat{\partial})_l q^T \cdot (\delta^l{}_j \omega_2 - \omega_j{}^l), \tag{27}$$

$$\mathcal{C}^{\bar{0}} \sim -\rho \mathrm{w}^T + \mu\rho q^T + 2^{-1}{}_2 e \omega^{\bar{l}j}\omega_{j\bar{l}} - 2^{-1}{}_2 e \omega_2^2 = 0, \tag{28}$$

$$\mathcal{C}_{\bar{l}} \sim -\rho\omega^T{}_{\bar{l}} - \mu\rho h^T{}_{\bar{l}} + {}_2 e(\omega^{\bar{l}j} - e^{j\bar{l}}\omega_2)\omega_j{}^{\bar{0}} = 0. \tag{29}$$

Observe that $e_{j\bar{l}}$ is given in terms of h^{TT}, h^{TL} as it is shown by eq. (23b). Consequently eq. (27) provides the value of $\omega_j{}^{\bar{0}}$ in terms of h^{TT}, h^{TL}, q^T, ω^{TT}, ω^{TL}, ω^{LT}, ω^{LL}. This is the role played by $\mathcal{D}_j$. We then go to $\mathcal{C}^{\bar{0}}$, eq. (27) which is regarded as an equation to solve for $q^T = q^T(h^{TT}, h^{TL}, \omega^{TT}, \omega^{TL}, \omega^{LT}, \omega^{LL})$. Once we obtain q^T, it is introduced back into $\mathcal{D}_j$ (27) which then yields ${\omega_j}^{\bar{0}} = {\omega_j}^{\bar{0}}(h^{TT}, h^{TL}, \omega^{TT}, \omega^{TL}, \omega^{LT}, \omega^{LL})$. This functional value of ${\omega_j}^{\bar{0}}$ is introduced back into $\mathcal{C}_{\bar{l}}$ (eq. (29)). From them we solve ω^{TT}, ω^{TL} in terms of the remaining four variables h^{TT}, h^{TL}, ω^{LT}, ω^{LL}. These values can also be substituted into the previous functionals q^T and ${\omega_j}^{\bar{0}}$ so we have ${\omega_j}^{\bar{0}}$, q^T, ω^{TT}, ω^{TL} expressed as functionals of (h^{TT}, h^{TL}, ω^{LT}, ω^{LL}). If we then insert all these expressions into $\mathcal{D}_{\bar{0}}$, we arrive to a functional equation which determines $\omega^{LL} = \omega^{LL}$ (h^{TT}, h^{TL}, ω^{LT}). And back again to the previous expressions we will get ${\omega_j}^{\bar{0}}$, q^T, ω^{TT}, ω^{TL}, ω^{LL} as functionals of (h^{TT}, h^{TL}, ω^{LT}).

After all this procedure is done we will have the unconstrained action in terms of the physical excitation h^{TT}, the canonical momenta $p \sim \mu h^{TL} - \omega^{LT}$ and an auxiliary variable ω^{LT}. Similarly to what we have explicitly seen in the linearized case, we conjeture that its field equation will be a constraint which can ve solved for ω^{LT} in terms of h^{TT} and the canonical momenta p, giving rise to the final, unconstrained canonical action. This point deserves a more detailed analysis.

Summing up, we have presented a second order, diffeomorphism invariant action containing a first order CS-term which contains one local degree of freedom corresponding to a propagating spin-2 massive excitation. There is a substancial difference between vector Chern-Simons gravity and topological massive gravity arising from the fact that here we have the Einstein action with the standard sign whereas in topological massive gravity the Einstein's action must be written with the opposite sign[1][3].

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