Matter collineations: The inverse "symmetry inheritance" problem

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Matter collineations, as a symmetry property of the energy-momentum tensor T_{ab}, are studied from the point of view of the Lie algebra of vector fields generating them. Most attention is given to space–times with a degenerate energy-momentum tensor. Some examples of matter collineations are found for dust fluids (including Szekeres's space–times), and null fluid space–times.

I. INTRODUCTION

Let (M,g) be a space–time, i.e., M is a four-dimensional, connected, Hausdorff, smooth manifold, and g is a smooth Lorentz metric of signature $(-+++)$ defined on M.

The usual component notation in local charts will often be used throughout this article. Then, associated with the metric g is the symmetric connection Γ, a covariant derivative with respect to Γ will be denoted by a semicolon and a partial derivative by a comma. Also, the components of the Ricci tensor and the Ricci scalar are, respectively,

$$R_{ab}=R^{c}{}_{acb}, \quad R=R_{ab}g^{ab},$$

with $R^{a}{}_{bcd}$ being the components of the Riemann tensor.

Einstein's equations in local coordinates are

$$G_{ab}=T_{ab}, \quad G_{ab}\equiv R_{ab}-\frac{R}{2}\, g_{ab},$$

with T_{ab} being the components of the energy-momentum tensor.

In this article we will be interested in properties of vector fields X satisfying

$$\mathscr{L}_X T_{ab}=0, \tag{1}$$

where $\mathscr{L}$ is the Lie derivative operator. Such a vector field will be called a "matter collineation."

The motivation for studying matter collineations is twofold; on one hand, there is the purely mathematical interest of studying the invariance properties of a given geometrical object, namely, the Einstein tensor, that arises quite naturally in the theory of general relativity and plays a significant role in that theory, since it is related, via the Einstein's field equations, to the material content of the space–time (represented by the energy-momentum tensor). On the other hand, one of the simplifications often used in finding exact solutions to the Einstein's field equations, is the assumption of certain symmetries of the space–time metric; this giving rise to the so-called "symmetry inheritance problem," which can be roughly stated as that of finding out how the physical fields, occurring in a certain region of the space–time, reflect the symmetries of the metric (see Ref. 1 and references cited therein). It appears then natural, as well as physically

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desirable, to look into the converse problem, i.e., what implies, for the metric tensor of a space–time, the existence of some sort of symmetry for the physical fields describing the material content of that space–time. This involves two steps, namely, (a) looking into the properties of the set of solutions to Eq. (1); and (b) studying the implications that the existence of such vector fields might have on the space–time metric (or other geometrical objects). We shall only deal here with step (a); the other one being currently under study.

As a final remark to these considerations, note that the present problem is not totally unrelated to that of the PUTH (postulate of uniform thermal histories, see Refs. 2 and 3); in this last reference it is shown that for perfect fluids with a barotropic equation of state, the PUTH is satisfied if and only if there exist three linearly independent vector fields $\xi_{(\alpha)}$, $\alpha=1,2,3$ satisfying

$$\mathscr{L}_{\xi_{(\alpha)}}u=\mathscr{L}_{\xi_{(\alpha)}}\mu=0, \tag{2}$$

with u and μ being, respectively, the fluid four-velocity and the energy density.

As we shall see in Sec. III, and in the particular case of a dust perfect fluid, this is readily verified by a well-defined subfamily of the whole family of matter collineations $\mathscr{T}(M)$.

In this article only smooth matter collineations will be considered, in order to ensure that the set of all such vector fields, together with the usual Lie bracket operation, is a Lie algebra—the Lie algebra of matter collineations $\mathscr{T}(M)$.

Also, there may exist "local" differentiable vector fields satisfying Eq. (1), i.e., vector fields defined on an open subset V of M which cannot be smoothly extended to M. However, we will restrict our attention to the case when X is a global vector field (defined on M).

As for the energy-momentum T_{ab}, it will be assumed that it is smooth and that its algebraic Segre type is the same at each point p in M. Thus,[4] the eigenvalues and eigenvectors of T_{ab} may be regarded as locally smooth on M, and the canonical Segre forms for T_{ab} at a point may be regarded as holding in any coordinate domain of any point p of M.

