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Scalar field cosmologies: a dynamical systems study

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Abstract

Inhomogeneous G_2 solutions whose material content can be modelled by a scalar field are considered with a view towards their application as cosmological models. Some generic features of these spacetimes are reviewed and discussed, and dynamical systems theory is used to analyse the qualitative behaviour of this class of spacetimes; giving special attention to the equilibrium (fixed) points of the dynamical system and their associated metrics, and discussing their inflationary behaviour.

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1. Introduction

The dynamical systems approach has been consistently used to study the dynamics of various classes of (homogeneous) cosmological models [1], and more recently, that of inhomogeneous G_2 perfect fluid models; see [2] for an excellent and exhaustive study.

Scalar field spacetimes have been widely used in general relativity to model certain situations of interest in both the astrophysical (see [3, 4] and references cited therein) and the cosmological context [5], especially the so-called *inflationary scenarios* [6, 7].

In the present paper we study diagonal, separable orthogonally transitive G_2 cosmologies with an energy–momentum tensor corresponding to a scalar field. We review some results presented in previous articles and characterize and classify the possible solutions of this kind; finally, we carry out an analysis of the families of solutions thus obtained using dynamical systems theory. In particular, the fixed points are obtained and dealt with in some detail, showing that all of them correspond to self-similar solutions (that is, metrics admitting a homothety), and studying their inflationary behaviour (i.e., whether they inflate or not and in what regions of the spacetime).

In what follows, we shall denote the spacetime as (M, g), with M a connected four-dimensional Hausdorff C^k, $k \geqslant 2$, manifold, and g a Lorentz metric of signature +2. Greek indices and lowercase latin indices (both running from 0 to 3) will stand for coordinate and tetrad indices respectively (the latter associated with a pseudo-orthonormal tetrad $\{\theta^a\}$ defined so that $ds^2 = \eta_{ab}\theta^a\theta^b$). Einstein's field equations (EFE) will be written in an arbitrary coordinate chart as $G_{\mu\nu} = T_{\mu\nu}$, where $G_{\mu\nu}$ and $T_{\mu\nu}$ denote the coordinate components of the Einstein and the energy–momentum tensors respectively, the latter being assumed of the form:

$$T_{\mu\nu} = \Phi_\mu \Phi_\nu - \left(\tfrac{1}{2}\Phi^\alpha \Phi_\alpha + V(\Phi)\right) g_{\mu\nu}, \tag{1}$$

where $\Phi = \Phi(x^\alpha)$ is a real function on M, the *scalar field*; $V(\Phi)$ is the *potential* describing some kind of self-interaction, and $\Phi_\alpha = \Phi_{,\alpha}$ stand for the covariant components of the gradient of Φ. Thus, a spacetime (M, g) (or one of its open submanifolds) whose energy–momentum tensor is given by (1) will be referred to as a *scalar field spacetime*.

The paper is structured as follows: section 2 contains a summary of previous results on scalar field spacetimes; in section 3 we give the field equations for separable, orthogonally transitive G_2 models, and discuss some issues concerning separability, classify the various possible cases and write the EFE for each case as a first-order autonomous system of ordinary differential equations. All this is done in a completely general way; i.e., without any extra assumptions, in this respect our work extends and generalizes some interesting developments in this field already existing in the literature (see [8, 9] and references cited therein). In section 4 we discuss the different classes found in the preceding section, and carry out a dynamical systems study of some of them (the physically relevant ones); in doing so, we show that it is always possible, within the present framework, to introduce new variables in which the phase space is compact. Phase portraits exhibiting some typical behaviour are obtained and all the equilibrium points are found and classified. Finally, in section 5, the solutions corresponding to the fixed points are dealt with; showing that they are all self-similar and studying their behaviour in connection with inflation.

2. Scalar field spacetimes: generalities

In this section we review some general results regarding scalar field spacetimes (Segre type of the energy–momentum tensor, energy conditions, symmetries, etc) which are relevant to later developments. We have omitted the proofs for the sake of conciseness, but the reader is referred to [10] where details can be found. The following results can then be proven:

Theorem 1. *Given a scalar field spacetime, the manifold M can be decomposed as the disjoint union of the following three sets $M = T \cup S \cup \mathcal{F}$, where T (T-region) is the open region on which Φ_α is timelike, S (S-region) is also open with Φ_α spacelike and $\mathcal{F}$ is a closed set on which $\Phi^\gamma\Phi_\gamma = 0$ (including Φ = constant). Further, Φ^α is always an eigenvector of the energy–momentum tensor with corresponding eigenvalue $\lambda \equiv \frac{1}{2}\Phi^\gamma\Phi_\gamma - V(\Phi)$, and the Segre type [11] is, on each region:*

- *T-region: $\{1, (111)\}$ with eigenvalues λ and $\sigma \equiv -\frac{1}{2}\Phi^\alpha\Phi_\alpha - V(\Phi)$ (degenerate).*
- *S-region: $\{(1, 11)1\}$ with eigenvalues $\sigma \equiv -\frac{1}{2}\Phi^\alpha\Phi_\alpha - V(\Phi)$ (degenerate) and λ.*
- *$\mathcal{F}$ is the union of two sets, one in which $\Phi^\gamma\Phi_\gamma = 0$ and $\Phi_\alpha \neq 0$, Segre type $\{(2, 11)\}$ with degenerate eigenvalue $\nu \equiv -V(\Phi)$; and the other one where $\Phi_\alpha = 0$ and $V(\Phi)$ = constant, the Segre type then being $\{(1, 111)\}$ (Λ-term) [11]). If the energy–momentum tensor is analytic, this region has empty interior.*

The reader is also referred to [12, 13] for some interesting developments regarding spacetime decomposition in terms of the allowed Segre types in general relativity theory. See also [15] for corresponding spacetime decompositions in terms of Petrov types.

We also note in passing that, over the T-region, a scalar field spacetime is formally equivalent to a perfect fluid one with non-twisting 4-velocity $u^\alpha = (-\Phi^\gamma\Phi_\gamma)^{-1/2}\Phi^\alpha$ and density and pressure given by

$$\rho = -\tfrac{1}{2}\Phi^\gamma\Phi_\gamma + V(\Phi) \qquad \text{and} \qquad p = -\tfrac{1}{2}\Phi^\gamma\Phi_\gamma - V(\Phi) \tag{2}$$

a fact which has been used to analyse inflation in this class of spacetimes (see [6, 8], etc). See also [16] for an early discussion of this equivalence.

In view of the results in the previous theorem, it is now immediate to see what the dominant energy condition implies on a scalar field spacetime (see, for instance, [17]), thus we have

Theorem 2. *Given a scalar field spacetime, the dominant energy condition is satisfied if and only if* $V(\Phi) \geqslant 0$ *all over* M.

Further results concern isometries and homotheties in scalar field spacetimes, recall that a spacetime is said to admit an isometry or a homothety (generated respectively by a Killing vector (KV) or a homothetic vector (HV) $\vec{\xi}$) if

$$\mathcal{L}_{\vec{\xi}}g_{\mu\nu} = 2kg_{\mu\nu}, \tag{3}$$

where k is a constant that vanishes in the case of a KV, and is non-zero if $\vec{\xi}$ is a (proper) HV (in which case one can always re-scale $\vec{\xi}$ so as to set $k = 1$). Note that in both cases (KV or proper HV)

$$\mathcal{L}_{\vec{\xi}}T_{\mu\nu} = 0. \tag{4}$$

Theorem 3. *If a scalar field spacetime admits a HV (possibly KV),* $\vec{\xi}$, *it then follows:*

(1) $\mathcal{L}_{\vec{\xi}}\Phi_\alpha = 0$ *(hence* $\mathcal{L}_{\vec{\xi}}\Phi^\alpha = -2k\Phi^\alpha$*).*
(2) $\mathcal{L}_{\vec{\xi}}V(\Phi) = -2kV(\Phi)$.
(3) If $\vec{\xi}$ *is a proper HV, the only possible form for the potential* $V(\Phi)$ *is* $V(\Phi) = \Lambda\exp(\kappa\Phi)$*, where* Λ *and* $\kappa(\neq 0)$ *are real constants, and* $\mathcal{L}_{\vec{\xi}}\Phi = -2k/\kappa$.

Thus, if a scalar field spacetime is self-similar (that is, if it admits a proper homothety) the potential *must* necessarily be of the exponential type (including zero as a special case); thus, in particular one cannot have potentials of the form/containing terms m^2. This result can also be easily generalized to the case of complex scalar fields.

So far, the results presented here are completely general in the sense that we do not pre-suppose any geometry, to close this section we recall though a result also proven in [10] which holds for diagonal orthogonally-transitive G_2 spacetimes (type *B-ii* in Wainwright's classification [18]). For such spacetimes, the line element can be written as [18]

$$\mathrm{d}s^2 = \exp(2f_1)(-\mathrm{d}t^2 + \mathrm{d}x^2) + \exp(f_2)(\exp(2f_3)\,\mathrm{d}y^2 + \exp(-2f_3)\,\mathrm{d}z^2) \tag{5}$$

with $f_A = f_A(t,x)$ for $A = 1, 2, 3$ and the two KV generating the G_2 being $\vec{\xi} = \partial_y$ and $\vec{\eta} = \partial_z$. Taking then an orthonormal tetrad adapted to the orbits of the isometry group and their orthogonal complements (i.e. θ_2, θ_3 on the orbits of the group G_2 and θ_0, θ_1 on their complements) we have

Theorem 4. *The necessary and sufficient condition for an OT-*G_2 *spacetime to represent a scalar field with energy–momentum tensor given by (1) is that equations*

$$(G_{00} + G_{22})(G_{11} - G_{22}) - G_{01}^2 = 0 \tag{6}$$

$$G_{22} - G_{33} = 0, \qquad G_{23} = 0 \tag{7}$$

and

$$\left[\theta_t^0(G_{00}+G_{22})^{1/2}+\theta_t^1(G_{11}-G_{22})^{1/2}\right]_{,x}=\left[\theta_x^0(G_{00}+G_{22})^{1/2}+\theta_x^1(G_{11}-G_{22})^{1/2}\right]_{,t} \tag{8}$$

are satisfied.

