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Conformally reducible $2+2$ spacetimes

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Abstract
Spacetimes which are conformal to $2+2$ reducible spacetimes are considered. We classify them according to their conformal algebra, giving in each case explicit expressions for the metric and conformal Killing vectors, and providing physically meaningful examples.

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1. Introduction

The solution of the field equations in general relativity is much simplified by assuming the existence of symmetries in the spacetime geometry. Knowledge of these symmetries is also useful in classifying spacetimes by the structure of the Lie algebra generated by these symmetries. The study of isometries and homotheties, and of the spacetimes admitting these symmetries is virtually complete. However, this is not true of conformals despite considerable interest in these symmetries in recent years. Our current interest is in the study of conformal symmetries in a special, but important, class of spacetimes, namely conformally reducible spacetimes (i.e., spacetimes conformal to reducible spacetimes).

Reducible (also called decomposable) spacetimes can be characterized by the existence of certain covariantly constant tensor fields. Thus, a spacetime admitting a nowhere vanishing, non-null covariantly constant vector field is said to be $1+3$ *reducible*, its manifold structure then being that of a product of a one-dimensional manifold and a three-dimensional manifold, and the line element simply the sum of the two line elements of the factor manifolds (product of metrics). On the other hand, if a rank-2 symmetric, covariantly constant tensor field is admitted, the spacetime is called $2+2$ *reducible* and it can be seen that, again, the manifold is the product of two (two-dimensional) manifolds and the line element is the sum of the line elements defined on the factor manifolds [1]. Alternatively, a spacetime is reducible if its holonomy group is non-degenerately reducible (see for instance [2] and references therein).

From a geometrical point of view, the structure of reducible spacetimes is rather simple and, at the same time, interesting.

In the case of $2+2$ reducible spacetimes, the Ricci tensor is easily seen to be of the Segre type $\{(1,1)(11)\}$ or its degeneracy, so these spacetimes are not very interesting from a physical point of view (no perfect fluids, null Einstein–Maxwell or pure radiation solutions exist, and the only non-null Einstein–Maxwell solution of this type is Bertotti–Robinson [1]). However, spacetimes conformally related to those do not have such a restriction so that they can, in principle, represent situations of physical interest and yet have a geometric structure which is relatively easy to investigate since their properties are, to a large extent, a consequence of those of the underlying reducible spacetime; hence their potential interest. We also note that all spherical, plane and hyperbolic symmetric spacetimes are conformally reducible $2+2$ spacetimes [3]. In fact they are a special type of spacetimes known as *warped spacetimes* [4].

In this paper we concentrate on the study of the conformally reducible $2+2$ spacetimes mostly from the point of view of their conformal Lie algebra but, in so doing, we also review and classify $2+2$ reducible spacetimes. Conformally reducible $1+3$ spacetimes will be the subject of a subsequent paper.

The paper is structured as follows. In section 2, the basic geometric features, such as invariant characterization, symmetries admitted, etc are dealt with. A number of results on Killing vectors and homothetic vectors in two-dimensional spaces with a metric of arbitrary signature are proved. Most of these results are well known and references can be found in the literature, whereas others are not; in any case we have collected them together with their proofs for the sake of completeness. Finally, a classification scheme for both reducible and conformally reducible spacetimes has been put forward based on the dimension of their conformal algebras.

Section 3 studies the fixed points of both homothetic and Killing vectors in two-dimensional spaces, deriving normal forms for both the line element and the generator of the symmetry (Killing or homothetic vector) in a neighbourhood containing the fixed point.

In section 4, conformally flat spacetimes of this class are dealt with, giving normal forms for the possible line elements and for the 15 conformal Killing vectors generating the conformal group of these spacetimes; this is done using different coordinate gauges.

In section 5, attention is given to non-conformally flat spacetimes of this class and the classification put forward in section 2 is implemented there. Again, normal forms for the line elements and the corresponding generators of the conformal algebra are given in all cases (often in more than one coordinate gauge). All the information has been collected and summarized in two tables for the sake of conciseness.

Finally, in section 6, some examples of perfect fluid and vacuum spacetimes are presented and references are made to others that already exist in the literature. Besides the explicit examples, a thorough study of the general case for perfect fluids is also provided (the only limiting assumption being that the velocity of the fluid is tangent to one of the two factor submanifolds).

2. Conformally reducible $2+2$ spacetimes

A spacetime (M, g) is said to be $2+2$ *conformally reducible* if there exists a coordinate chart $\{x^a\}$ such that the line element takes the form

$$\mathrm{d}s^2 = \exp(2\mu(x^a))\left(\mathrm{d}\sigma_1^2 + \mathrm{d}\sigma_2^2\right), \tag{1}$$

where

$$d\sigma_1^2 = g_{\alpha\beta}(x^\gamma)\, dx^\alpha\, dx^\beta \qquad \alpha, \beta, \ldots = 0, 1, \tag{2}$$

$$d\sigma_2^2 = g_{AB}(x^C)\, dx^A\, dx^B \qquad A, B, \ldots = 2, 3, \tag{3}$$

with $d\sigma_1^2$ of signature zero and $d\sigma_2^2$ of signature +2. That is, (M, g) is conformally related to a 2 + 2 reducible (locally decomposable) spacetime, say $(M, \hat{g})$ whose associated line element, from now on, shall be written as

$$d\Sigma^2 = d\sigma_1^2 + d\sigma_2^2. \tag{4}$$

The above decomposition effectively says that M has (locally) the product manifold structure, say $M = V_1 \times V_2$; V_1 and V_2 are the factor submanifolds coordinated respectively by $\{x^\alpha\}$ and $\{x^A\}$ and endowed with two-dimensional metrics, say h_1 and h_2 (associated line elements $d\sigma_1^2$ and $d\sigma_2^2$), $\hat{g}$ then being $\hat{g} = h_1 \oplus h_2$. In the remainder of this paper, and whenever needed, we shall refer to those as the *2-spaces* (V_1, h_1) *and* (V_2, h_2).

A 2 + 2 reducible spacetime $(M, \hat{g})$ is of the Petrov type D or O, and therefore any conformally related spacetime will also be of one of those types. Further, 2 + 2 reducible spacetimes can be invariantly characterized by the existence of two null recurrent vector fields, say $\hat{l}^a$ and $\hat{k}^a$, such that $\hat{l}^a \hat{k}_a = -1$, which can always be scaled in such a way that the recurrence vector is parallel to one of them, say $\hat{l}^a$ (see for instance [5]), i.e.,

$$\hat{l}_{a/b} = \alpha \hat{l}_a \hat{l}_b, \qquad \hat{k}_{a/b} = -\alpha \hat{k}_a \hat{l}_b, \tag{5}$$

where a 'slash' denotes the covariant derivative with respect to the connection associated with $\hat{g}$, and α is some real function of the coordinates associated with the integral distribution spanned by $\hat{l}^a$ and $\hat{k}^a$ (x^α in the above notation). This invariant characterization of $(M, \hat{g})$ in turn provides an invariant characterization of (M, g). To see this, define the following vector fields in (M, g): $l^a \equiv e^{-\mu} \hat{l}^a$ and $k^a \equiv e^{-\mu} \hat{k}^a$; clearly they are both null and satisfy $l^a k_a = -1$. Computing their covariant derivatives in (M, g) (see for instance [1] for the relationship between the connections associated with g and $\hat{g}$), one obtains

$$l_{a;b} = \alpha\, e^{-\mu} l_a l_b - \mu_{,a} l_b + (\mu_{,c} l^c) g_{ab}, \qquad k_{a;b} = -\alpha\, e^{-\mu} k_a l_b - \mu_{,a} k_b + (\mu_{,c} k^c) g_{ab}, \tag{6}$$

so that l^a and k^a are geodesic (although not affinely parametrized), shearfree and hypersurface orthogonal. Thus we have shown

Theorem 1. *Let* (M, g) *be a spacetime. If there exists a function* $\mu : M \to \mathbb{R}$ *and null vectors* l^a *and* k^a ($l^a k_a = -1$) *satisfying (6), then* (M, g) *is conformally related to a* 2 + 2 *reducible spacetime with conformal factor* $\exp(2\mu)$.