Finally, the article is organized as follows: Sec. II presents some general results and sets up the distinction between nondegenerate and degenerate energy-momentum tensors. Section III deals with the problem of the dimension of $\mathscr{T}(M)$ for degenerate energy-momentum tensors and provides a few results towards its characterization in the two physically significant cases, namely, dust and null fluids. Finally, Sec. IV contains some examples of dust and null fluid space–times admitting nontrivial matter collineations.

II. GENERAL RESULTS

Let X be a matter collineation, i.e., a solution of Eq. (1) for X where T_{ab} satisfies the conditions referred to above.

All Killing vector fields (KVF), proper homothetic vector fields (HVF), and proper special conformal Killing vector fields (SCKV) are, naturally, matter collineations. However, the converse is not always true (in this case, the matter collineation will be called "proper" or "nontrivial"). Assuming that $X\in\mathscr{T}(M)$, the following results can be easily obtained:

(1) Assume that (M,g) is a space–time such that $R\neq 0$ (R being the Ricci scalar), then X is a Ricci collineation ($\mathscr{L}_X R_{ab}=0$) iff X is an SCKV [i.e., $\mathscr{L}_X g_{ab}=2\lambda g_{ab}$ with $\lambda(x^c)$ such that $\lambda_{;ab}=0$, see, for instance, Refs. 1, 5, and 6], the conformal factor then being $\lambda\equiv-\frac{1}{2}\mathscr{L}_X(\ln R)$. If $R=0$, then matter collineations and Ricci collineations obviously coincide.

(2) Assuming that f is a (smooth) scalar function defined on M, then $Y\equiv fX$ is also a matter collineation iff either f is constant on M or X satisfies $T_{ab}X^b=0$, in which case T_{ab} is necessarily degenerate (i.e., its rank is less than 4) and X is an eigenvector of the Ricci tensor with eigenvalue $R/2$. This has also been noted in Ref. 7.

(3) If T_{ab} is nondegenerate, then a matter collineation X will satisfy $\mathscr{L}_X T^a{}_b=0$ (or $\mathscr{L}_X T^{ab}=0$) iff X is a KVF.

When studying the Lie algebra $\mathscr{T}(M)$ of matter collineations, two cases arise naturally, according to whether T_{ab} is nondegenerate or degenerate.

If T_{ab} is nondegenerate, then rank $T_{ab}=4$ and the Lie algebra of matter collineations $\mathscr{T}(M)$ is finite-dimensional, its maximal dimension being 10, 9, being excluded by Fubini's theorem.

If T_{ab} is degenerate, then rank $T_{ab}<4$ and we cannot guarantee the finite dimensionality of $\mathscr{T}(M)$.

In fact, from remark (2) above it is easy to show that in many cases this algebra is infinite dimensional.

From now on, we will restrict ourselves to the study of the degenerate case. Therefore, it will be assumed that rank $T_{ab}<4$, in which case, the only physically significant space–times are either dust fluids (perfect fluids with $p=0$) or null fluids (radiation and null Einstein–Maxwell fields);[8] the rank of T_{ab} being in both cases 1. Their study will be the purpose of the next section.

III. MATTER COLLINEATIONS FOR DEGENERATE ENERGY-MOMENTUM TENSORS

As we already pointed out at the end of the last section, there are only two cases of algebraically degenerate energy–momentum tensors, which are physically relevant: both of them of rank 1, namely, dust and null fluids.

The energy-momentum tensor representing a dust fluid is given by

$$T_{ab}=\mu u_a u_b, \tag{3}$$

where μ is a positive function representing the energy density as measured by an observer comoving with the fluid, and u^a is a timelike, unit vector field ($u^a u_a=-1$) representing the four-velocity of the fluid. From the contracted Bianchi identities it follows that $\dot{u}_a=0$, i.e., geodesic flow.

Let now $X\in\mathscr{T}(M)$, (M,g) being a dust-fluid space–time. It is then easy to see that this is equivalent to

$$\mathscr{L}_X\phi_a=0, \quad \text{with} \quad \phi_a\equiv\mu^{1/2}u_a, \tag{4}$$

that is, $\mathscr{L}_X u_a=-(2\mu)^{-1}(\mathscr{L}_X\mu)u_a$ and $\mathscr{L}_X\mu=2\mu(u^b\mathscr{L}_X u_b)$.