See [19], [20] for a corresponding result in the case of perfect fluids, although note that in that case, inequations are needed besides equations similar to (6), (7) in order to fully characterize a perfect fluid satisfying the dominant energy condition.

3. Diagonal, separable OT-G_2 cosmologies

In this section we shall look into a subclass of the diagonal OT-G_2 scalar field cosmologies such as the ones described above (see (5)); namely those which are separable in coordinates adapted to the Killing vectors. Separability of the metric functions in a given geometric context is a concept which can be given a precise and invariant definition (see, for instance, [19]), in the current context this simply means that

$$f_A(t,x)=T_A(t)+X_A(x), \qquad A=1,2,3 \tag{9}$$

in the above metric (5).

3.1. Field equations: separability of the scalar field

As was shown in a previous work [10], the separability of the metric functions implies the separability of the scalar field (note that this was assumed as an extra condition in the existing literature, see [8, 9, 5] and references therein, whereas we actually showed that it must be so). Thus, from now on we put

$$\Phi_{,t}=f(t), \qquad \text{and} \qquad \Phi_{,x}=h(x) \tag{10}$$

and following [10] we write the field equations as

$$\dot\tau_1=\tfrac14\tau_2^2-\tau_3^2-\tfrac12 f^2-\tfrac12(k_1-k_2) \tag{11}$$

$$\dot\tau_2=2\tau_1\tau_2-2\tau_3^2-\tfrac12\tau_2^2-f^2-(k_1+k_2) \tag{12}$$

$$\dot\tau_3+\tau_2\tau_3=K \tag{13}$$

$$\lambda_1'=\tfrac14\lambda_2^2-\lambda_3^2-\tfrac12 h^2-\tfrac12(k_1-k_2) \tag{14}$$

$$\lambda_2'=2\lambda_1\lambda_2-2\lambda_3^2-\tfrac12\lambda_2^2-h^2+(k_1+k_2) \tag{15}$$

$$\lambda_3'+\lambda_2\lambda_3=K \tag{16}$$

$$\dot\tau_2+\tau_2^2-\left(\lambda_2'+\lambda_2^2\right)=2V(\Phi)\,\mathrm{e}^{2f_1} \tag{17}$$

$$\tau_1\lambda_2+\tau_2\left(\lambda_1-\tfrac12\lambda_2\right)-2\tau_3\lambda_3=fh, \tag{18}$$

where $\tau_A\equiv\dot T_A$, $\lambda_A\equiv X_A'(A=1,2,3)$, dots and primes stand for derivatives with respect to t and x respectively, and k_1, k_2 are constants.

Note the $t\leftrightarrow x$ discrete symmetry that these equations (and therefore their solutions) possess, that is, if one gets a solution to the above set of field equations for certain τ_A and λ_A, then another solution is obtained exchanging the τ_A with the λ_A (and (k_1,k_2) with $(-k_2,-k_1)$); note though that both solutions behave quite differently as the T- and S-regions are exchanged when t and x are exchanged. This discrete symmetry is peculiar to all OT-G_2 models (not only the separable ones), see [20] for details.

3.2. Three possibilities

Turning now our attention to (18) which is the 01 field equation; and assuming $h(x) \neq 0$, we can divide through by $h(x)$ and differentiate next with respect to x to get

$$\tau_1(\lambda_2/h)' + \tau_2\left(\lambda_1/h - \tfrac{1}{2}\lambda_2/h\right)' - 2\tau_3(\lambda_3/h)' = 0. \tag{19}$$

Thus, three possibilities arise depending on the maximum number of linearly independent functions of t amongst τ_1, τ_2 and τ_3: three independent functions of t and just one of x, two independent functions of t and two of x, and just one linearly independent function of t and three of x; the first and last cases referred to being equivalent on account of the $t \leftrightarrow x$ discrete symmetry that these solutions possess (although physically the solutions are very different in behaviour, see remark above).

Also note that the contracted Bianchi identity $\left(T^{\mu\nu}_{\ ;\nu} = 0\right)$ implies

$$\Box\Phi - \frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi} = 0 \qquad \text{i.e.,} \qquad \frac{\mathrm{d}V(\Phi)}{\mathrm{d}\Phi} = \mathrm{e}^{-2f_1}[-(\dot{f} + \tau_2 f) + (h' + \lambda_2 h)]. \tag{20}$$

Case 1. τ_1, τ_2 and τ_3 are three linearly independent functions of t, therefore

$$(\lambda_2/h)' = \left(\lambda_1/h - \tfrac{1}{2}\lambda_2/h\right)' = (\lambda_3/h)' = 0 \tag{21}$$

which is equivalent to

$$\lambda_A(x) = c_A h(x), \qquad c_A \in \mathbb{R} \quad \text{for} \quad A = 1, 2, 3. \tag{22}$$

Substitution of these into equations (16)–(18) yields

$$h' + c_2 h^2 = K/c_3 \tag{23}$$

$$2V(\Phi)\,\mathrm{e}^{2(T_1+X_1)} = \dot{\tau}_2 + \tau_2^2 - c_2(h' + c_2 h^2) \tag{24}$$

$$c_2\tau_1 + \left(c_1 - \tfrac{1}{2}c_2\right)\tau_2 - 2c_3\tau_3 = f, \tag{25}$$

substituting now (23) into (24), differentiating it with respect to x and recalling that $X'_A = \lambda_A$ and that $V_{,x} = V_{,\Phi}h$ we get $V_{,\Phi} + 2c_1 V = 0$ which readily implies

$$V(\Phi) = V_0 \exp(-2c_1\Phi), \qquad V_0 = \text{constant} \tag{26}$$

thus, we have shown that in this case the potential must have the exponential form.

Equations (14) and (15) yield relations amongst the constants c_A, k_1, k_2 and K; whereas the contracted Bianchi identity (20) is now identically satisfied.

Before we proceed, it is worthwhile noting that whenever $c_2 \neq 0$ one can always, by scaling t and x appropriately, set it equal to 1, i.e., $c_2 = 1$. In order to see this, define new variables $x' \equiv c_2 x$ and $t' \equiv c_2 t$, the line element (5) will read

$$\mathrm{d}s^2 = \frac{1}{c_2^2}\exp(2f_1)(-\mathrm{d}t'^2 + \mathrm{d}x'^2) + \exp(f_2)(\exp(2f_3)\,\mathrm{d}y^2 + \exp(-2f_3)\,\mathrm{d}z^2) \tag{27}$$

and a trivial redefinition of $f_1 = T_1 + X_1$ brings the metric to the form (5) with respect to the primed coordinates. We shall assume this in the following, but drop primes for convenience; the field equation (23) becomes then

$$h' + h^2 = K/c_3 \equiv \pm a^2, \tag{28}$$

where the constant a can be zero. This equation can be readily integrated out to give

$$h(x) = \begin{cases} a\tanh a(x - x_0) \\ (x - x_0)^{-1} \\ -a\tan a(x - x_0) \end{cases} \tag{29}$$

depending on whether we take the positive sign, $a = 0$ or the negative sign, respectively. Also note that x_0 can be set equal to zero by a trivial shift in the origin of the x coordinate, and we shall do so in what follows.

Assuming $c_2 = 1$, it follows from (14), (15):

$$c_1 = c_3^2 + \tfrac{1}{4}, \qquad c_2 = 1, \qquad k_1 = \mp a^2\left(c_3^2 - \tfrac{1}{4}\right), \qquad k_2 = \pm a^2\left(c_3^2 + \tfrac{3}{4}\right) \tag{30}$$

and the remaining field equations (those coming from (11)–(13), (17)) take the form

$$\dot\tau_1 = -\tfrac{1}{2}\tau_1^2 + \tfrac{1}{4}\left(\tfrac{7}{8} + c_3^2 - 2c_3^4\right)\tau_2^2 - \left(2c_3^2 + 1\right)\tau_3^2 - \left(c_3^2 - \tfrac{1}{4}\right)\tau_1\tau_2 \\ + 2c_3\tau_1\tau_3 + 2c_3\left(c_3^2 - \tfrac{1}{4}\right)\tau_2\tau_3 \pm a^2\left(c_3^2 + \tfrac{1}{4}\right) \tag{31}$$

$$\dot\tau_2 = -\tau_1^2 - \tfrac{1}{2}\left(\tfrac{9}{8} - c_3^2 + 2c_3^4\right)\tau_2^2 - 2\left(2c_3^2 + 1\right)\tau_3^2 - 2\left(c_3^2 - \tfrac{5}{4}\right)\tau_1\tau_2 \\ + 4c_3\tau_1\tau_3 + 4c_3\left(c_3^2 - \tfrac{1}{4}\right)\tau_2\tau_3 \mp a^2 \tag{32}$$

$$\dot\tau_3 + \tau_2\tau_3 = \pm a^2 c_3 \tag{33}$$

$$\dot\tau_2 + \tau_2^2 \mp a^2 = 2V(\Phi)\,\mathrm{e}^{2f_1}. \tag{34}$$

Note now that (31)–(33) constitute an autonomous system of first-order ODEs, whereas (34) gives, upon substitution of $\dot\tau_2$ by its value in terms of τ_A as given by (32), an algebraic equation containing τ_A and $V(\Phi)$. This allows the treatment of the system formed by (31)–(33) as a dynamical system, while (34) will then provide an equation for the phase space. We shall deal with this in the next section.

