# Covariant Duality Symmetric Actions

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## Abstract

A manifestly Lorentz and diffeomorphism invariant form for the abelian gauge field action with local duality symmetry of Schwarz and Sen is given. Some of the underlying symmetries of the covariant action are further considered. The Noether conserved charge under continuous local duality rotations is found. The covariant couplings with gravity and the axidilaton field are discussed. 11.10-z, 11.30-j, 03.50-z

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The equations of motion of the four dimensional low energy effective field theory for the bosonic sector of the heterotic string, which can be obtained from dimensional reduction of $N = 1$ supergravity theory coupled to gauge fields in ten dimensions [1], are invariant under the $SL(2, R)$ non linear duality transformations of the massless fields involved. This has been used to find new interesting black hole solutions carrying both electric and magnetic charges in string theory [2]. Actually, $SL(2, Z)$ a subgroup of $SL(2, R)$, has been conjectured to be an exact symmetry of the full string theory [3]. This duality symmetry, called S-duality, which also inverts the coupling constant, together with the "target space duality" or T-duality, have brought out new perspectives to the understanding of non perturbative features in string theory.

Recently, Schwarz and Sen [4] have developed a method which permits to achieve $SL(2, R)$ duality symmetry at the level of the action by introducing extra auxiliary gauge fields. However, in their formulation, explicit Lorentz and general coordinate invariances are missing. It is only after eliminating the auxiliary fields through their equations of motion that the usual transformation rules for the remaining fields in a specific gauge are recovered. In this paper we will consider the manifestly Lorentz, gauge and diffeomorphism invariant generalizations of the local duality symmetric actions of Schwarz and Sen for abelian fields. Furthermore, some of the underlying symmetries of the proposed actions are discussed and the connection with previous works is established.

The simplest model of a duality symmetric action presented in Ref. [4], deals with an appealing generalization of the Maxwell electromagnetic theory in which two abelian gauge fields $A_m^\alpha$ ($\alpha = 1, 2$) are considered. Let us briefly discuss the main ideas. The non-covariant action proposed by Schwarz and Sen is

$$I_{SS} = -\frac{1}{2} \int d^4x (B^{i\alpha} \mathcal{L}_{\alpha\beta} E_i^\beta + B^{i\alpha} B_i^\alpha),$$ (1)

where

$$E_i^\alpha = F_{oi}^\alpha = \partial_o A_i^\alpha - \partial_i A_o^\alpha, \quad B^{i\alpha} = \frac{1}{2} \epsilon^{ijk} F_{jk}^\alpha = \epsilon^{ijk} \partial_j A_k^\alpha,$$ (2)

and

$$\mathcal{L} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{3}$$

which has the following properties

$$det\mathcal{L} = 1, \quad \mathcal{L}^T = \mathcal{L}^{-1}, \quad \mathcal{S}\mathcal{L}\mathcal{S}^T = \mathcal{L}, \tag{4}$$

where $\mathcal{S}$ is a $SL(2,R)$ matrix. The action is invariant under the following gauge transformations

$$\delta A_0^\alpha = \Psi^\alpha, \quad \delta A_i^\alpha = \partial_i \Lambda^\alpha \tag{5}$$

and the discrete duality transformations

$$A_m^\alpha \to \mathcal{L}_{\alpha\beta} A_m^\beta. \tag{6}$$

At the classical level, equivalence with the usual Maxwell action in the temporal gauge $A_0^\alpha = 0$ follows after elimination of the fields $A_i^2$ using their equations of motion. The duality transformation Eq. (6) reduces, on shell, to the well known transformation $\vec{E} \to \vec{B}$ and $\vec{B} \to -\vec{E}$. It is worth recalling that the fields $A_0^\alpha$ do not play the usual role of Lagrange multipliers and that the Gauss law constraint is consequence of $\partial_i B^{\alpha i} = 0$. This equivalence holds also at the quantum level as has been shown in Ref. [5], where the canonical quantization procedure reveals the presence of second class constraints which may lead to serious problems in supergravity models.