On the other hand, if (M,g) is a null fluid space–time, its energy-momentum tensor is given by

$$T_{ab}=e^{\gamma}k_a k_b, \tag{5}$$

where γ is a function and k^a is a null vector field ($k^a k_a=0$), which is geodesic also as a consequence of the contracted Bianchi identities.

In this case it is also easy to show that $X\in\mathscr{T}(M)$ is equivalent to

$$\mathscr{L}_X\phi_a=0, \quad \text{with} \quad \phi_a\equiv e^{\gamma/2}k_a, \tag{6}$$

that is, $\mathscr{L}_X k_a=-\frac{1}{2}(\mathscr{L}_X\gamma)k_a$.

Therefore, the problem of finding the matter collineations in a space–time whose energy-momentum tensor is rank 1, can be formally reduced to that of finding those vector fields X that leave a given one-form ϕ invariant.

In order to solve this problem let us consider the following (differentiable) distributions:

$$\mathscr{K}:p \to \mathscr{K}_p\subseteq T_pM:Z_p\in\mathscr{K}_p, \quad \text{iff} \quad \langle \mathscr{Z},\phi\rangle|_p=0, \quad (\mathscr{K}:\text{Kernel of } \phi),$$

$$\mathscr{T}:p \to \mathscr{T}_p\subseteq T_pM:X_p\in\mathscr{T}_p, \quad \text{iff} \quad (\mathscr{L}_X\phi)|_p=0,$$

$$\mathscr{F}\equiv\mathscr{K}\cap\mathscr{T},$$

where T_pM is the tangent space to M at a point p.

Obviously, $\mathscr{T}$ is integrable, and one can show quite easily that so is $\mathscr{F}$. As for $\mathscr{K}$, the integrability condition is $d\phi \wedge \phi = 0$ (or, in coordinate language $\epsilon^{abcd}\phi_b\phi_{c,d}=0$; i.e., $\phi_a \propto f_{,a}$ for some function f). Assuming now that $\dim \mathscr{T}_p$ and $\dim \mathscr{F}_p$ are constant over M (or over the open region of interest) one has the following different possibilities:

	I	II	III	IV	V	VI	VII
$\dim \mathscr{K}_p$	3	3	3	3	3	3	3
$\dim \mathscr{T}_p$	0	1	1	2	2	3	3
$\dim \mathscr{F}_p$	0	0	1	1	2	2	3
$\dim \mathscr{T}$	0	1	∞	∞	∞	∞	∞

Notice that, whenever $\dim \mathscr{F}_p \neq 0$, the dimension of the Lie algebra of matter collineations $\mathscr{T}(M)$ is necessarily ∞, for, given a vector field X in $\mathscr{F}$ [and therefore $X \in \mathscr{T}(M)$] and any function f on M, the vector field $Y \equiv fX$ is also in $\mathscr{F}$ and hence is another matter collineation (this corresponds to remark 2 in the previous section). Thus, for instance, in case IV, a general matter collineation will look like

$$X = \alpha X_{(1)} + fX_{(2)}, \tag{7}$$

where $X_{(1)p}$, $X_{(2)p}$ span $\mathscr{T}_p$, $X_{(2)p}$ spans $\mathscr{F}_p$, α is an arbitrary constant, and f is a (smooth) function on M, also arbitrary.

Case VII can be easily seen to be that corresponding to $\mathscr{K}$ integrable, i.e., ϕ_a being proportional to a gradient.

We can now specify these general results (which hold for any symmetric, second order tensor of rank 1) to each one of the cases we are interested in, namely, dust and null fluid.

In the case of dust, some interesting results may be derived for those matter collineations in $\mathscr{F}$. We collect them in the following:

Proposition: Let $Y \in \mathscr{F}$, therefore it satisfies

(a) $\mathscr{L}_Y \mu = \mathscr{L}_Y u_a = 0$,

(b) $\omega_{ac} Y^c = 0$, ω_{ac} being the vorticity of the fluid.

The proof is straightforward: if $Y \in \mathscr{F}$ then it satisfies $\mathscr{L}_Y \phi_a = \phi_a Y^a = 0$ with $\phi_a = \mu^{1/2} u_a$. From these two equalities, (a) follows immediately. On the other hand, taking the covariant derivative of $Y^a u_a = 0$, skew-symmetrizing it and bearing in mind that $\dot{u}_a = 0$, part (b) follows.