If $c_2 = 0$, from the field equations it is straightforward to see that

$$\tau_1 = -\frac{1}{4}\left(1 - \frac{c_1^2}{c_3^2}\right)\frac{1}{t} + \frac{1}{2}k_2 t, \qquad \tau_2 = \frac{1}{t}, \qquad \tau_3 = \frac{c_1}{2c_3}\frac{1}{t}, \qquad \text{and} \qquad f = 0 \tag{35}$$

and also that

$$k_1 = 0, \qquad h^2 = \frac{k_2}{2c_3^2 + 1}, \qquad \lambda_1 = c_1 h, \qquad \lambda_2 = 0 \tag{36}$$

$$\lambda_3 = c_3 h, \qquad V(\Phi) = 0, \qquad \Phi = \Phi_0 + \sqrt{\frac{k_2}{2c_3^2 + 1}}\,x \qquad (\Phi_0 = \text{constant}) \tag{37}$$

whence the line element can be easily written; further the regions T and $\mathcal{F}$ are empty and the energy–momentum tensor is everywhere of the Segre type $\{(1, 11)1\}$.

Case 2. Suppose now that one of the τ_As depends linearly on the other two, say,

$$\tau_1 = \left(a + \tfrac{1}{2}\right)\tau_2 + b\tau_3 \tag{38}$$

where a and b are arbitrary constants.

Substituting this into (18), dividing both members by $h(x)$ and differentiating with respect to x we get, after some trivial algebra,

$$f = k\tau_2 + k'\tau_3, \qquad \lambda_1 = \left(b\frac{k}{k'} - a\right)\lambda_2 - 2\frac{k}{k'}\lambda_3, \qquad h = \frac{b}{k'}\lambda_2 - \frac{2}{k'}\lambda_3, \tag{39}$$

where k, k' are constants of integration that we shall assume in principle to be both non-vanishing, and will study separately the special cases when one or the other constant (or both) is zero.

Recalling the expressions (17) and (20) which give the potential $V(\Phi)$ and its derivative with respect to the scalar field $\mathrm{d}V(\Phi)/\mathrm{d}\Phi$ respectively, and taking into account that $V_{,t} = V_{,\Phi} f$ and $V_{,x} = V_{,\Phi} h$ it readily follows $a = 0$ and $k' = 2bk$; hence,

$$f = k\tau_2 + k'\tau_3, \qquad h = \frac{1}{2k}\left(\lambda_2 - 4\frac{k}{k'}\lambda_3\right), \qquad \tau_1 = \frac{1}{2k} f, \qquad \lambda_1 = kh \tag{40}$$

and also

$$\frac{1}{2}\left(\dot{\tau}_2 + \tau_2^2\right)^{\cdot} = \left[\frac{1-2k^2}{2k}\left(\dot{\tau}_2 + \tau_2^2\right) + K\frac{k'^2 - 2}{k'}\right] f \tag{41}$$

$$\frac{1}{2}\left(\lambda_2' + \lambda_2^2\right)' = \left[\frac{2k^2-1}{2k}\left(\lambda_2' + \lambda_2^2\right) - K\frac{k'^2 - 2}{k'}\right] h \tag{42}$$

which can be integrated out to give

$$\dot{\tau}_2 + \tau_2^2 = \frac{2k}{1-2k^2} C\,\mathrm{e}^{2(1-2k^2)T_1} - K\frac{2k(k'^2-2)}{k'(1-2k^2)} \tag{43}$$

$$\lambda_2' + \lambda_2^2 = \frac{2k}{2k^2-1} C'\,\mathrm{e}^{\frac{2k^2-1}{k^2}X_1} - K\frac{2k(k'^2-2)}{k'(1-2k^2)}, \tag{44}$$

where C and C' are constants of integration (both positive), $k^2 \neq 1/2$, and we have taken into account equation (40).

Substituting now the above expressions in (17) and taking into account that from (10) and (40) it follows that

$$\Phi(t,x) = 2kT_1(t) + \frac{1}{k}X_1(x) + \Phi_0 \qquad (\Phi_0 = \text{constant})$$

one has

$$V(\Phi) = \frac{k}{1-2k^2}\left[V_0\,\mathrm{e}^{-2k\Phi} + V_1\,\mathrm{e}^{-\frac{1}{k}\Phi}\right], \tag{45}$$

where V_0 and V_1 are constants that depend on the previously defined C and C'. It is perhaps worthwhile noting at this point that potentials of the above type (i.e., combination of exponential potentials) have been considered in connection with assisted inflation in supergravity theories (see [28–30]).

One can now substitute this information back into the field equations (11)–(16) to get

$$k_1 = 2K\frac{k}{k'}, \qquad k_2 = \frac{1}{2}K\frac{k'}{k} \tag{46}$$

$$\dot{\tau}_2 = \frac{1}{2}(1-2k^2)\tau_2^2 - (2+k'^2)\tau_3^2 + \left(\frac{k'}{k} - 2kk'\right)\tau_2\tau_3 - \left(2K\frac{k}{k'} + \frac{K}{2}\frac{k'}{k}\right) \tag{47}$$

$$\dot{\tau}_3 + \tau_2\tau_3 = K \tag{48}$$

$$\lambda_2' = \frac{1}{4}\frac{2k^2-1}{k^2}\lambda_2^2 - 2\frac{k'^2+2}{k'^2}\lambda_3^2 - 2\left(2\frac{k}{k'} - \frac{1}{kk'}\right)\lambda_2\lambda_3 + \left(2K\frac{k}{k'} + \frac{K}{2}\frac{k'}{k}\right) \tag{49}$$

$$\lambda_3{}' + \lambda_2\lambda_3 = K. \tag{50}$$

In this case one gets two systems of autonomous differential equations; one formed by τ_2 and τ_3 and the other by λ_2 and λ_3. Also note that the phase spaces, one corresponding to the

τ_A and the other to the λ_A, can be obtained by substituting $\dot{\tau}_2$ and λ_2' in (43) and (44) by the right-hand sides of the above equations.

If one assumes now $k = 0$ and $k' \neq 0$ the analysis of all possible subcases, although straightforward, turns out to be rather tedious and lengthy; we shall omit the details and give only the conclusions, which can be summarized in the following list:

(1) $\tau_1 = \tau_3 = 0$, $\tau_2 =$ constant and $\lambda_1, \lambda_2, \lambda_3$ are also constant. In this case $V(\Phi) =$ constant.
(2) $\tau_2 = (t - t_0)^{-1}$ and τ_1, τ_3 are multiples of τ_2 whereas $\lambda_1, \lambda_2, \lambda_3$ are constant, then again one has $V(\Phi) =$ constant, with $V(\Phi) = 0$ in some well-defined cases.
(3) τ_1, τ_3 are multiples of τ_2 which is either one of the following

$$\tau_2 = c \tanh c(t - t_0), \qquad c \tan c(t_0 - t), \qquad \text{or else} \qquad (t - t_0)^{-1}$$

where c is a constant. The functions λ_1, λ_3 are multiples of λ_2 which has an expression analogous to that of τ_2, namely,

$$\lambda_2 = m \tanh m(x - x_0), \qquad m \tan m(x_0 - x), \qquad \text{or else} \qquad (x - x_0)^{-1} \tag{51}$$

m being also constant. In this case $V(\Phi)$ has a form similar to (45).
(4) τ_1, τ_3 are multiples of $\tau_2 = (t - t_0)^{-1}$ with $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 =$ constant, in this case the potential is zero, $V(\Phi) = 0$.
(5) $\lambda_1 = \lambda_2 = 0$ and $\lambda_3 =$ constant and $\tau_1 = \frac{1}{2}\tau_2 + b\tau_3$ and τ_2 and τ_3 satisfy

$$\dot{\tau}_2 = \tfrac{1}{2}\tau_2^2 + 2b\tau_2\tau_3 - (k'^2 + 2)\tau_3^2 \qquad \dot{\tau}_3 = -\tau_2\tau_3.$$

The potential satisfies in this case $V(\Phi) =$ constant.

Note that in all of the above cases τ_2 and τ_3 are no longer linearly independent, therefore being particular instances of case 3 below.

Assuming $k \neq 0$ and $k' = 0$ leads to conclusions similar to those above, while assuming $k = k' = 0$ leads directly to special cases of case 1; namely, the functions λ_A are proportional amongst themselves, λ_2 satisfying $\lambda_2' + \lambda_2^2 =$ constant, and therefore

$$\lambda_2(x) = \begin{cases} -a \tan a(x - x_0) \\ \frac{1}{x - x_0} \\ a \tanh a(x - x_0) \end{cases}$$

depending on whether the constant is negative, zero or positive; further, $V(\Phi) =$ constant and also $f(t)h(x) = 0$, and one then gets explicit expressions for $f(t)$ and $h(x)$. Again, the constants x_0 and t_0 can be trivially set equal to zero by suitably redefining x and t.

Case 3. There is just one linearly independent function amongst τ_1, τ_2 and τ_3. If this is the case, then λ_1, λ_2 and λ_3 are three linearly independent functions, and because of the $t \leftrightarrow x$ symmetry of the field equations, this case is formally equivalent to case 1 (see the remarks at the end of the previous section). The potential also has the exponential form.