Likewise, most of the properties regarding the symmetries of (M, g) can be deduced from those of $(M, \hat{g})$, providing a classification scheme for this class of spacetimes. Thus, consider the reducible spacetime $(M, \hat{g})$ with line element given by (4), Coley and Tupper [6] showed that $(M, \hat{g})$ cannot admit a proper CKV unless it is conformally flat (CF), in which case $d\sigma_1^2$ and $d\sigma_2^2$ must be of constant curvature k_1, k_2, respectively, with $k_1 + k_2 = 0$. Thus, by the Defrise–Carter theorem [7], the CKV (proper or otherwise) of a non-CF spacetime (M, g) correspond to the Killing vectors (KV) or homothetic vectors (HV) of $(M, \hat{g})$; i.e., we need to find only the KV and HV of the reducible spacetime (4) in order to find the conformal Killing vectors (CKV) of spacetime (1). Also, it is known that if $(M, \hat{g})$ is not CF, then its KV are just the KV of the 2-spaces with metrics $d\sigma_1^2$ and $d\sigma_2^2$, i.e., if $\zeta^a = (\zeta^0, \zeta^1)$ is a KV of $d\sigma_1^2$, then $\xi^a = (\zeta^0, \zeta^1, 0, 0)$ is a KV of $(M, \hat{g})$, etc. Also $(M, \hat{g})$ will admit a HV iff each of $d\sigma_1^2$ and $d\sigma_2^2$ admit a HV, i.e. if $\kappa^a = (\kappa^0, \kappa^1)$ and $\lambda^a = (\lambda^2, \lambda^3)$ are HV of the 2-spaces (adjusted to the same numerical values of the respective homothetic scalars), then $\eta^a = (\kappa^0, \kappa^1, \lambda^2, \lambda^3)$ is a HV of $(M, \hat{g})$ with the same value for its homothetic scalar [9, 6].

The following theorems summarize some results concerning two-dimensional spaces and their isometries and homotheties:

Theorem 2. *Let V be a 2-space with metric h of arbitrary signature; the following statements can then be made regarding its isometries and generating KV.*

(1) A 2-space of constant curvature admits the maximum number of three KV.
(2) No 2-space can admit only two KV, i.e., if it admits two KV it admits a third KV and is then of constant curvature.
(3) A 2-space which is not of constant curvature may admit one non-null KV only. Then, provided that the KV $\vec{\xi}$ has no fixed points, coordinates may be chosen so that the metric is diagonal and independent of one coordinate, and the KV is in the direction of that coordinate, i.e., $\xi^a = \delta^a_i$, for some i, and the metric reads

$$\mathrm{d}\sigma^2 = \exp(2g(x^j))(\varepsilon_1(\mathrm{d}x^i)^2 + \varepsilon_2(\mathrm{d}x^j)^2), \tag{7}$$

where $g(x^j)$ is an arbitrary function of x^j and ε_1 and ε_2 are either $+1$ or one is $+1$ and the other is -1, depending on the signature of V.
(4) A 2-space admitting one null KV is necessarily of constant zero curvature.

Proof. Statements (1) and (2) above are well known, and we refer the reader to [1], where standard forms for both the metric and KV of 2-spaces of constant curvature can be found.

As for statement (3), consider the line element associated with h written as

$$\mathrm{d}\sigma^2 = \Omega^{-2}(\epsilon\, \mathrm{d}(x^1)^2 + \mathrm{d}(x^2)^2), \qquad \epsilon = \pm 1; \tag{8}$$

Killing's equations imply for $\vec{\xi}$

$$\xi^1_{,1} = \xi^2_{,2}, \qquad \xi^1_{,2} = -\epsilon \xi^2_{,1}. \tag{9}$$

In order for coordinates $x^{1'} = \alpha(x^1, x^2)$ and $x^{2'} = \beta(x^1, x^2)$ to exist satisfying $\vec{\xi} = \partial_{1'}$ and $h^{1'2'} = 0$ ($h_{a'b'}$ then being diagonal) it must be that

$$\alpha_{,1}\xi^1 + \alpha_{,2}\xi^2 = 1, \tag{10}$$

$$\beta_{,1}\xi^1 + \beta_{,2}\xi^2 = 0, \tag{11}$$

$$\epsilon\alpha_{,1}\beta_{,1} + \alpha_{,2}\beta_{,2} = 0, \tag{12}$$

and from the elementary theory of partial differential equations, it readily follows that the above system always has a solution[3].

In order to prove statement (4) above, let $\vec{\xi}$ be a null KV in a two-dimensional spacetime, and consider Killing's equations written as $\xi_{a;b} = F_{ab}$, $F_{ab} = -F_{ba}$; the bivector F_{ab} is then necessarily zero or timelike simple, but since $\vec{\xi}$ is null, it follows $\xi^a F_{ab} = 0$, hence F_{ab} cannot be timelike and is therefore zero. One can then set up a null diad $\vec{l}, \vec{n}$ such that $h_{ab} = l_a n_b + n_a l_b$(i.e., $l^a l_a = n^a n_a = 0, l^a n_a = 1$) where $\vec{\xi} = \vec{l}$, hence $l_{a;b} = 0$ and then $n_{a;b} = 0$ also, from where it immediately follows that V must be flat.

It is worth noticing that, since $F_{ab} = 0$, it follows that $\vec{\xi}$ cannot have any fixed point; i.e., a point p at which $\vec{\xi}(p) = 0$ since otherwise $\vec{\xi} = 0$ at every point $p \in V$. □

Theorem 3. *Let V be a 2-space with metric h of arbitrary signature; the following statements can then be made regarding its (proper) HV.*

(1) A 2-space of constant curvature admits a HV iff it is of curvature zero.

[3] Notice that the above equations imply $\alpha_{,1} = \epsilon F \xi^1$ and also $\alpha_{,2} = F\xi^2$, with $F = (\epsilon(\xi^1)^2 + (\xi^2)^2)^{-1}$ and it is then immediately seen that the condition $\alpha_{,12} = \alpha_{,21}$ is identically satisfied. Similar remarks also apply to function β.

(2) *A 2-space which is not of constant curvature may admit one non-null HV, $\vec{\eta}$, in which case, provided that the HV has no fixed points, coordinates may be chosen such that $\eta^a = \delta^a_i$ for some i, and the metric is of the form*

$$\mathrm{d}\sigma^2 = \exp(2\psi x^i)\exp(2g(x^j))(\varepsilon_1(\mathrm{d}x^i)^2 + \varepsilon_2(\mathrm{d}x^j)^2), \tag{13}$$

where ψ = constant is the homothetic scalar, $g(x^j)$ is an arbitrary function of x^j, and ε_1 and ε_2 are either +1 or one is +1 and the other is −1.