In the following we will consider the action

$$I = -\frac{1}{2} \int d^4x \; (u_n \; \mathcal{F}^{\alpha mn} \Phi_{mp}^\alpha u^p + \Lambda^{\alpha mp} \Phi_{mp}^\alpha), \tag{7}$$

where

$$\Phi_{mp}^\alpha \equiv \mathcal{F}_{mp}^\alpha + \mathcal{L}_{\alpha\beta} F_{pm}^\beta, \tag{8}$$

$u$ is an otherwise arbitrary vector field that satisfies

$$u_m u^m = -1, \tag{9}$$

$F$ and $\mathcal{F}$ are the gauge invariant field strengths and their duals

$$F^\alpha_{mn} = \partial_m A^\alpha_n - \partial_n A^\alpha_m, \quad \mathcal{F}^{\alpha mn} = \frac{1}{2}\varepsilon^{mnpq} F^\alpha_{pq}, \tag{10}$$

and $\Lambda$ is an auxiliary antisymmetric field. Note that $\Phi$ is a self-dual tensor, i.e.,

$$\Phi^\alpha_{mn} \equiv \frac{1}{2}\varepsilon^{mnpq}\mathcal{L}_{\alpha\beta}\Phi^{\beta pq} \tag{11}$$

and hence $\Lambda$ is an anti self-dual tensor

$$\Lambda^\alpha_{mn} \equiv -\frac{1}{2}\varepsilon^{mnpq}\mathcal{L}_{\alpha\beta}\Lambda^{\beta pq}. \tag{12}$$

Our conventions are $\eta_{mn} = diag(-1, 1, 1, 1)$, $\varepsilon^{ijk} = \varepsilon^{0ijk}$ and $\varepsilon^{0123} = 1$. Clearly, the action Eq.(7) is Lorentz and gauge invariant and manifestly invariant under the duality transformations provided $\Lambda$ transforms as follows:

$$\Lambda^\alpha_{mn} \to \mathcal{L}_{\alpha\beta}\Lambda^\beta_{mn}. \tag{13}$$

The equations of motion obtained from the action Eq.(7) are:

$$\frac{\delta}{\delta A^\alpha_m} I = 0 \quad \Rightarrow \quad G^{\alpha m} \equiv \varepsilon^{mnpq}\partial_p[u_n u^r \Phi^\alpha_{qr} - \Lambda^\alpha_{nq}] = 0, \tag{14}$$

$$\frac{\delta}{\delta u_n} I = 0 \quad \Rightarrow \quad H_n \equiv \mathcal{F}^\alpha_{mn}\Phi^{\alpha mp} u_p + \mathcal{F}^\alpha_{mp}\Phi^{\alpha mn} u^p = 0, \tag{15}$$

and

$$\frac{\delta}{\delta \Lambda^\alpha_{mn}} I = 0 \quad \Rightarrow \quad \Phi^\alpha_{mn} = 0. \tag{16}$$

Note that $\partial_m \Phi^{\alpha mn} = 0$ implies $\partial_m F^{\alpha mn} = 0$ which are just the Maxwell equations for the $A^\alpha$ fields. One can show that $\Lambda = 0$ on-shell, i.e. is a non-dynamical variable. After eliminating one of the two abelian gauge fields (for example $A^2$) with the use of Eq. (16) and taking into account that $u_n u^p \mathcal{F}^{1mn} F^2_{pm} = -u_n u^p \mathcal{F}^{2mn} F^1_{pm}$ up to a total divergence we obtain the gauge invariant Maxwell covariant action $I = -\frac{1}{4}\int d^4 x F^1_{mn} F^{1mn}$ for the other gauge field.