4. Case 1. A dynamical systems approach

We concentrate now on case 1 above, that is, the three functions τ_A, $A = 1, 2, 3$ are assumed to be linearly independent, whereas $\lambda_1(x) = c_1 h(x)$, $\lambda_2(x) = h(x)$, $\lambda_3(x) = c_3 h(x)$ with $c_1 = c_3^2 + 1/4$ and $h(x)$ given by (29), the τ_A satisfy then the autonomous system (31)–(33) to be complemented with (34). Assume $c_3 \neq 0$ and define the following variables:

$$\tau \equiv \frac{1}{c_3}\tau_3, \qquad \psi \equiv 2\tau_1 - \tau_2 - 2\left(c_1 + \frac{1}{2}\right)\tau \tag{52}$$

the system (31)–(33) can then be rewritten in terms of the new variables as

$$\dot{\psi} + \tau_2 \psi = 0 \tag{53}$$

$$\dot{\tau} + \tau_2 \tau = \pm a^2 \tag{54}$$

$$\dot{\tau}_2 = -\left[(1 - c_1)(\tau_2 - \tau) - \tfrac{1}{2}\psi\right]^2 - 2c_3^2(\tau_2 - \tau)^2 + \tau_2^2 \mp a^2 \tag{55}$$

whereas (34) takes the form, on account of (55), of

$$\tfrac{1}{2}\left[(1 - c_1)(\tau_2 - \tau) - \tfrac{1}{2}\psi\right]^2 + c_3^2(\tau_2 - \tau)^2 + V\,\mathrm{e}^{2f_1} = \tau_2^2 \mp a^2. \tag{56}$$

Note that this defines an unbounded phase space in the variables (ψ, τ, τ_2).

If $c_3 = 0$ then $c_1 = 1/4$, $f = \tau_1 - \frac{1}{4}\tau_2$ and we can proceed in a similar way defining

$$\bar{\psi} \equiv 2\tau_1 - \tau_2, \tag{57}$$

the system (31)–(33) can now be rewritten as

$$\dot{\bar{\psi}} + \tau_2 \bar{\psi} = \pm\tfrac{3}{2}a^2 \tag{58}$$

$$\dot{\tau}_3 + \tau_2 \tau_3 = 0 \tag{59}$$

$$\dot{\tau}_2 = -\tfrac{1}{4}\left(\bar{\psi} - \tfrac{3}{2}\tau_2\right)^2 + \tau_2^2 - 2\tau_3^2 \mp a^2, \tag{60}$$

now, on substituting (60) above into (34) we get

$$\tfrac{1}{2}\left[\tfrac{1}{2}\left(\bar{\psi} - \tfrac{3}{2}\tau_2\right)\right]^2 + \tau_3^2 + V\,\mathrm{e}^{2f_1} = \tau_2^2 \mp a^2 \tag{61}$$

which, as in the general case $c_3 \neq 0$, defines an unbounded phase space in the variables $(\bar{\psi}, \tau_3, \tau_2)$.

In order to proceed we have to distinguish the three possible cases; namely $h' + h^2 = a^2$, $h' + h^2 = -a^2$ or $h' + h^2 = 0$ (negative sign, positive sign or zero respectively in equation (56) or (61)).

4.1. Case $h' + h^2 = a^2$

Consider now the system (53)–(55) along with equation (56) whenever the plus sign holds in (54) (the minus sign holding then in (55) and (56)). Define next a new time variable, say T as follows:

$$\frac{\mathrm{d}}{\mathrm{d}T} \equiv \frac{1}{\tau_2}\frac{\mathrm{d}}{\mathrm{d}t} \tag{62}$$

and denote derivatives with respect to this new variable with a prime (not to be confused with a derivative with respect to x, which does not appear any longer in this context, as the x-dependence of the metric functions and scalar field is completely fixed), the system (53)–(55) becomes

$$\psi' + \psi = 0 \tag{63}$$

$$\tau' + \tau = \frac{a^2}{\tau_2} \tag{64}$$

$$\frac{\tau_2'}{\tau_2} = -\frac{1}{\tau_2^2}\left[(1 - c_1)(\tau_2 - \tau) - \frac{1}{2}\psi\right]^2 - 2c_3^2\frac{1}{\tau_2^2}(\tau_2 - \tau)^2 + 1 - \frac{a^2}{\tau_2^2} \tag{65}$$

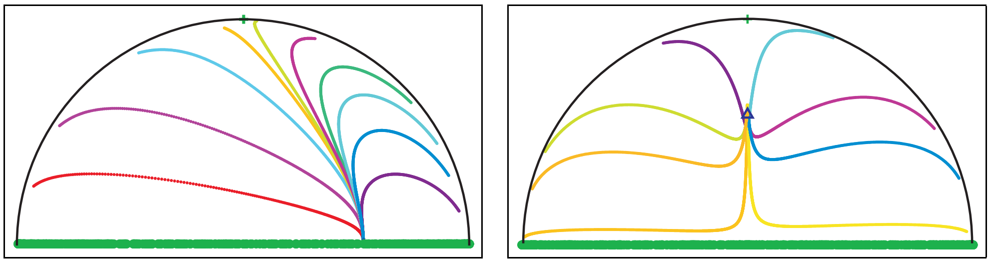


Figure 1. These figures show some phase-space diagrams for two different values of the parameter c_1: $c_1 = 0.26$ (left), $c_1 = 1$ (right). The variable E is plotted against the vertical axis, whereas the horizontal axis represents a projection of the plane AB. The solid black curve represents the surface of the physical phase space $(A^2 + B^2 + E^2 = 1)$ and the grey line at the bottom the set of fixed points $A^2 + B^2 = 1$.

dividing now (56) through by τ_2^2 and rearranging terms one gets

$$\frac{1}{2}\frac{\left[(1-c_1)(\tau_2-\tau)-\frac{1}{2}\psi\right]^2}{\tau_2^2}+\frac{c_3^2(\tau_2-\tau)^2}{\tau_2^2}+\frac{V\mathrm{e}^{2f_1}}{\tau_2^2}+\frac{a^2}{\tau_2^2}=1. \tag{66}$$

One can now define new variables as

$$A \equiv \frac{1}{\sqrt{2}}\frac{\left[(1-c_1)(\tau_2-\tau)-\frac{1}{2}\psi\right]}{\tau_2}, \qquad B \equiv \frac{c_3(\tau_2-\tau)}{\tau_2}, \qquad E \equiv \frac{a}{\tau_2} \tag{67}$$

now, since the potential is always positive, we have that our phase space is contained in the sphere

$$A^2 + B^2 + E^2 \leqslant 1 \tag{68}$$

and is therefore compact; whereas the system, in terms of these new variables, reads

$$A' = \sqrt{2}(1 - c_1 - \sqrt{2}A)[1 - A^2 - B^2 - E^2] - AE^2 \tag{69}$$

$$B' = 2(c_3 - B)[1 - A^2 - B^2 - E^2] - BE^2 \tag{70}$$

$$E' = -2E[1 - A^2 - B^2 - E^2] + E(1 - E^2). \tag{71}$$

Note the symmetry $E \to -E$, thus we shall restrict our analysis to the case $E \geqslant 0$.

It is easily verified that the subset $A^2 + B^2 + E^2 = 1$ constitutes an invariant set of the system. The metrics corresponding to this set are

$$\begin{aligned}\mathrm{d}s^2 = {} & (\cosh at - 1)^{2\alpha} \sinh^{2(c_1+1-\alpha)} at \cosh^{2c_1} ax[-\mathrm{d}t^2 + \mathrm{d}x^2] \\ & + \sinh at \cosh ax[(\cosh at - 1)^{2\beta} \sinh^{2(c_3-\beta)} at \cosh^{2c_3} ax\, \mathrm{d}y^2 \\ & + (\cosh at - 1)^{-2\beta} \sinh^{-2(c_3-\beta)} at \cosh^{-2c_3} ax\, \mathrm{d}z^2],\end{aligned} \tag{72}$$

where α, β are constants. The scalar field corresponding to these solutions is

$$\Phi(t, x) = \ln \Phi_0(\cosh at - 1)^{\alpha-2c_3\beta} \sinh^{1-(\alpha-2c_3\beta)} at \cosh ax \tag{73}$$

and the potential is $V(\Phi) = 0$.

We next perform the usual fixed-point analysis and collect the results in table 1.

In figure 1, we show a few projections of the orbits of the above dynamical system for some particular values of the parameters so as to illustrate the evolution of the trajectories in the phase space.

Table 1. We shall label the points I, II, III and IV respectively. Note that the points I to III are isolated and that, for the point II, it must be $c_1^2 \leqslant \frac{3}{2}$ since otherwise we would be outside the phase space defined by equation (68). The fixed point set IV consists of non-isolated fixed points.