The converse result also holds: given a vector field Y orthogonal to the velocity field of the fluid u and satisfying (a) and (b), it follows that $Y \in \mathscr{F}$.

Since ω_{ac} (vorticity of the fluid) is a skew-symmetric tensor, its rank (in a space–time) can only be 0 or 2 (it cannot be 4 since $\omega_{ac} u^c = 0$); therefore if, at a given point p (and hence at every point, according to our hypothesis), $\dim \mathscr{F}_p$ happens to be greater than or equal to 2, the fluid is then irrotational ($\omega_{ac} = 0$). If, on the other hand $\omega_{ac} \neq 0$, then $\mathscr{F}_p$ is at most of dimension 1.

In view of all this, we can give the following algebraic recipe for finding $\mathscr{F}$:

(1) Solve $\omega_{ac} Z^c = 0$ for Z (ω_{ac} being the vorticity of u).
(2) Get $Y^a = h^a{}_{.c} Z^c$ (where $h^a{}_{.c} = \delta^a{}_{.c} + u^a u_c$ is the orthogonal projector to the velocity field of the fluid).
(3) Solve $\mu_{,c} Y^c = 0$ for Y^c.

Then, the dimension of $\mathscr{F}_p$ will be equal to the number of linearly independent solutions to the last equation; and any linear combination of these with smooth functions as coefficients, will be a vector field in $\mathscr{F}$.

As a final remark, notice that $Y \in \mathscr{F}$ cannot be a proper HVF (since for these $\mathscr{L}_Y u_a = u_a$).

As for a null fluid, with energy-momentum tensor given by Eq. (5), some specific conditions can also be derived for those matter collineations in $\mathscr{F}$. In order to work them out, let us choose a real null tetrad adapted to the null eigenvector of the Ricci tensor, i.e.,$\{k_a, l_a, x_a, y_a\}$ satisfying $-k_a l^a = x^a x_a = y^a y_a = 1$ and all other inner products zero. Then, with the above notation, we can state

Proposition: Let $Y \in \mathscr{F}$, therefore it satisfies

$$\omega x_{[a} y_{c]} Y^c = 0, \quad (\omega\text{: vorticity of } k^a)$$

[or equivalently: $\mathrm{Re}(\theta + i\sigma) m_{[a} \bar{m}_{c]} Y^c = 0$ in a complex null tetrad]. As in the previous case the proof is straightforward and follows almost directly from $\mathscr{L}_Y \phi_a = \phi_a \; Y^a = 0$ (with $\phi_a = e^{\gamma/2} k_a$).

There are then two possibilities:

(1) $\omega \neq 0$, therefore $Y^c = A k^c$ and hence $\theta = -\frac{1}{2}\gamma_{;c} k^c = 0$, $\mathscr{L}_Y \gamma = 0$ and $\sigma\bar{\sigma} = \omega^2$ (see, for instance, Ref. 8, p. 81) where θ and σ stand, respectively, for the expansion and shear of the null congruence tangent to k^a. In particular, this implies (see Ref. 8, p. 384)

(a) There do not exist null Einstein–Maxwell fields satisfying these conditions.
(b) Pure radiation fields satisfying this and such that k is aligned with a principal null direction (p.n.d.) of the Weyl tensor, can only be of the Petrov type N.
(c) Pure radiation fields with k nonaligned with a p.n.d. of the Weyl tensor cannot be of the Petrov type N.

(2) $\omega = 0$. Then $Y^c = A k^c + B x^c + C y^c$ (or $Y^c = A k^c + D m^c + \bar{D} \bar{m}^c$ in terms of a complex null tetrad). Nonrotating, null electromagnetic fields belong to the Robinson–Trautman class if they are diverging ($\theta \neq 0$), and to Kundt's class if they are nondiverging ($\theta = 0$) (see Ref. 8). As for pure radiation fields, we refer the reader to the above reference (pp. 384–385).