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Contact with the non-covariant formulation of Schwarz and Sen can be made as follows. Our action has an additional gauge symmetry:

$$\delta u_m = \epsilon_m(x), \quad \delta\Lambda_{mn}^\alpha = -\frac{1}{2}[\epsilon^r(x)\mathcal{F}_{mr}^\alpha u_n - u^r\mathcal{F}_{mr}^\alpha\epsilon_n(x) - (m \leftrightarrow n)], \tag{17}$$

while the vector fields $A_m^\alpha$ are inert under these transformations. This symmetry allows one to fix the vector field $u$ to a constant vector, e.g. $u_m = -\delta_m^0$. With this choice the action of Schwarz and Sen is obtained.

The action Eq. (7) is not only invariant under the discrete duality transformations; actually, it is invariant under the continuous duality rotations

$$A^{\alpha m} \to \mathcal{D}_{\alpha\beta}A^{\beta m}, \tag{18}$$

where $\mathcal{D}$ is a $SO(2, R)$ matrix

$$\mathcal{D} = \begin{pmatrix} cos\omega & sin\omega \\ -sin\omega & cos\omega \end{pmatrix}, \tag{19}$$

which implies the existence of conserved currents. If we consider the infinitesimal duality rotations

$$\delta A_m^\alpha = \delta\omega\mathcal{L}_{\alpha\beta}A_m^\beta, \tag{20}$$

the Noether conserved current associated to this invariance is

$$j^m = -\frac{1}{2}\mathcal{F}^{\alpha mn}A_n^\alpha, \tag{21}$$

which satisfies $\partial_m j^m = 0$ on shell.This current is not gauge invariant, but it changes under gauge transformations by the divergence of an antisymmetric tensor. The corresponding conserved generator is

$$G = -\frac{1}{4}\int d^3x\varepsilon^{ijk}F_{jk}^\alpha A_i^\alpha, \tag{22}$$

which can be rewritten as

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$$G = \int d^3 x A_i^\alpha \mathcal{L}_{\alpha\beta} \Pi^{\beta i}, \tag{23}$$

where $\Pi^{\beta i} = -\frac{1}{2}\mathcal{L}_{\alpha\beta} B^{i\alpha}$ are the canonical conjugate momenta. The integrand in Eq. (22) is just the sum of two abelian Chern-Simons 3-form and $G$ as given by this equation is gauge invariant. Note that, in spite of the topological nature of the Chern-Simons forms, the generator Eq. (22) is a genuine Noether charge. Hence the Chern-Simons 3-forms constructed out with the fiels $A_i^\alpha$ are the generators of the continuous duality rotations, i.e., $\delta A_i^\alpha = \delta\omega\{G, A_i^\alpha\}$. At this stage, Eq. (16) can be used to eliminate $A_m^2$ from Eq. (22) to obtain

$$G = \frac{1}{2}\int d^3 x (-\vec{A}\cdot\nabla\times\vec{A} + \vec{E}\cdot\nabla^{-2}\nabla\times\vec{E}), \tag{24}$$

which turns out to be the generator of the non-local duality transformations in the usual canonical Maxwell theory [6].

Starting out with the covariant action the coupling with gravity keeping the duality invariance is now straighforward. Following the minimal coupling prescription we have

$$I_c = -\frac{1}{2}\int d^4 x\sqrt{-g}\,(u_n\mathcal{F}^{\alpha mn}\Phi_{pm}^\beta g^{pq}u_q + g^{mn}g^{pq}\Lambda_{mp}^\alpha\Phi_{nq}^\alpha), \tag{25}$$

where now

$$\mathcal{F}^{\alpha mn} = \frac{1}{2\sqrt{-g}}\varepsilon^{mnpq}F_{pq}^\alpha. \tag{26}$$

This action is manifestly invariant under general coordinate transformations and manifestly gauge and duality rotations invariant. It is also invariant under conformal transformations

$$g_{mn} \to \Omega^2 g_{mn}, \quad F_{mn}^\alpha, \to F_{mn}^\alpha, \quad \Lambda_{mn}^\alpha \to \Lambda_{mn}^\alpha \tag{27}$$

provided

$$u_m \to \Omega u_m. \tag{28}$$

The constraint imposed by $\Lambda$ turns out to be the same as in Eq. (16). This can be used to eliminate the field $A_m^2$ and the covariant Maxwell action in curved space-time is obtained.