Fixed point	$A^2+B^2+E^2$	Eigenvalues	Type
$A=0$ $B=0$ $E=1$	1	-1 -1 2	Saddle
$A=\frac{1}{\sqrt{2}}(1-c_1)$ $B=c_3$ $E=0$	$\frac{1}{2}\left(\frac{1}{2}+c_1^2\right)$	$c_1^2-\frac{3}{2}$ $c_1^2-\frac{1}{2}$	If $\frac{3}{2}\geqslant c_1^2>\frac{1}{2}$ saddle, if $c_1^2<\frac{1}{2}$ sink
$A=\sqrt{2}\frac{1-c_1}{1+2c_1^2}$ $B=\frac{\sqrt{4c_1-1}}{1+2c_1^2}$ $E=\sqrt{\frac{2c_1^2-1}{2c_1^2+1}}$	$\frac{c_1^2}{\frac{1}{2}+c_1^2}$	-1 $\frac{1}{2}\left(-1+\sqrt{\frac{9-14c_1^2}{2c_1^2+1}}\right)$ $\frac{1}{2}\left(-1-\sqrt{\frac{9-14c_1^2}{2c_1^2+1}}\right)$	If $c_1^2>\frac{1}{2}$ sink, if $c_1^2\leqslant\frac{1}{2}$ $\nexists$
$A^2+B^2=1$ $E=0$	1	0 1 $2\sqrt{2}\sqrt{1-B^2}(c_1-1)-2B\sqrt{4c_1-1}$	Sources

The evolution can be described as follows:

- $c_1^2 \leqslant \frac{1}{2}$. The trajectories set off from any point and get to the sink (fixed point III).
- $c_1^2 > \frac{1}{2}$. A sink appears (fixed point II) where all the trajectories get to as t evolves. In this case, the fixed point II turns out to be a saddle point.

For the special case $c_3 = 0$, we turn our attention to equations (58)–(60) for the case in which the plus sign holds in (58) and the minus sign in (60) and (61), and proceed in a similar way: define a new time variable T as in (62) and variables P, Q, E as follows:

$$P \equiv \frac{1}{2\sqrt{2}}\frac{\bar{\psi}-\frac{3}{2}\tau_2}{\tau_2}, \qquad Q \equiv \frac{\tau_3}{\tau_2}, \qquad E \equiv \frac{a}{\tau_2} \tag{74}$$

the system in these new variables reads:

$$P' = \frac{3}{4\sqrt{2}}E^2 + \left(P + \frac{3}{4\sqrt{2}}\right)(2P^2 + 2Q^2 + E^2 - 1) \tag{75}$$

$$Q' = Q(2P^2 + 2Q^2 + E^2 - 2) \tag{76}$$

$$E' = E(2P^2 + 2Q^2 + E^2 - 1) \tag{77}$$

which also exhibits the symmetry $E \to -E$ and $Q \to -Q$ as well. All the fixed points are isolated in this case and they turn out to be: I′: ($P = \pm 1/\sqrt{2}$, $Q = 0$, $E = 0$) (saddles), II′: ($P = -3/(4\sqrt{2})$, $Q = 0$, $E = 0$) (sink) and III′: ($P = -3/(4\sqrt{2})$, $Q = \pm\sqrt{23/32}$, $E = 0$) (sources).

In all of the above developments, we have been implicitly assuming that $\tau_2 \neq 0$. If $\tau_2 = 0$, the functions τ_A are not linearly independent; further, it follows from the original system (11)–(18) that τ_1 and τ_3 are linearly dependent and this case is included in case 2 above and has to be treated separately. Nevertheless, if it is assumed that the system (53)–(55) holds, one has $\psi = \tau = 0$ and $a = 0$; which in turn implies $\tau_1 = \tau_2 = \tau_3 = 0$ (the metric is then static), $\Phi = \Phi(x)$ and also $V(\Phi) = 0$.

4.2. Case $h' + h^2 = 0$

This case can be solved exactly up to a quadrature. We rewrite the system for this case as

$$\dot{\psi} + \tau_2\psi = 0$$
$$\dot{\tau} + \tau_2\tau = 0$$
$$\dot{\tau_2} = -\left((1 - c_1)(\tau_2 - \tau) - \frac{\psi}{2}\right)^2 - 2c_3^2(\tau_2 - \tau)^2 + \tau_2^2.$$

From the first two equations above we get $\psi = c\tau$ and the last equation becomes

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}t} = A\tau_2^2 + B\tau_2\tau + D\tau^2 \tag{78}$$

with

$$A = \tfrac{1}{2} - c_1^2, \qquad B = 1 + 2c_1^2 + c - cc_1, \qquad D = -\left(c + \tfrac{1}{2} + \left(c_1 - \tfrac{1}{2}c\right)^2\right). \tag{79}$$

Now, since $\dot{\tau} = -\tau_2\tau$, it follows

$$\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau} = -A\frac{\tau_2}{\tau} - B - D\frac{\tau}{\tau_2} \tag{80}$$

and then

$$\frac{\mathrm{d}}{\mathrm{d}(\ln\tau)}\left(\frac{\tau_2}{\tau}\right) = \left(\frac{\mathrm{d}\tau_2}{\mathrm{d}\tau} - \frac{\tau_2}{\tau}\right)$$

thus, equation (80) implies

$$\frac{\mathrm{d}\left(\frac{\tau_2}{\tau}\right)}{\mathrm{d}(\ln\tau)} = -(A+1)\frac{\tau_2}{\tau} - B - D\frac{\tau}{\tau_2}. \tag{81}$$

Putting now $A + 1 \equiv M$ and $y \equiv \tau_2/\tau$, we get from the above equation

$$\int \frac{y\,\mathrm{d}y}{My^2 + By + D} = -\int \mathrm{d}(\ln\tau)$$

and then

$$\frac{1}{2M}\ln(My^2 + By + D) - \frac{B}{2M}\int \frac{\mathrm{d}y}{My^2 + By + D} = \ln\frac{\tau_0}{\tau}. \tag{82}$$

Now, the integral appearing in the left-hand side of the above equation has a value which depends on the sign of the discriminant $\Delta \equiv B^2 - 4MD$; as it turns out, $\Delta = 2\left(\frac{5}{4} - c_1\right)c^2 + 8(1 - c_1)c + 4\left(1 + 2c_1^2\right)$, which can have either sign for different values of $c_1 (\geqslant 1/4)$ and c. If $\Delta > 0$ then

$$\int \frac{\mathrm{d}y}{My^2 + By + D} = \frac{1}{\sqrt{\Delta}}\ln\frac{2My + B - \sqrt{\Delta}}{2My + B + \sqrt{\Delta}}$$

while $\Delta < 0$ yields

$$\int \frac{\mathrm{d}y}{My^2 + By + D} = \frac{2}{\sqrt{-\Delta}}\arctan\frac{2My + B}{\sqrt{-\Delta}}.$$

The special case $\Delta = 0$, which corresponds to

$$c = \frac{2}{4c_1 - 5}\left[4 - 4c_1 \pm \sqrt{6 - 24c_1 - 4c_1^2 + 16c_1^3}\right]$$

leads to

$$\int \frac{\mathrm{d}y}{My^2 + By + D} = \int \frac{y\,\mathrm{d}y}{M\left(y + \frac{B}{2M}\right)^2} = -\int \mathrm{d}(\ln\tau)$$

that is

$$\frac{B}{2My + B} + \ln\frac{2My + B}{2M} = M\ln\frac{\tau_0}{\tau}.$$

4.3. Case $h' + h^2 = -a^2$

Consider again the system (53)–(55) along with equation (56) whenever the minus sign holds in (54) and the plus sign holds in both (55) and (56). Rewriting (56) as

$$\tfrac{1}{2}\left[(1 - c_1)(\tau_2 - \tau) - \tfrac{1}{2}\psi\right]^2 + c_3^2\tau_2^2 + c_3^2\tau^2 + V\,\mathrm{e}^{2f_1} = \tau_2^2 + 2c_3^2\tau_2\tau + a^2, \quad (83)$$

one has that both sides of the equation are positive and then, dividing through by $\tau_2^2 + 2c_3^2\tau_2\tau + a^2$ and taking into account that $V\,\mathrm{e}^{2f_1} \geqslant 0$ one finally gets for the phase space

$$A^2 + B^2 + E^2 \leqslant 1, \quad (84)$$

that is, it is compact in the new variables A, B and E which in this case are defined as

$$A = \frac{\frac{1}{\sqrt{2}}\left[(1 - c_1)(\tau_2 - \tau) - \frac{1}{2}\psi\right]}{\sqrt{\tau_2^2 + 2c_3^2\tau_2\tau + a^2}}, \qquad B = \frac{c_3\tau}{\sqrt{\tau_2^2 + 2c_3^2\tau_2\tau + a^2}},$$
$$E = \frac{c_3\tau_2}{\sqrt{\tau_2^2 + 2c_3^2\tau_2\tau + a^2}}. \quad (85)$$

Following now a similar procedure to that in the case $+a^2$, we define a new time variable as follows:

$$\frac{\mathrm{d}}{\mathrm{d}T} \equiv \frac{1}{\sqrt{\tau_2^2 + 2c_3^2\tau_2\tau + a^2}}\frac{\mathrm{d}}{\mathrm{d}t} \quad (86)$$

and denoting again with a prime the derivative with respect to this new time variable T, the autonomous system reads in the new (compact phase space) variables:

$$A' = \frac{1}{\sqrt{2}}\left\{(1 - c_1)\left[2 - 2A^2 - 2B^2 - 2E^2 + \frac{1}{2c_3^2}E(B - E)\right] - \frac{1}{\sqrt{2}}\frac{1}{c_3}AE\right\}$$
$$- A\left\{\frac{1}{c_3}E[1 + EB - 2A^2 - 2B^2 - E^2] + c_3B[1 + 2EB - 2A^2$$
$$- 2B^2 - E^2]\right\} - c_3EA(EB - 1), \quad (87)$$

$$B' = c_3[EB(3 - EB) - 1] - \frac{1}{c_3}E(B - E) - B\left\{\frac{1}{c_3}E[1 + EB - 2A^2 - 2B^2 - E^2]\right.$$
$$\left. + c_3B[1 + 2EB - 2A^2 - 2B^2 - 2E^2]\right\}, \quad (88)$$

$$E' = c_3(1 - EB)[1 + 2EB - 2A^2 - 2B^2 - 2E^2]$$
$$- \frac{1}{c_3}E^2\left[1 + EB\left(1 + c_3^2\right) - 2A^2 - 2B^2 - E^2 - c_3^2\right]. \quad (89)$$

We will not take the study of this case any further because the metric is spatially periodic; note $\lambda_2 = -a\tan(ax)$ and therefore $\mathrm{e}^{X_2} = \cos(ax)$, and this also implies signature problems; thus we regard this case as not physically interesting.