(3) *A two-dimensional spacetime admitting a null HV is necessarily of constant zero curvature.*

(4) *If a 2-space admits a (proper) HV and a KV, then it is either of zero constant curvature, and the KV and HV commute, or they do not commute and the line element can be written as*

$$\mathrm{d}\sigma^2 = (x^2)^{2(\psi-1)}(\epsilon\,\mathrm{d}(x^1)^2 + \mathrm{d}(x^2)^2), \qquad \epsilon = \pm 1, \tag{14}$$

the HV then being $\vec{\eta} = x^1\partial_1 + x^2\partial_2$ and the KV $\vec{\xi} = \partial_1$.

Proof. Statement (1) follows from the fact that, for any HV $\vec{\eta}$ with homothetic constant ψ, one has that $\mathcal{L}_{\vec{\eta}} R = -2\psi R$, R being the Ricci scalar; thus if R is constant, the above equation implies that it must in fact be zero (provided the $\psi \neq 0$, i.e., $\vec{\eta}$ is a proper HV); the converse (i.e., a 2-space of zero constant curvature admits a proper HV) holds trivially for considering the canonical form of the metric for one such space; that is, $\mathrm{d}\sigma^2 = [\epsilon\,\mathrm{d}(x^1)^2 + \mathrm{d}(x^2)^2]$, $\epsilon = \pm 1$, it is then immediately seen that the vector field $\vec{\eta} = \psi(x^1\partial_1 + x^2\partial_2)$ is a proper HV with homothetic constant ψ. The proof of statement (2) runs much along the same lines as that of statement (3) in theorem 2, that is, one shows that coordinates always exist such that $\vec{\eta}$ is aligned with one of them (i.e., $\eta^{a'} = \delta^{a'}_{i'}$ for some i') and the metric is diagonal; imposing the homothetic equation $\mathcal{L}_{\vec{\eta}} h = 2\psi h$ written in those coordinates and taking into account the remaining freedom in the definition of the other coordinate $x^{j'}$, yields the desired result.

Statement (3) can be proved either by coordinate methods, as in the previous case, or using an approach similar to that in the proof of statement (4) in theorem 2. Thus, consider the homothetic equation $\eta_{a;b} = \psi h_{ab} + F_{ab}$, where $F_{ab} = -F_{ba}$ is the homothetic bivector; it follows (see for instance [8]) that

$$F_{ab;c} = R_{abcd}\eta^d. \tag{15}$$

Since $\vec{\eta}$ is null, it satisfies $\eta^a\eta_{a;b} = 0$ and therefore $F_{ab}\eta^b = \psi\eta_a$ and one can then set up a null diad $\{\vec{l}, \vec{n}\}$ with $\vec{\eta} = \vec{l}$ such that $h_{ab} = l_a n_b + n_a l_b$ and $F_{ab} = \psi(l_a n_b - n_a l_b)$. From $l_{a;b} = \psi h_{ab} + F_{ab}$ and taking into account the above diad forms for h_{ab} and F_{ab}, it follows that $n_{a;b} = -2\psi n_a n_b$. A direct computation shows that $F_{ab;c} = 0$, and therefore, on account of (15) and the expression for the Riemann tensor of a two-space ($R_{abcd} = R/2(h_{ac}h_{bd} - h_{ad}h_{bc})$) it follows $R = 0$.

In order to prove statement (4) above, suppose that the KV $\vec{\xi}$ is aligned with, say x^1, so that (7) holds and one has $\vec{\xi} = \partial_1$ and $\mathrm{d}\sigma^2 = \exp(2g(x^2))[\epsilon\,\mathrm{d}(x^1)^2 + \mathrm{d}(x^2)^2]$. There exist then two inequivalent Lie algebra structures for the homothetic algebra spanned by $\vec{\xi}$ and the proper HV $\vec{\eta}$, namely[4]

$$[\vec{\xi}, \vec{\eta}] = 0, \qquad [\vec{\xi}, \vec{\eta}] = a\vec{\xi}, \tag{16}$$

where $a(\neq 0)$ is a constant (which can be set equal to 1 by rescaling conveniently $\vec{\eta}$). In the first case, $\vec{\eta} = \eta^1(x^2)\partial_1 + \eta^2(x^2)\partial_2$ and imposing the homothetic equations on the above line element (and rescaling coordinates conveniently), it readily follows

$$\mathrm{d}\sigma^2 = \mathrm{e}^{(2\psi x^2)}[\epsilon\,\mathrm{d}(x^1)^2 + \mathrm{d}(x^2)^2], \qquad \vec{\eta} = \partial_2, \qquad \vec{\xi} = \partial_1,$$

[4] The Lie bracket of a KV and a proper HV must necessarily be a KV.

and the two-space is then of curvature zero. In the second case, $\vec{\eta} = (ax^1 + F(x^2))\partial_1 + \eta^2(x^2)\partial_2$ and, following the same procedure as in the previous case, one gets, after some trivial rescaling of the coordinates,

$$d\sigma^2 = (x^2)^{2m}[\epsilon\, d(x^1)^2 + d(x^2)^2], \qquad \vec{\eta} = x^1\partial_1 + x^2\partial_2, \qquad \vec{\xi} = \partial_1,$$

the homothetic constant being $\psi = m + 1$.

Exchanging the coordinates x^1 and x^2 above, one gets another line element (and its corresponding Lie algebra of homotheties) thus completing the list of all possible metrics and homothetic vectors in this case. □

When dealing with 2-spaces of zero signature (such as those with line element $d\sigma_1^2$) it is sometimes useful to use null coordinates, say u and v, so that the line element takes the form

$$d\sigma_1^2 = -2G(u, v)\, du\, dv, \tag{17}$$

where G is a function of its arguments. The next theorem contains the analogues of theorems 2 and 3 specialized to this case:

Theorem 4. *Let V be a 2-space with metric h of zero signature, and suppose that null coordinates u, v are chosen so that the line element takes the form (17). The following statements can then be made regarding its KV and (proper) HV.*

(1) (V, h) is of constant curvature k if and only if it admits the maximum of three KV, in which case the metric and KV read, for $k = \pm 1$ ($\neq 0$),

$$d\sigma_1^2 = -4\frac{du\, dv}{k(u + v)^2}, \qquad \vec{\xi}_1 = u^2\partial_u - v^2\partial_v, \qquad \vec{\xi}_2 = u\partial_u + v\partial_v, \qquad \vec{\xi}_3 = \partial_u - \partial_v \tag{18}$$

and, for $k = 0$,

$$d\sigma_1^2 = -2\, du\, dv, \qquad \vec{\xi}_1 = \partial_u, \qquad \vec{\xi}_2 = \partial_v, \qquad \vec{\xi}_3 = u\partial_u - v\partial_v. \tag{19}$$

In this case ($k = 0$) it also admits the proper HV

$$\vec{\eta} = u\partial_u + v\partial_v, \tag{20}$$

with homothetic constant $\psi = 1$.