IV. EXAMPLES

A. Szekeres' solutions

Using coordinates $(x^1, x^2, x^3, x^4) \equiv (t, r, y, z)$ and according to Ref. 9 the general form of these metrics can be written using two arbitrary functions λ and σ as follows:

$$ds^2 = -dt^2 + e^{\lambda}\, dr^2 + e^{\sigma}(dx^2 + dy^2), \tag{8}$$

where λ and σ depend in principle on all four coordinates. This family of solutions satisfies Einstein's field equations for a dust perfect fluid with four-velocity $u^a = (1,0,0,0)$, and therefore $\omega_{ac} = 0$ (nonrotating).

It is a well-known fact that they do not admit KVF's;[8] it can also be proven that they do not admit proper HVF, CKV, affine collineations, or curvature collineations. However, they *do* admit infinitely many matter collineations. In fact, following the procedure outlined in Sec. III for finding those matter collineations $Y \in \mathscr{F}$ (i.e., matter collineations which are also orthogonal to the four-velocity of the fluid); and since $\omega_{ac} = 0$, it suffices for Y to satisfy

$$Y^a u_a = 0, \quad \mu_{,a} Y^a = 0. \tag{9}$$

These conditions define at each point $p \in M$ a two-plane (as it follows from the functional dependence of the energy density μ, see Ref. 9 for further details). Hence $\dim \mathscr{F}_p = 2$ so that there exist two linearly independent vector fields spanning $\mathscr{F}$ and therefore any combination of these with arbitrary (smooth) functions will also be a proper matter collineation (also in $\mathscr{F}$).

B. Dust solutions admitting a G_3 on a V_2

Using coordinates $(x^1,x^2,x^3,x^4)\equiv(\theta,\varphi,r,t)$, the general form of these metrics can be written as[8]

$$ds^2=H^2\{d\theta^2+L^2\,d\varphi^2\}+e^{2\lambda}\,dr^2-e^{2\nu}\,dt^2, \tag{10}$$

where H and λ are independent of θ and φ, L can only depend on θ, and ν is a function of t alone.

Two different families of metrics are to be considered:

(i) $H_{,r}=0$; in which case $\nu\neq 0$.
(ii) $H_{,r}\neq 0$; in which case $\nu=0$.

The expressions for the metric functions corresponding to these two cases can be found in Ref. 8, pp. 159–161.

For the purpose of finding matter collineations of these metrics, the actual calculation shows that the results for case (ii) follow straightforwardly from those in case (i) if one imposes the condition $\nu=0$. Therefore, special attention will be given to case (i).

Following again the procedure given in the previous section for the determination of those matter collineations in $\mathscr{F}$, and noticing that for the metrics under consideration one has $\mu_{,\theta}=\mu_{,\varphi}=0$, $u^a=(0,0,0,e^{-\nu})$, and $\omega_{ab}=0$, it is found that the general form of a matter collineation $Y\in\mathscr{F}$ is

$$Y=(Y^1,Y^2,Y^3,0),\quad \text{if}\quad \mu_{,r}=0, \tag{11}$$

$$Y=(Y^1,Y^2,0,0),\quad \text{if}\quad \mu_{,r}\neq 0, \tag{12}$$

where Y^i ($i=1,2,3$ or $i=1,2$) are completely arbitrary functions.

C. Null fluids

Using coordinates $(x^1,x^2,x^3,x^4)\equiv(z,v,u,w)$ we now consider a null Einstein–Maxwell field with the following metric:[10]

$$ds^2=dz^2+dv^2+2A\,dv\,dw+2D\,du\,dw+B\,dw^2. \tag{13}$$

Here $A=P(v,w)u+Q(v,w)$; $B=\frac{1}{2}L^2(v,w)u^2+S(v,w)u+T(v,w)$; and $D=D(v,w)$. The energy-momentum corresponding to this metric is that given by Eq. (5) with $k^a=\partial_u$.

A particular solution is obtained setting $P=L=0$ and $D=f(w)$ (and then $\omega=0$); thus, the energy density of the electromagnetic field becomes

$$e^{\gamma}=-\frac{1}{2f^3}\{2f_{,w}Q_{,v}-2fQ_{,vw}-SQ_{,v}+fT_{,vv}\}, \tag{14}$$

and $Y^a=k^a$ becomes then a nontrivial matter (or Ricci) collineation in $\mathscr{F}$.

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