5. The fixed-point solutions

In this section we shall study in some detail the solutions corresponding to the fixed points I, II, III (isolated) and IV (non-isolated) (see table 1); and also those corresponding to I′, II′ and III′ (special case $c_3 = 0$).

First of all, note that the points II and IV (and also I′, II′ and III′) have $E = 0$, and it is then easy to see, from (63)–(65) together with (67), that $a = 0$ necessarily and therefore $h(x) = x^{-1}$; thus, strictly speaking, they are included in the subcase 4.2, but we shall study them here as they are fixed points of the system arising in case 4.1.

Next, for the isolated points I and III, the functions τ_A are, in fact, (non-zero) constants. Putting $\tau_1 = \alpha, \tau_2 = \beta$ and $\tau_3 = \gamma$, and taking (5) into account; the line element (after rescaling coordinates) can be written as

$$\begin{aligned} \mathrm{d}s^2 = \mathrm{e}^{2\alpha t} \cosh^{2c_1} ax(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{e}^{2\beta t} \cosh ax (\mathrm{e}^{2\gamma t} \cosh^{2c_3} ax\, \mathrm{d}y^2 \\ + \mathrm{e}^{-2\gamma t} \cosh^{-2c_3} ax\, \mathrm{d}z^2). \end{aligned} \tag{90}$$

It is then very easy to see from the homothetic equations (3) specialized to this metric that the vector field

$$\vec{X} = \frac{k}{2\alpha}\partial_t + k\left[1 - \frac{1}{\alpha}(\beta + 2\gamma)\right] y\partial_y + k\left[1 - \frac{1}{\alpha}(\beta - 2\gamma)\right] z\partial_z \tag{91}$$

is a homothetic vector field for it with homothetic constant $k \neq 0$. This is so for any value of the constants α, β and γ. Also recall (see section 3.2) that the potential is $V(\Phi) = V_0 \exp(-2c_1\Phi)$, and the scalar field is given by

$$\Phi(t, x) = \ln \Phi_0\, \mathrm{e}^{f_0 t} \cosh ax, \qquad f_0 = \alpha + \left(c_1 - \tfrac{1}{2}\right)\beta - 2c_3\gamma \qquad \text{and} \qquad \Phi_0 = \text{constant}.$$

This solution coincides precisely with that given in equation (37) of [14] (see also [12]) with the following identifications $H = \exp(c_2 X), 2\alpha = 2\psi_0 + \phi_0, 2\beta = \phi_0, \gamma = \varphi_0$, and $c_1 = \bar{c}_1/c_2, c_3 = \bar{c}_3/c_2$, where the barred constants stand for those appearing in [14] without bar.

For the fixed point II we have

$$\tau_1 = \frac{1}{\left(2c_1^2 - 1\right)t}, \qquad \tau_2 = \frac{2}{\left(2c_1^2 - 1\right)t}, \qquad \tau_3 = 0, \qquad \text{and} \qquad h = \frac{1}{x} \tag{92}$$

and the line element reads[1] (after suitably re-scaling coordinates)

$$\mathrm{d}s^2 = A^2 t^{\frac{1}{c_1^2 - 1/2}} x^{2c_1} \left\{[-\mathrm{d}t^2 + \mathrm{d}x^2] + x^{1-2c_1}\left[x^{2\sqrt{c_1 - 1/4}}\,\mathrm{d}y^2 + x^{-2\sqrt{c_1 - 1/4}}\,\mathrm{d}z^2\right]\right\} \tag{93}$$

where A is a scaling constant and c_1 is a parameter restricted only by $c_1 \geqslant 1/4$. The scalar field and the potential are

$$\ln \Phi_0 x\left[\left(c_1^2 - 1/2\right)t\right]^{\frac{2c_1}{2c_1^2 - 1}}, \qquad V = V_0 \exp(-2c_1\Phi)$$

Φ_0, V_0 being constants.

This solution is also self-similar, with homothetic vector (corresponding to homothetic constant k) given by

$$\vec{X} = \frac{2k}{1 + 2c_1}\left(t\partial_t + x\partial_x + \left(c_1 - \sqrt{c_1 - 1/4}\right)y\partial_y + \left(c_1 + \sqrt{c_1 - 1/4}\right)z\partial_z\right).$$

[1] Note a typo following equation (61) in [10]: instead of $\ln a(x - x_0)$ it should just be $a(x - x_0)$ and a can be absorbed by the rescaling of the coordinates.

Also, if one considers the following static spacetime, with a metric conformal to the above given by

$$\mathrm{d}\bar{s}^2 = A^2\big[x^{2c_1}(-\mathrm{d}t^2 + \mathrm{d}x^2) + x\big(x^{2\sqrt{c_1-1/4}}\,\mathrm{d}y^2 + x^{-2\sqrt{c_1-1/4}}\,\mathrm{d}z^2\big)\big] \qquad (94)$$

it is easy to see that it also corresponds to a scalar field $\Phi(x) = \ln \Phi_0 x$ (Φ_0 = constant) with potential identically zero.

This solution (its T-region only, see below) is included, except for the scale factor A, in a two-parameter family of solutions that appears in [20], and also in [14] (see equation (39) of that reference). In both cases the metric was found assuming a perfect fluid content for the spacetime and the existence of a conformal Killing vector (two in fact, besides the HV) in the former case. As a perfect fluid it has no barotropic equation of state (we refer the reader to [20] for further details). The two conformal Killing vectors that the metric admits are $\vec{X}_1 = \partial_t$ which commutes with the two KVs generating the G_2, and $\vec{X}_2 = 2t\partial_t + 2x\partial_x + (2c_1 - 2\sqrt{c_1 - 1/4} + 1)y\partial_y + (2c_1 + 2\sqrt{c_1 - 1/4} + 1)z\partial_z$ which does not.

Note that for the special value $c_1 = 1/4$ the spacetime admits a further Killing vector (namely $\vec{\zeta} = y\partial_z - z\partial_y$) which acts on the same (flat) two-dimensional orbits as $\vec{\xi} = \partial_y$ and $\vec{\eta} = \partial_z$ (i.e. it is a LRS spacetime). The line element reads in this case

$$\mathrm{d}s^2 = A^2 t^{-16/7}[-x^{1/2}\,\mathrm{d}t^2 + x^{1/2}\,\mathrm{d}x^2 + x(\mathrm{d}y^2 + \mathrm{d}z^2)] \qquad (95)$$

and although the metric functions exist for both positive and negative values of t it only has the correct signature for $t > 0$, as $x > 0$ necessarily. For further comments on this solution, see [10].

For the set (IV) of non-isolated fixed points $A^2 + B^2 = 1$, the constant a is necessarily 0 (see remarks above). We put $A = \cos\varphi$ and $B = \sin\varphi$ for some parameter $\varphi \in [0, 2\pi)$ and get, from (31)–(34)

$$\tau_1 = \frac{\alpha}{t}, \qquad \tau_2 = \frac{1}{t}, \qquad \tau_3 = \frac{\beta}{t}, \qquad \text{and} \qquad h = \frac{1}{x} \qquad (96)$$

with $\alpha = 5/4 + c_3^2 - 2c_3\sin\varphi - \sqrt{2}\cos\varphi$ and $\beta = c_3 - \sin\varphi$. It follows then from (5) that the line element for these solutions is of the form

$$\mathrm{d}s^2 = A^2\{t^{2\alpha}x^{2c_1}[-\mathrm{d}t^2 + \mathrm{d}x^2] + tx[t^{2\beta}x^{2c_3}\,\mathrm{d}y^2 + t^{-2\beta}x^{-2c_3}\,\mathrm{d}z^2]\} \qquad (97)$$

where A is a constant and the coordinates have been scaled so as to absorb other non-essential constants.

This metric is also self-similar, its corresponding HVF being:

$$X = \frac{k}{1 + c_1 + \alpha}[t\partial_t + x\partial_x + (c_1 - c_3 + \alpha - \beta)y\partial_y + (c_1 + c_3 + \alpha + \beta)z\partial_z].$$

The corresponding scalar field and potential are:

$$\Phi(t, x) = \ln \Phi_0 x t^{1-\sqrt{2}\cos\varphi}, \qquad V(\Phi) = 0.$$

As for the fixed point I′ ($P = \pm 1/\sqrt{2}$, $Q = E = 0$), it is easy to see that $\tau_1 = \tau_2 = \tau_3 = 0$, and since $a = c_3 = 0$, $c_1 = 1/4$ one has (recall $h = \lambda_2 = 1/x$, $\lambda_1 = c_1 h$)

$$\mathrm{d}s^2 = A^2[x^{-1/2}(-\mathrm{d}t^2 + \mathrm{d}x^2) + x(\mathrm{d}y^2 + \mathrm{d}z^2)] \qquad (98)$$

and then $\Phi(t, x) = \ln \Phi_0 x$, $V(\Phi) = 0$, where A and Φ_0 are constants. Note that this metric coincides with (94) for $c_1 = 1/4$; it is therefore static (∂_t is a hypersurface orthogonal timelike KV) and has got, besides ∂_t, another KV $\vec{\zeta} = y\partial_z - z\partial_y$ (which acts on the flat two-dimensional orbits coordinated by y and z; that is, the spacetime is LRS). Further, the spacetime also admits the HVF (homothetic constant k) $\vec{X} = k/5(4t\partial_t + 4x\partial_x + 3y\partial_y + 3z\partial_z)$ and it consists of a single S-region ($f^2 - h^2 = -1/x^2 < 0$). We shall not take its study any further, as it is uninteresting as a cosmological solution.