(2) (V, h) may admit one KV $\vec{\xi}$ and one (proper) HV $\vec{\eta}$ with homothetic constant $\psi = m + 1$ (with $m \neq -1$, else V is of constant zero curvature), and one then has in the chosen coordinates

$$d\sigma_1^2 = -2(u + v)^{2m}\, du\, dv, \qquad \vec{\xi} = \partial_u - \partial_v, \qquad \vec{\eta} = u\partial_u + v\partial_v. \tag{21}$$

(3) (V, h) admits one KV only, then the line element and the KV read

$$d\sigma_1^2 = -2G(u + v)\, du\, dv, \qquad \vec{\xi} = \partial_u - \partial_v, \tag{22}$$

where $G(u + v)$ is any function of $u + v$ excluding the one appearing in (18).

(4) (V, h) may admit just one proper HV, $\vec{\eta}$, with homothetic constant ψ, then,

$$d\sigma_1^2 = -2\, e^{-\psi(u-v)}\, e^{2g(u+v)}\, du\, dv, \qquad \vec{\eta} = \partial_u - \partial_v, \tag{23}$$

where $g(u + v)$ is an arbitrary function.

Proof. The proofs of statements (2), (3) and (4) above are rather straightforward and run much along the same lines as those of the corresponding statements in the previous theorems, therefore, we shall omit them. As for statement (1), consider the line element (17); it will be of constant curvature $\pm\lambda^2$ if and only if

$$R_{0101} = -G_{,uv} + G^{-1}G_{,u}G_{,v} = \mp\lambda^2 G^2,$$

that is,

$$(\ln G)_{,uv} = \pm\lambda^2 G. \tag{24}$$

This is Liouville's equation, the solution of which, for $k \neq 0$, is

$$G = \frac{2U_{,u}(u)V_{,v}(v)}{k(U(u) + V(v))^2}, \tag{25}$$

where U, V are arbitrary functions of u and v, respectively. In this case the 2-space metric becomes

$$\mathrm{d}\sigma_1^2 = -4\frac{U_{,u}(u)V_{,v}(v)}{(\pm\lambda^2)(U(u) + V(v))^2}\,\mathrm{d}u\,\mathrm{d}v, \tag{26}$$

i.e.,

$$\mathrm{d}\sigma_1^2 = -4\frac{\mathrm{d}U\,\mathrm{d}V}{(\pm\lambda^2)(U + V)^2}, \tag{27}$$

which corresponds to (17) with $G = 2(\pm\lambda^2)^{-1}(U + V)^{-2}$. Henceforth, we shall revert to using lower case letters for the coordinates.

When $\lambda^2 = 0$, the above equation leads to $G(u, v) = P(u)Q(v)$ and a trivial redefinition of the coordinates u and v leads to the flat 2-space:

$$\mathrm{d}\sigma_1^2 = -2\,\mathrm{d}u\,\mathrm{d}v. \tag{28}$$

□

The cases when the KV or the HV have fixed points, will be treated in section 3.

Taking into account these results, we see that, if $(M, \hat{g})$ is not CF (or flat), the possible symmetries are

(1) 6 KV, which occurs if each of $\mathrm{d}\sigma_1^2$, $\mathrm{d}\sigma_2^2$ are of constant curvature k_1, k_2 with $k_1 + k_2 \neq 0$, i.e., each 2-space admits the maximum 3 KV;
(2) 4 KV and 1 HV, whenever one of the 2-spaces is of constant zero curvature (thus also admitting a HV) and the other admitting one HV and one KV;
(3) 4 KV, which occurs when one 2-space is of constant curvature and the other admits one KV only;
(4) 3 KV and 1 HV, which occurs when one 2-space is of constant zero curvature and the other is not of constant curvature but admits a HV and no KV;
(5) 3 KV, when one of the 2-spaces is of constant curvature and the other has neither KV nor HV;
(6) 2 KV and 1 HV, occurring when both 2-spaces admit one KV and one HV each;
(7) 2 KV, which occurs when each 2-space admits one KV only;
(8) 1 KV and 1 HV, whenever one of the 2-spaces admits a KV and both of them a HV;
(9) 1 KV, which occurs when only one 2-space admits a KV;
(10) 1 HV, which occurs when each 2-space admits a HV and no KV;
(11) neither of 2-space admits any KV or HV.

Notice that in the first five cases, at least one of the two 2-spaces is of constant curvature; this corresponds to the well-known cases of spherical, plane or hyperbolic symmetries.

The case in which the 2 + 2 reducible spacetime $(M, \hat{g})$ is CF (or flat) will be dealt with separately in section 4, where some new (non-standard) coordinate systems will be introduced and their utility discussed.

3. Isometries and homotheties with fixed points in 2-spaces

Given a one-parameter Lie group of transformations $\{\varphi_t\}$ acting on a manifold M, any point $p \in M$ such that $\varphi_t(p) = p$, $\forall t$ is called a *fixed point* of the transformation group. Denoting by $\vec{\xi}$ the infinitesimal generator of that group in any coordinate chart, it is easy to see that the set of fixed points coincides precisely with the set of zeros of $\vec{\xi}$, i.e. $p \in M$ is a fixed point under the transformation group $\{\varphi_t\}$ iff $\vec{\xi}(p) = 0$. A thorough discussion of the properties and structure of the sets of fixed points of these and other more general types of transformations, can be found in [8], [10] and also [11].

In this section we are only concerned with fixed points of isometries and homotheties on two-dimensional manifolds V, endowed with a positive definite or a Lorentz metric h. We refer the reader to [12], and references cited therein, for a detailed discussion of this case while we next present a summary of the basic facts concerning this problem.

Consider a HV $\vec{\xi}$ with a fixed point $p \in V$. The homothetic equation can be written as $\xi_{a;b} = \psi h_{ab} + F_{ab}$, where ψ is the homothetic constant (zero if $\vec{\xi}$ is a KV), and the bivector F_{ab} satisfies [8] equation (15), i.e., $F_{ab;c} = R_{abcd}\xi^d$. It then follows that, at p, $\xi^a(p) = F_{ab;c}(p) = 0$ in any coordinate chart; if $\vec{\xi}$ is a KV, then $F_{ab}(p) \neq 0$ necessarily. Furthermore, at the fixed point p, the differential map φ_{t*} is a map of T_pV onto itself which can be seen to be $\varphi_{t*} = \exp(tA)$, A being a matrix whose elements are $A^a{}_b = \xi^a{}_{;b}(p) = \xi^a{}_{,b}(p)$ (see, for instance, [13]); furthermore, it follows that

$$\chi \circ \varphi_{t*} = \varphi_t \circ \chi, \tag{29}$$

where χ is the exponential diffeomorphism from some open neighbourhood of $\vec{0} \in T_pV$ to some open neighbourhood U of $p \in V$.[5] As a consequence, in the resulting normal exponential coordinate system in U, the integral curves of the HV, $x^a(t)$, satisfy $\dot{x}^a = A^a{}_b x^b$; i.e., $\xi^a = A^a{}_b x^b$ hence the components of the HV, in this particular coordinate system, are linear functions of the coordinates. This allows one to find coordinate expressions for both the HV $\vec{\xi}$ and the metric h in the neighbourhood U of p as well as to study the structure of the fixed point set in U, for it is easy to see from (29) that such a set is a submanifold of V (which can be shown to be totally geodesic [11]) of dimension two-rank A. In the case of a KV, since $A^a{}_b = F^a{}_b$ it follows that the dimension of the fixed point set is 0, hence the fixed point is necessarily isolated. For a HV, $A^a{}_b = \psi\delta^a{}_b + F^a{}_b$, and the dimension can then be 0 (isolated fixed point) or 1, in which case it is part of a null geodesic [8, 12] (and in particular this can only happen when h is a Lorentz metric). One can now classify the different possibilities according to the signature of the metric and the canonical type of the matrix A or, equivalently, the nature of the fixed point p from the point of view of the dynamical systems theory. The following situations may then arise (see [8, 12]).