Furthermore, a traceless on shell energy-momentum tensor can be derived through the usual definition $T^{mn} \equiv \frac{2}{\sqrt{-g}}\frac{\delta I_c}{\delta g_{mn}}$, in agreement with the conformal invariance of the action. ¿From this and after eliminating the field $A_m^2$ using Eq. (16), the Maxwell energy-momentum tensor is obtained. For $u_m = -n\delta_m^0$ it follows from Eq. (9) that $n^2 = (-g^{00})^{-1}$ and contact with the usual ADM slicing of the manifold [7] is made. In this way, the local duality invariant action of Schwarz and Sen in curved-space [4]is recovered.

Finally, let us introduce the complex scalar axidilaton field $\lambda \equiv \lambda_1 + i\lambda_2$, which combines two entities of stringy nature: the dilaton $\lambda_2 = exp\{-2\phi\}$ and the axion $\lambda_1 = \psi$ defined through the equations of motion of the antisymmetric two-form $B_{mn}$ in four dimensions. Introducing the symmetric $SL(2,R)$ matrix [4]

$$
\mathcal{M} = \frac{1}{\lambda_2}\begin{pmatrix} 1 & \lambda_1 \\ \lambda_1 & |\lambda|^2 \end{pmatrix},
\tag{29}
$$

the covariant action for the coupling between the axidilaton and the gauge fields $A_m^\alpha$ in a curved space is

$$
I_{ca} = -\tfrac{1}{2}\int d^4x\sqrt{-g}\,[u_n\mathcal{F}^{\alpha mn}(\mathcal{L}_{\alpha\beta}F_{pm}^\beta + (\mathcal{L}^T\mathcal{M}\mathcal{L})_{\alpha\beta}\mathcal{F}_{mp}^\beta)g^{pq}u_q
$$
$$
+ g^{mn}g^{pq}\Lambda_{mp}^\alpha(\mathcal{L}_{\alpha\beta}F_{nq}^\beta + (\mathcal{L}^T\mathcal{M}\mathcal{L})_{\alpha\beta}\mathcal{F}_{qn}^\beta)],
\tag{30}
$$

which is invariant under the $SL(2,R)$ transformations

$$
\mathcal{M} \to w\mathcal{M}w^T, \quad A_m^\alpha \to (w)_{\alpha\beta}A_m^\beta.
\tag{31}
$$

In fact, after eliminating the field $A_m^2$ through the corresponding duality condition which arise from the constraint imposed by $\Lambda$

$$
\mathcal{F}^{2mn} = \lambda_2 F^{1nm} + \lambda_1\mathcal{F}^{1mn},
\tag{32}
$$

the result is

$$
I_{ca} = -\frac{1}{4}\int d^4x\sqrt{-g}\,F_{mn}^1(\lambda_2 F^{1mn} + \lambda_1\mathcal{F}^{1mn}),
\tag{33}
$$

which exhibits the standard coupling of the axidilaton field with an abelian gauge field in a curved space.

Summarizing, introducing auxiliary fields, we have seen that local duality transformations in the sense of Schwarz and Sen can be implemented in a manifestly Lorentz invariant way as a symmetry of the action, the field $\Lambda$ playing the role of a multiplier whose associated constraint turns out to be the covariant duality condition. The equivalence with the covariant Maxwell theory follows after solving this constraint in order to eliminate one of the gauge fields from the original action. The connection with the non-covariant approach is also established as a result of an additional symmetry which permits to fix $u$ appropriately. The generator for the continuous duality transformations was found to be given in terms of Chern-Simons 3-forms. From the proposed covariant action, the coupling with gravity was obtained in a straighforward way. In addition, the presence of the axidilaton field coupled with the abelian gauge fields in a curved space was considered.

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