In the case of the fixed point II′, one has $\tau_1 = -8/7t^{-1}$, $\tau_2 = -16/7t^{-1}$ and $\tau_3 = 0$, and the line element takes then the form (95); i.e., it is a special case of the fixed point II.

Finally, for the fixed point III′ one has

$$\tau_1 = \frac{1}{2t}, \qquad \tau_2 = \frac{1}{t}, \qquad \tau_3 = \sqrt{\frac{23}{32}}\frac{1}{t}, \qquad h = \frac{1}{x} \tag{99}$$

and the metric reads then

$$ds^2 = A^2\left[tx^{1/2}(-dt^2 + dx^2) + tx\left(t^{\sqrt{23/8}}\,dy^2 + t^{-\sqrt{23/8}}\,dz^2\right)\right] \tag{100}$$

which is also self-similar with HVF, of associated homothetic constant k, $\vec{X} = k/7(4t\partial_t + 4x\partial_x + 3y\partial_y + 3z\partial_z)$. The scalar field and potential corresponding to this solution are $\Phi(t,x) = \ln \Phi_0 t^{1/4} x$, $\Phi_0 =$ constant and $V(\Phi) = 0$.

5.1. Inflation

We next address the question of whether any of these solutions inflates or not. Inflation in scalar field G_2-models was discussed in [10], showing that in both T- and S-regions there is a geometrically well-defined unique timelike congruence; namely, in the T-region the one parallel to the gradient of the scalar field; and in the S-region the unique timelike eigendirection of the energy–momentum tensor (in this case there is a whole hypersurface of timelike eigenvectors) that is orthogonal to the group orbits and also to the gradient of the scalar field $\Phi_{,\alpha}$. Further, it was shown that the strong energy condition is generically violated in the S-region (except in the case $V(\Phi) = 0$).

We shall designate by u^α the unit vector tangent to those congruences, thus we will have in the T-region:

$$u^\alpha = \frac{\Phi^\alpha}{(-\Phi_\gamma \Phi^\gamma)^{1/2}}, \qquad \text{i.e.}$$

$$\vec{u} = e^{-T_1 - X_1}\left[\frac{-f(t)}{\sqrt{-(h(x)^2 - f(t)^2)}}\partial_t + \frac{h(x)}{\sqrt{-(h(x)^2 - f(t)^2)}}\partial_x\right] \tag{101}$$

and in the S-region

$$\vec{u} = e^{-T_1 - X_1}\left[\frac{h(x)}{\sqrt{h(x)^2 - f(t)^2}}\partial_t + \frac{-f(t)}{\sqrt{h(x)^2 - f(t)^2}}\partial_x\right]. \tag{102}$$

Now, a given model is said to inflate if there is some timelike, physically meaningful, congruence $\vec{u}$ (such as the 4-velocity of the fluid whenever this is the material content of the spacetime) such that its deceleration parameter $q = -3\Theta^2(\dot{\Theta} + \Theta^2/3)$ is negative, where $\Theta = u^\gamma_{;\gamma}$ is the congruence expansion, and $\dot{\Theta} = \Theta_{,\gamma}u^\gamma$; thus $q < 0 \Leftrightarrow \dot{\Omega} - 1/3 < 0$ where $\Omega = \Theta^{-1}$ and a straightforward calculation yields, for the T-region ($f^2 - h^2 > 0$)

$$\dot{\Omega} = \left[\left(c_3^2 + \frac{1}{4}\right)h^2 - \tau_1 f - \frac{\dot{f}f^2 - h^4 + a^2h^2}{f^2 - h^2}\right]\Sigma + \Sigma_{,x}h - \Sigma_{,t}f \tag{103}$$

with

$$\Sigma \equiv \left[\left(c_3^2 + \frac{5}{4}\right)h^2 - (\tau_1 + \tau_2)f + \frac{(\dot{f} - f^2)h^2 + a^2 f^2}{f^2 - h^2}\right]^{-1} \tag{104}$$

and for the S-region ($f^2 - h^2 < 0$):

$$\dot{\Omega} = \left[-\left(c_3^2 + \frac{1}{4} \right) hf + \tau_1 h - hf \frac{\dot{f} - h^2 + a^2}{h^2 - f^2} \right] \Pi + \Pi_{,t} h - \Pi_{,x} f \tag{105}$$

with

$$\Pi \equiv \left[-\left(c_3^2 + \frac{5}{4} \right) hf + (\tau_1 + \tau_2) h + hf \frac{\dot{f} - h^2 + a^2}{h^2 - f^2} \right]^{-1}, \tag{106}$$

where we put $h(x) = h$, $f(t) = f$, etc in order to simplify notation.

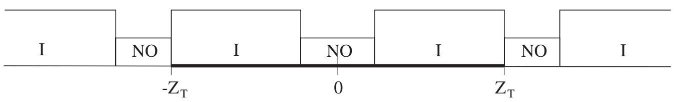
We shall next consider the above expressions for the fixed points I–IV, and I′–III′. In order to follow a unified approach, we introduce the following notation and conventions.

First of all, we shall designate as M the function $M(t, x) = f^2(t) - h^2(x)$ (with $f = \tau_1 + (c_1 - 1/2)\tau_2 - 2c_3\tau_3$, and $h(x) = a\tanh(ax)$ for points I and III and $h(x) = 1/x$ for the rest of the fixed points). Thus, the T and S-regions correspond to $M > 0$ and $M < 0$, respectively.

Next, note that for the points I, III and I′, $\tau_A \in \mathbb{R}$ and therefore $f(t) = f_0(c_1)$ (=constant), and the T- and S-regions will have the form $x \in (-x_T, x_T)$ and $x \in (-\infty, -x_T) \cup (x_T, +\infty)$ respectively, $x_T = x_T(c_1)$ being some constant depending on c_1.

For the rest of the points (II, IV and II′, III′) one has that $\tau_A \propto 1/t$, hence $f(t) = f_0/t$ where again $f_0 = f_0(c_1)$ is some constant depending on c_1. Setting now $z \equiv t/x$, it follows that the T- and S-regions take forms similar to those above with z in the place of x; that is, $z \in (-z_T, z_T)$ for the T-region and $z \in (-\infty, -z_T) \cup (z_T, +\infty)$ for the S-region, with $z_T (= |f_0|)$ some constant depending on c_1.

Finally, we shall represent the behaviour of the solution in the different regions by means of diagrams, such as



where the horizontal line represents the x (points I, III or I′) or the z (II, IV, II′ or III′) axis; the thicker segment, limited by $-x_T, x_T$ or $-z_T, z_T$ respectively, corresponds to the T-region (and the rest of the horizontal line to the S-region); and the behaviour, inflation (I) or non-inflation (No), is represented by 'boxes' (higher and lower respectively) drawn over the relevant sub-regions.

In all the subsequent developments, it is useful to recall that, over the T-region, the scalar field can be interpreted as a perfect fluid.

Point I. For this point we have

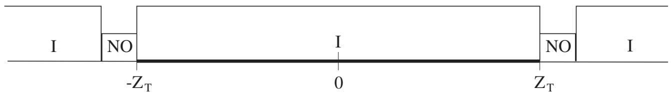
$$\tau_1 = \left(c_3^2 + \tfrac{5}{4} \right) a, \qquad \tau_2 = a, \qquad \tau_3 = c_3 a, \qquad \text{and} \qquad h = a\tanh ax \tag{107}$$

it turns then out that $M > 0$ and therefore the spacetime consists entirely of a T-region. Using (103) above, it follows that $\dot{\Omega} - 1/3 = 2/3 > 0$ always and therefore the solution never inflates.

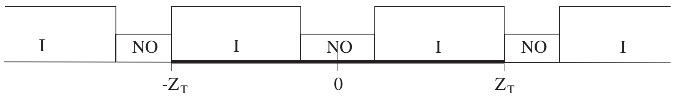
The line element for this case can be written as

$$\mathrm{d}s^2 = A\,\mathrm{e}^{2a(c_1+1)t} \cosh^{2c_1} ax(-\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{e}^{at}\cosh ax \times (\mathrm{e}^{2ac_3t}\cosh^{2c_3} ax\,\mathrm{d}y^2 + \mathrm{e}^{-2ac_3t}\cosh^{-2c_3} ax\,\mathrm{d}z^2). \tag{108}$$

Point II. The metric in this case is that given by (93) and the τ_A those given in (92). It turns out that for all the possible values of c_1, there exist non-empty T- and S-regions. There are two different inflationary patterns, depending on the range of values of the parameter c_1; thus, for $\frac{1}{4} < c_1 < \frac{1}{\sqrt{2}}$ one has



whereas for the rest of allowed values of c_1 one gets



Point III. For this fixed point we have

$$\tau_1 = \frac{a}{2}[\delta + (2c_1 + 1 - 2\sqrt{4c_1 - 1})\delta^{-1}], \qquad \tau_2 = a\delta, \qquad \tau_3 = ac_3\delta^{-1},$$
$$\text{and} \qquad h = a \tanh ax \tag{109}$$

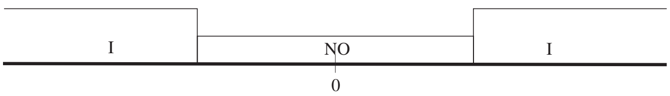
and the metric is

$$\mathrm{d}s^2 = A^2 \exp[a(\delta + (2c_1 + 1)\delta^{-1})t] \cosh^{2c_1} ax(-\mathrm{d}t^2 + \mathrm{d}x^2)$$
$$+ \mathrm{e}^{a\delta t} \cosh ax \left(\mathrm{e}^{2ac_3\delta^{-1}t} \cosh^{2c_3} ax\, \mathrm{d}y^2 + \mathrm{e}^{-2ac_3\delta^{-1}t} \cosh^{-2c_3} ax\, \mathrm{d}z^2\right) \tag{110}$$

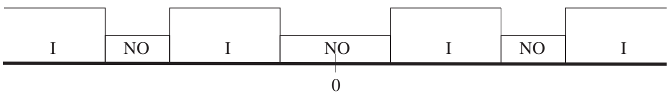
where $\delta = \sqrt{\frac{2c_1^2+1}{2c_1^2-1}}$.