(1) h is positive definite and $\vec{\xi}$ is a proper HV. The fixed point p is then necessarily isolated and asymptotically stable, and V is flat in some neighbourhood of p, thus corresponding to one of the cases treated later.

(2) h is positive definite and $\vec{\xi}$ is a KV. Again, p is necessarily isolated, a centre in fact, its orbits being closed periodic curves.

(3) h is a Lorentz metric and $\vec{\xi}$ is a proper HV. The homothetic bivector is then necessarily timelike and three different cases may arise, depending on its eigenvalues:

   (a) p is isolated and asymptotically stable; it then follows that V is flat in some neighbourhood of p;

[5] This result and the previous one on φ_{t*} at a fixed point, hold for affine collineations of which, both HV and KV are particular cases.

(b) p is part of a null geodesic of zeros of $\vec{\xi}$. It can then be shown that V must again be flat in some neighbourhood of p;

(c) p is isolated and is a saddle point of the autonomous plane system $\dot{x}^a = A^a{}_b x^b$.

(4) h is a Lorentz metric and $\vec{\xi}$ is a KV.

Cases 1, (3a) and (3b) above are trivial in the sense that they correspond to flat 2-space. Case (2) corresponds to axial symmetry, and one can show (see [14] and references cited therein for a discussion on axial symmetry in the context of GR) that coordinates $\{x, y\}$ exist such that the KV takes the form $\vec{\xi} = y\partial_x - x\partial_y$ and p is then the origin (0, 0). Alternatively, 'polar' coordinates $\{\rho, \phi\}$ can be introduced such that $\vec{\xi} = \partial_\phi, \phi \in [0, 2\pi)$ and the metric then takes the form $\mathrm{d}\sigma^2 = \mathrm{d}\rho^2 + A(\rho)^2\,\mathrm{d}\phi^2$, but one should carefully note that one such coordinate chart does not cover the fixed point p (see [14] for details).

As for case (3c), from $\xi^a = A^a{}_b x^b$ together with the condition that p is a saddle point of the above autonomous plane system (this means that the matrix $A^a{}_b$ has two non-zero real eigenvalues of different sign), it follows that coordinates x^0, x^1 exist in that exponential neighbourhood such that the HV takes the form

$$\vec{\xi} = \psi((1+\alpha)x^0\partial_0 + (1-\alpha)x^1\partial_1), \qquad |\alpha| > 1, \tag{30}$$

and the line element is then

$$\mathrm{d}\sigma^2 = A(\mathrm{d}x^0)^2 + 2B\,\mathrm{d}x^0\,\mathrm{d}x^1 + C(\mathrm{d}x^1)^2, \tag{31}$$

with

$$A = F\left(\frac{(x^0)^{1/(1+\alpha)}}{(x^1)^{1/(1-\alpha)}}\right)(x^0)^{\frac{2\alpha}{(1+\alpha)}}(x^1)^{\frac{2\alpha}{(1-\alpha)}}, \tag{32}$$

$$B = B\left(\frac{(x^0)^{1/(1+\alpha)}}{(x^1)^{1/(1-\alpha)}}\right), \tag{33}$$

$$C = G\left(\frac{(x^0)^{1/(1+\alpha)}}{(x^1)^{1/(1-\alpha)}}\right)(x^0)^{-2\alpha/(1+\alpha)}(x^1)^{-2\alpha/(1-\alpha)}. \tag{34}$$

It is then easy to see that the coordinate changes preserving the form (30) of the HV, and therefore the above expression of the metric, are

$$x^{0'} = K\left(\frac{(x^0)^{1/(1+\alpha)}}{(x^1)^{1/(1-\alpha)}}\right)(x^0)^{\frac{1}{2}}(x^1)^{\frac{1+\alpha}{2(1-\alpha)}}, \qquad x^{1'} = L\left(\frac{(x^0)^{1/(1+\alpha)}}{(x^1)^{1/(1-\alpha)}}\right)(x^0)^{\frac{1-\alpha}{2(1+\alpha)}}(x^1)^{\frac{1}{2}}; \tag{35}$$

but no such coordinate change exists, with a non-vanishing Jacobian on a neighbourhood of the fixed point that transforms some metric coefficient to zero.

In the case (4) above ($\vec{\xi}$ a KV, h a Lorentz metric and the fixed point is then isolated and, again, turns out to be a saddle point), similar arguments to those above show that coordinates x^0, x^1 exist in a neighbourhood of the fixed point p such that the KV [6] and the line element read, respectively,

$$\vec{\xi} = x^0\partial_0 - x^1\partial_1, \tag{36}$$

$$\mathrm{d}s^2 = \frac{x^1}{x^0}A(x^0x^1)(\mathrm{d}x^0)^2 + 2B(x^0x^1)\,\mathrm{d}x^0\,\mathrm{d}x^1 + \frac{x^0}{x^1}C(x^0x^1)(\mathrm{d}x^1)^2, \tag{37}$$

[6] Alternatively, another set of coordinates exists on U, say t, x such that $\vec{\xi} = x\partial_t + t\partial_x$, i.e., $\vec{\xi}$ is the familiar generator of a boost in the x direction.

where A, B and C are arbitrary functions of their arguments. The coordinate changes preserving the above forms of $\vec{\xi}$ and the metric are

$$x^{0'} = K(x^0x^1)\left(\frac{x^0}{x^1}\right)^{\frac{1}{2}}, \qquad x^{1'} = L(x^0x^1)\left(\frac{x^1}{x^0}\right)^{\frac{1}{2}}. \tag{38}$$

In this case though, it is not difficult to see that one can always, by means of one such change (of a nowhere-vanishing Jacobian in the neighbourhood of p), set $h_{0'0'} = h_{1'1'} = 0$ and therefore the line element reads (dropping primes for convenience)

$$\mathrm{d}s^2 = 2B(x^0x^1)\,\mathrm{d}x^0\,\mathrm{d}x^1. \tag{39}$$

The above considerations on fixed points lead to the following 'curious' example. Consider a reducible spacetime with line element of the form (4) where $\mathrm{d}\sigma_1^2$ is of constant curvature (and therefore possesses 3 KV) and $\mathrm{d}\sigma_2^2$ is axially symmetric, i.e., $\mathrm{d}\sigma_2^2 = \mathrm{d}\rho^2 + A(\rho)^2\,\mathrm{d}\phi^2$ and admits (for an arbitrary $A(\rho)$) just one KV, namely $\vec{\xi} = \partial_\phi$. The whole spacetime then admits 4 KV which act everywhere on three-dimensional orbits, *except* at points with $\rho = 0$ ($x = y = 0$ in Cartesian-like coordinates) which form a two-dimensional surface. Thus we have an example of a G_4 of isometries acting on a two-dimensional orbit. This does not contradict Fubini's theorem (see [1]) as it would seem at first sight[7].