In this case one can distinguish three different regions for the value of the parameter c_1 (recall $c_1 > 1/\sqrt{2}$, see table 1), each of them corresponding to a different inflationary behaviour and/or a different splitting of the spacetime in T- and S-regions.

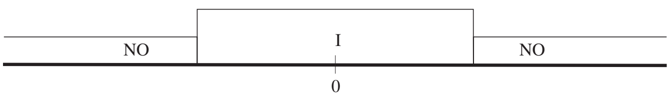
- $c_1 \in (1/\sqrt{2}, 0.925\,669\ldots)$. The spacetime consists of a single T-region and two different kinds of behaviour are possible: for values of c_1 close to $1/\sqrt{2}$ one gets



For bigger values of c_1, one has a sequence of open regions, symmetrically placed with respect to $x = 0$, where inflationary and non-inflationary behaviours alternate; i.e.,

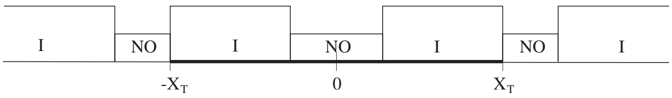


- $c_1 \in (2.048\,756\ldots, +\infty)$. The spacetime also consists of a single T-region and, again, there are two different patterns; namely, for $c_1 \leqslant 3.193\,91\ldots$ one has

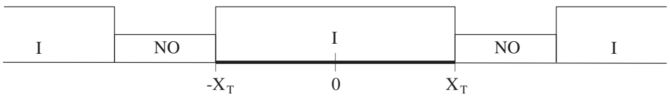


whereas for values bigger than the one above, the spacetime does not inflate; that is,

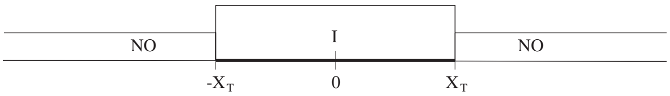
- $c_1 \in (0.925\,669\ldots, 2.048\,756\ldots)$. For this range of values of c_1 the spacetime splits into a T- and an S-region, and three different inflationary patterns occur; for values of c_1 close to the lower bound of the interval one has



for intermediate values of c_1 (e.g. $c_1 = 1.2$);

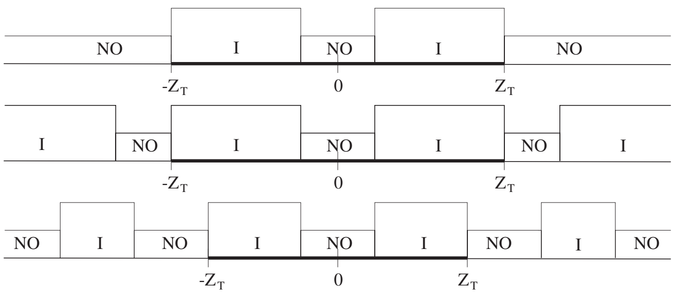


and for values close to the upper limit of the interval,



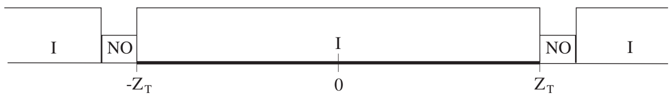
Point IV. For this set of non-isolated fixed points the τ_A and the metric are given by (96) and (97). It then turns out that $f(t) = f_0/t$ with $f_0 = 1 - \sqrt{2}\cos\varphi$; i.e., f_0 does not depend on c_1 but only on φ which singles out one point within the set $A^2 + B^2 = 1$; hence, for one chosen fixed point, the T- and S-regions are determined, unlike the previous cases which depended on the parameter c_1.

In this case, the inflationary behaviour is more complicated to describe as there are two parameters, φ and c_1. However, it is relatively easy to see that the behaviour in the T-region is essentially the same for all the values of the parameters involved, namely; there is always a non-inflationary sub-region centred about $z = 0$ surrounded by (symmetrical) inflationary ones which occupy the rest of the T-region. Also, it can be shown that in the S-region there is always a non-inflationary zone of the form $(-z_1, -z_T) \cup (z_T, z_1)$, where $z_1 = z_1(c_1, \varphi) > 0$ is a positive constant which depends on the two parameters c_1 and φ. For $|z| > z_1$ the behaviour varies depending on the values that these parameters assume. Systematic numerical experiments strongly suggest that there are essentially three possibilities; namely, $z_1 = \infty$ and therefore the entire S-region is non-inflationary; or there are two symmetric inflationary subregions $z_1 < |z| < z_2$ (where again, $z_2 = z_2(c_1, \varphi) > 0$) followed by non-inflationary ones stretching out to $|z| = \infty$; or else, there is inflation for $z_1 < |z| \leqslant \infty$. In diagram form this can be summarized as

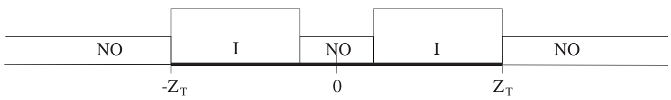


Point I′. As discussed previously, this point corresponds to a static solution whose associated energy–momentum tensor is of the Segre type $\{(1, 11)1\}$ everywhere (S-region); and therefore has very little interest from the point of view of cosmology.

Point II′. This is a special case ($c_1 = 1/4$) of the solutions corresponding to the fixed point II, which have been dealt with previously. Only note in passing that the inflationary pattern can be represented as



Point III′. For this fixed point, the functions τ_A and the metric are given respectively by (99) and (100). The spacetime has both T- and S-regions; the T-region being just $z \in (-1/2, +1/2)$ and the S-region then $z \in (-\infty, -1/2) \cup (+1/2, +\infty)$. The behaviour of this solution can be described by



6. Conclusion

We have studied scalar field spacetimes as sources of inhomogeneous G_2 cosmological models, concentrating then on diagonal G_2 orthogonally transitive models which are separable in coordinates adapted to the Killing vectors, and showing that this kind of separability implies separability of the scalar field [10]. Also, and following a previous paper [10], we have discussed the splitting of a general scalar field spacetime (i.e. irrespective of the assumed underlying geometry) into various regions of constant Segre type; namely: T, S and $\mathcal{F}$; the latter being closed (with no interior whenever the Einstein tensor is analytic) and the other two, which are open, being non-empty in general. It is then pointed out that the dominant energy condition is always satisfied provided the potential is non-negative; further, it is recalled that over the T-region, a scalar field spacetime is always formally equivalent to (i.e. can be interpreted as) a perfect fluid with non-twisting velocity.

Next, we have shown that three cases arise from the field equations depending on the maximum number of linearly independent functions: three independent functions of t and just one of x, two independent functions of t, or just one linearly independent function of t and three of x; the first and last cases referred to being equivalent on account of the $t \leftrightarrow x$ discrete symmetry that these solutions possess. Again, all this is completely general and does not depend on any further simplifying assumption; thus, the families of solutions considered in the references cited in sections 1 and 3 are shown to be special instances of one of the above cases.

Following this, a dynamical systems study of the first case described in the preceding paragraph is carried out. Typical phase portraits are obtained and the fixed points, as well as one invariant set, are found and classified (sinks, sources and/or saddle points). It is shown that the phase space can always be compactified if one defines appropriate variables; it should be noted that this is always done starting from the same equation, namely, the one involving the potential in the separated field equations system.

We have studied the equilibrium points; showing that they all correspond to self-similar solutions and giving explicit expressions for the metric, homothetic vector field and scalar field. It is worth mentioning that some static solutions appear and that, in one case, there is a whole family of solutions admitting conformal Killing vectors. Also, the inflationary behaviour of these solutions is analysed, showing that in the T-region some of the solutions corresponding to these fixed points always inflate while some others never inflate; the typical (most common) behaviour, though, is that in which a 'central' inflationary sub-region (see section 5) exists surrounded by symmetrical non-inflationary sub-regions; the reciprocal behaviour (central non-inflationary sub-region surrounded by inflationary ones) also occurs.

Regarding the S-region, it is argued that in G_2 scalar field spacetimes, a geometrically distinguished timelike congruence always exists; putting forward the idea that one could, in that region, check whether such a congruence inflates or not. Such an analysis is also carried out for the fixed points referred to above, with similar results to those obtained for the T-region. All the information regarding inflation is collected and summarized in the form of diagrams.

Acknowledgments

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