4. Spherical, plane and hyperbolic symmetric CF spacetimes

We first consider the cases in which $\mathrm{d}\sigma_2^2$ is of constant curvature, i.e., the spacetime is spherically symmetric, plane symmetric or hyperbolic symmetric. MacCallum [15] has shown that, given the general spherically symmetric metric,

$$\mathrm{d}\Sigma^2 = -\mathrm{e}^{2\nu(t,r)}\,\mathrm{d}t^2 + \mathrm{e}^{2\omega(t,r)}\,\mathrm{d}r^2 + Y^2(t,r)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \tag{40}$$

if $Y_{,a}Y^{,a} \neq 0$, one can always introduce isotropic coordinates such that $Y = r\,\mathrm{e}^{\omega(t,r)}$. Precisely the same argument applies in the cases of plane and hyperbolic symmetry. It follows that, when $Y_{,a}$ is neither null nor zero, the reducible spacetime $(M, \hat{g})$ with $\mathrm{d}\sigma_2^2$ of constant curvature can be written in the form

$$\mathrm{d}\Sigma^2 = -\mathrm{e}^{2(\nu-\omega)}r^{-2}\,\mathrm{d}t^2 + r^{-2}\,\mathrm{d}r^2 + \mathrm{d}\Omega^2, \tag{41}$$

where $\nu = \nu(t,r)$, $\omega = \omega(t,r)$ and

$$\mathrm{d}\Omega^2 = \mathrm{d}y^2 + \eta^2(y,k)\,\mathrm{d}z^2, \tag{42}$$

with

$$\eta(y,k) = \begin{cases} \sin y & \text{if} \quad k = +1 \quad \text{(spherical symmetry)} \\ 1 & \text{if} \quad k = 0 \quad \text{(plane symmetry)} \\ \sinh y & \text{if} \quad k = -1 \quad \text{(hyperbolic symmetry).} \end{cases} \tag{43}$$

The original spherically, plane or hyperbolic symmetric spacetime metric is then recovered from (41) by multiplying it by the conformal factor $r^2\,\mathrm{e}^{2\omega}$.

Notice that the above choice of $\eta(y,k)$ is not the standard one for $k = 0$ (as it appears, for instance, in [1], i.e., $\eta(y,0) = y$), but it is easy to see that both are related by a trivial coordinate change.

The constant curvature k_2 of $\mathrm{d}\Omega^2$ (i.e. $\mathrm{d}\sigma_2^2$) is equal to k and so $(M, \hat{g})$ will be CF if

$$\mathrm{d}\sigma_1^2 = -\mathrm{e}^{2(\nu-\omega)}r^{-2}\,\mathrm{d}t^2 + r^{-2}\,\mathrm{d}r^2 \tag{44}$$

[7] The authors are very grateful to Professor G S Hall for enlightening discussions over this point.

has constant curvature $k_1 = -k$. If $d\sigma_1^2$ is of constant curvature $k_1 \neq -k$, then $d\sigma_1^2$ admits 3 KV to add to the 3 KV of $d\sigma_2^2$.

Before proceeding, let us comment on the use of isotropic coordinates. If one uses standard coordinates as in (40), the condition for $d\sigma_1^2$ to be of constant curvature contains derivatives of ν, ω and Y with respect to t and with respect to r [3] and is impossible to solve. On the other hand, in isotropic coordinates, the condition contains r derivatives only and is easy to integrate. One could, of course, use the very simple expressions for constant curvature 2-spaces, as in Petrov [16], but the isotropic coordinates are more useful in discussing perfect fluid and other physical spacetimes, as will be shown later.

If $Y^{,a}Y_{,a} = 0$, then either $Y^{,a}$ is a null vector or $Y =$ constant. If $Y^{,a}$ is a null vector, the corresponding metric can be written in the following form:

$$d\Sigma^2 = -2G(u, v)\, du\, dv + u^2\, d\Omega^2, \tag{45}$$

where u, v are null coordinates. If $Y_{,a} = 0$, the metric takes the form

$$d\Sigma^2 = -2G(u, v)\, du\, dv + d\Omega^2. \tag{46}$$

In the first case, a redefinition of the function $G(u, v)$ given by $G(u, v) \to u^2 G(u, v)$ shows that in either case the underlying reducible spacetime $(M, \hat{g})$ is of the above form (46), i.e.,

$$d\Sigma^2 = -2G(u, v)\, du\, dv + d\Omega^2.$$

This situation will be discussed in section 4.4.

The 2-space $d\sigma_1^2$ given by (44) has constant curvature k_1 iff the following condition holds:

$$\nu_{,rr} - \omega_{,rr} - (\nu_{,r} - \omega_{,r})^2 - \frac{1}{r}(\nu_{,r} - \omega_{,r}) + \frac{k_1 + 1}{r^2} = 0. \tag{47}$$

Putting $U = e^{(\nu-\omega)}$, equation (47) becomes

$$r^2 U_{,rr} - rU_{,r} + (k_1 + 1)U = 0. \tag{48}$$

If $k_1 = -k^2$, then

$$U = e^{(\nu-\omega)} = a(t)r^{1-k} + b(t)r^{1+k}. \tag{49}$$

If $k_1 = 0$, then

$$U = e^{(\nu-\omega)} = r[a(t) + b(t)\ln r]. \tag{50}$$

If $k_1 = k^2$, then

$$U = e^{(\nu-\omega)} = r[a(t)\cos(k \ln r) + b(t)\sin(k \ln r)]. \tag{51}$$

In each case the coordinate t can be chosen so that $b(t) = 1$ unless $b(t) = 0$ in which case $a(t)$ can be incorporated into the definition of t and the metric then becomes static. It follows that conformally flat 2 + 2 reducible spacetimes have metrics of the following forms:

$$d\Sigma^2 = -r^{-2}[a(t) + r^2]^2\, dt^2 + r^{-2}\, dr^2 + d\theta^2 + \sin^2\theta\, d\phi^2 \tag{52}$$

for spherical symmetry (i.e. $k_1 = -1$, $k = 1$),

$$d\Sigma^2 = -[a(t) + \ln r]^2\, dt^2 + r^{-2}\, dr^2 + dy^2 + dz^2 \tag{53}$$

for plane symmetry (i.e. $k_1 = 0$),

$$d\Sigma^2 = -[a(t)\cos(\ln r) + \sin(\ln r)]^2\, dt^2 + r^{-2}\, dr^2 + d\theta^2 + \sinh^2\theta\, d\phi^2 \tag{54}$$

for hyperbolic symmetry (i.e. $k_1 = +1$, $k = 1$).

4.1. Spherically symmetric CF spacetimes

In this case the metric is given by equation (52). Solving the CKV equations leads to

$$\begin{aligned}
\xi^0 &= r^2[a(t)+r^2]^{-2}\{\sin\theta(A_t\sin\phi - B_t\cos\phi) - C_t\cos\theta\} + D + \tfrac{1}{2}[a(t)+r^2]^{-1}D_{tt},\\
\xi^1 &= r^2\sin\theta(-A_r\sin\phi + B_r\cos\phi) + r^2C_r\cos\theta - rD_t,\\
\xi^2 &= \cos\theta(A\sin\phi - B\cos\phi) + C\sin\theta + a_1\sin\phi + a_2\cos\phi,\\
\xi^3 &= \operatorname{cosec}\theta(A\cos\phi + B\sin\phi) + \cot\theta(a_1\cos\phi - a_2\sin\phi) + a_3,
\end{aligned} \tag{55}$$

where A, B, C are functions of t, and r is given by

$$A, B, C = m_i(t)r + n_i(t)r^{-1}, \qquad (i = 1, 2, 3) \tag{56}$$

with

$$\dddot{m}_i + 4a\dot{m}_i + 2\dot{a}m_i = 0, \tag{57}$$

$$n_i = -\tfrac{1}{2}\ddot{m}_i - am_i. \tag{58}$$

The function $D(t)$ also satisfies equation (57), i.e.,

$$\dddot{D}_i + 4a\dot{D}_i + 2\dot{a}D_i = 0. \tag{59}$$

Note that equations (57) and (59) can be partially integrated to give

$$m_i\ddot{m}_i - \tfrac{1}{2}\dot{m}_i^2 + 2am_i^2 = \alpha_i, \tag{60}$$

$$D\ddot{D} - \tfrac{1}{2}\dot{D}^2 + 2aD^2 = \alpha_4, \tag{61}$$

where α_i, α_4 are constants.

The constants a_1, a_2, a_3 give rise to the three spherical KV. The conformal scalar ψ is given by

$$\psi = -A\sin\theta\sin\phi + B\sin\theta\cos\phi + C\cos\theta, \tag{62}$$

so the solutions for A, B, C (i.e., for m_i), give rise to nine constants representing the nine proper CKV while the three constants arising from equation (59), equivalently (61) lead to three further KV. Restoring the conformal factor $r^2\,e^{2\omega}$ will, in general, transform these three KV into three proper CKV, thus giving the twelve CKV and three KV of a general CF spherical spacetime.

4.2. Plane symmetric CF spacetimes

We use the coordinate transformation $x = \ln r$ to change metric (53) into the form

$$d\Sigma^2 = -[a(t)+x]^2\,dt^2 + dx^2 + dy^2 + dz^2, \tag{63}$$

which is, in fact, a flat spacetime metric. Solving the CKV equations leads to

$$\begin{aligned}
\xi^0 &= (a+x)^{-1}\left[\tfrac{1}{2}(\alpha_1 e^t - \alpha_2 e^{-t})(y^2+z^2) + (\gamma_1 e^t - \gamma_2 e^{-t} + \dot{f})y + (\delta_1 e^t - \delta_2 e^{-t} + \dot{g})z\right.\\
&\quad \left. + \tfrac{1}{2}a^2(\alpha_1 e^t - \alpha_2 e^{-t}) - (\epsilon_1 e^t - \epsilon_2 e^{-t} + \dot{h})\right]\\
&\quad + \tfrac{1}{2}(\alpha_1 e^t - \alpha_2 e^{-t})(a+x) - \int \dot{a}(\alpha_1 e^t - \alpha_2 e^{-t})\,dt + \omega_1,\\
\xi^1 &= -\tfrac{1}{2}(\alpha_1 e^t + \alpha_2 e^{-t})(y^2+z^2-x^2) - 2(\beta_1 y + \beta_2 z)x - (\gamma_1 e^t + \gamma_2 e^{-t} + f)y - (\delta_1 e^t\\
&\quad + \delta_2 e^{-t} + g)z + \epsilon_1 e^t + \epsilon_2 e^{-t} + x\int a(\alpha_1 e^t - \alpha_2 e^{-t})\,dt + \omega_2 x + h,
\end{aligned} \tag{64}$$

$$\xi^2 = \beta_1(x^2 - y^2 + z^2) - 2\beta_2 yz - \beta_3 z + (\alpha_1 e^t + \alpha_2 e^{-t})xy + (\gamma_1 e^t + \gamma_2 e^{-t} + f)x$$
$$+ y\int a(\alpha_1 e^t - \alpha_2 e^{-t})\,dt + \int a(\gamma_1 e^t - \gamma_2 e^{-t} + \dot f)\,dt + \omega_2 y + \omega_3,$$
$$\xi^3 = \beta_2(x^2 + y^2 - z^2) - 2\beta_1 yz + \beta_3 y + (\alpha_1 e^t + \alpha_2 e^{-t})xz + (\delta_1 e^t + \delta_2 e^{-t} + g)x$$
$$+ z\int a(\alpha_1 e^t - \alpha_2 e^{-t})\,dt + \int a(\delta_1 e^t - \delta_2 e^{-t} + \dot g)\,dt + \omega_2 z + \omega_4,$$
$$\psi = (\alpha_1 e^t + \alpha_2 e^{-t})x + \int a(\alpha_1 e^t - \alpha_2 e^{-t})\,dt - 2\beta_1 y - 2\beta_2 z + \omega_2, \tag{65}$$

where $\alpha_1, \alpha_2, \beta_1, \beta_2, \beta_3, \gamma_1, \gamma_2, \delta_1, \delta_2, \epsilon_1, \epsilon_2, \omega_1, \omega_2, \omega_3, \omega_4$ are 15 arbitrary constants and f, g, h are functions of t only satisfying

$$\ddot f - f + 2\beta_1 a = 0 \qquad (f = 0 \quad \text{if} \quad \beta_1 = 0), \tag{66}$$
$$\ddot g - g + 2\beta_2 a = 0 \qquad (g = 0 \quad \text{if} \quad \beta_2 = 0), \tag{67}$$
$$\ddot h - h = \dot a \int a(\alpha_1 e^t + \alpha_2 e^{-t})\,dt + a\int \dot a(\alpha_1 e^t + \alpha_2 e^{-t})\,dt + \dot a\omega_1 - a\omega_2, \tag{68}$$

with $h = 0$ if $\alpha_1, \alpha_2, \omega_1, \omega_2$ are all zero.

Note that ω_2 corresponds to a HV, $\alpha_1, \alpha_2, \beta_1, \beta_2$ correspond to SCKV and the others correspond to the ten KV of flat spacetime. Restoring the conformal factor $r^2 e^{2\omega}$, i.e. $e^{2(x+\omega)}$, gives rise to twelve CKV in general plus the three plane symmetric KV, which are the conformal symmetries of a general CF plane symmetric spacetime in isotropic coordinates.

4.3. Hyperbolic symmetric CF spacetimes

The coordinate transformation $x = \ln r$ and a re-labelling of the coordinates changes the metric (54) into the form

$$d\Sigma^2 = -[a(t)\cos x + \sin x]^2 dt^2 + dx^2 + dy^2 + \sinh^2 y\, dz^2. \tag{69}$$

Solving the CKV equations leads to

$$\xi^0 = (a\cos x + \sin x)^{-2}[\sinh y(A_t \sin z - B_t \cos z) - \cosh y C_t]$$
$$+ F - (a^{-1}F_t)_t \cos x(a\cos x + \sin x)^{-1},$$
$$\xi^1 = \sinh y(-A_x \sin z + B_x \cos z) + C_x \cosh y + a^{-1}F_t, \tag{70}$$
$$\xi^2 = \cosh y(A\sin z - B\cos z) - C\sinh y + a_1 \cos z + a_2 \sin z,$$
$$\xi^3 = \operatorname{cosech} y(A\cos z + B\sin z) + \coth y(-a_1 \sin z + a_2 \cos z) + a_3;$$
$$\psi = A\sinh y \sin z - B\sinh y\cos z - C\cosh y, \tag{71}$$

where A, B, C are functions of t and x given by

$$A, B, C = m_i(t)\cos x + n_i(t)\sin x \qquad (i = 1, 2, 3) \tag{72}$$

with

$$a\dddot n_i - \dot a\ddot n_i - a(1 + a^2)\dot n_i + \dot a n_i = 0, \tag{73}$$
$$am_i = \ddot n_i - n_i. \tag{74}$$

The function $F(t)$ satisfies the third-order equation

$$(a^{-1}\dot F)^{\cdot\cdot} - (a + a^{-1})\dot F - \dot a F = 0. \tag{75}$$

Equations (73) and (74) give rise to nine constants corresponding to nine CKV. The three constants in the solution of equation (75) lead to three KV and these twelve symmetries are, in general, twelve proper CKV when the conformal factor $r^2 e^{2\omega}$ (i.e. $e^{2(x+\omega)}$) is restored. The remaining three constants a_1, a_2, a_3 correspond to the KV of hyperbolic symmetry.

4.4. Spherically, plane and hyperbolic symmetric CF spacetimes in null coordinates

We now consider the case of the metric (46) in which the 2-space $\mathrm{d}\sigma_1^2$ is of constant curvature $k_1 = \pm 1$ or 0. If $k_1 = 0$ the flat metric (19) applies and the plane symmetric $2+2$ spacetime is just Minkowski spacetime expressed in null coordinates. The KV, HV and CKV of this are easily obtained from the standard Minkowski symmetries by a simple coordinate transformation. We will thus confine our attention to the spherical and hyperbolic cases.

For the case of spherical symmetry, the metric of the CF $2+2$ reducible spacetime is (see theorem 4 and equations (42) and (43))

$$\mathrm{d}\Sigma^2 = 4(u+v)^{-2}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2, \tag{76}$$

and solving the CKV equations yields

$$\begin{aligned}
\xi^0 &= \tfrac{1}{2}[(\alpha_1 u^2 - 2\alpha_2 u - \alpha_3)\sin\theta\sin\phi + (\beta_1 u^2 - 2\beta_2 u - \beta_3)\sin\theta\cos\phi \\
&\qquad + (\gamma_1 u^2 - 2\gamma_2 u - \gamma_3)\cos\theta] - \epsilon_1 u^2 + \epsilon_2 u - \epsilon_3, \\
\xi^1 &= \tfrac{1}{2}[(\alpha_1 v^2 - 2\alpha_2 v - \alpha_3)\sin\theta\sin\phi + (\beta_1 v^2 - 2\beta_2 v - \beta_3)\sin\theta\cos\phi \\
&\qquad + (\gamma_1 v^2 - 2\gamma_2 v - \gamma_3)\cos\theta] + \epsilon_1 v^2 + \epsilon_2 v + \epsilon_3, \\
\xi^2 &= (u+v)^{-1}[-(\alpha_1 uv + \alpha_2(u-v) + \alpha_3)\cos\theta\sin\phi - (\beta_1 uv + \beta_2(u-v) + \beta_3)\cos\theta\cos\phi \\
&\qquad + (\gamma_1 uv + \gamma_2(u-v) + \gamma_3)\sin\theta] + \delta_1\cos\phi + \delta_2\sin\phi, \\
\xi^3 &= (u+v)^{-1}[-(\alpha_1 uv + \alpha_2(u-v) + \alpha_3)\operatorname{cosec}\theta\cos\phi - (\beta_1 uv \\
&\qquad + \beta_2(u-v) + \beta_3)\operatorname{cosec}\theta\sin\phi] - \delta_1\cot\theta\sin\phi + \delta_2\cot\theta\cos\phi + \delta_3.
\end{aligned} \tag{77}$$

The constants $\alpha_1, \beta_i, \gamma_i$ $(i = 1, 2, 3)$ give rise to the nine proper CKV while the six KV are given by the constants ϵ_i and δ_i. The conformal scalar is given by

$$\begin{aligned}
\psi &= (u+v)^{-1}[(\alpha_1 uv + \alpha_2(u-v) + \alpha_3)\sin\theta\sin\phi + (\beta_1 uv + \beta_2(u-v) + \beta_3)\sin\theta\cos\phi \\
&\qquad + (\gamma_1 uv + \gamma_2(u-v) + \gamma_3)\cos\theta].
\end{aligned} \tag{78}$$

Also from theorem 4 and equations (42) and (43) it follows that the line element of the CF $2+2$ reducible spacetime in the case of hyperbolic symmetry is

$$\mathrm{d}\Sigma^2 = -4(u+v)^{-2}\,\mathrm{d}u\,\mathrm{d}v + \mathrm{d}y^2 + \sinh^2 y\,\mathrm{d}z^2, \tag{79}$$

and the CKV components are

$$\begin{aligned}
\xi^0 &= \tfrac{1}{2}[(\alpha_1 u^2 - 2\alpha_2 u - \alpha_3)\sinh y\sin z + (\beta_1 u^2 - 2\beta_2 u - \beta_3)\sinh y\cos z \\
&\qquad + (\gamma_1 u^2 - 2\gamma_2 u - \gamma_3)\cosh y] - \epsilon_1 u^2 + \epsilon_2 u - \epsilon_3, \\
\xi^1 &= \tfrac{1}{2}[(\alpha_1 v^2 - 2\alpha_2 v - \alpha_3)\sinh y\sin z + (\beta_1 v^2 - 2\beta_2 v - \beta_3)\sinh y\cos z \\
&\qquad + (\gamma_1 v^2 - 2\gamma_2 v - \gamma_3)\cosh y] + \epsilon_1 v^2 + \epsilon_2 v + \epsilon_3, \\
\xi^2 &= (u+v)^{-1}[-(\alpha_1 uv + \alpha_2(u-v) + \alpha_3)\cosh y\sin z - (\beta_1 uv + \beta_2(u-v) + \beta_3)\cosh y\cos z \\
&\qquad + (\gamma_1 uv + \gamma_2(u-v) + \gamma_3)\sinh y] + \delta_1\cos z + \delta_2\sin z, \\
\xi^3 &= (u+v)^{-1}[-(\alpha_1 uv + \alpha_2(u-v) + \alpha_3)\operatorname{cosech} y\cos z - (\beta_1 uv \\
&\qquad + \beta_2(u-v) + \beta_3)\operatorname{cosech} y\sin z] - \delta_1\coth y\sin z + \delta_2\coth y\cos z + \delta_3.
\end{aligned} \tag{80}$$

As in the previous case the nine proper CKV correspond to $\alpha_i, \beta_i, \gamma_i$ while the six KV correspond to ϵ_i, δ_i $(i = 1, 2, 3)$. The conformal scalar is given in this case by

$$\begin{aligned}
\psi &= (u+v)^{-1}[(\alpha_1 uv + \alpha_2(u-v) + \alpha_3)\sinh y\sin z + (\beta_1 uv + \beta_2(u-v) + \beta_3)\sinh y\cos z \\
&\qquad + (\gamma_1 uv + \gamma_2(u-v) + \gamma_3)\cosh y].
\end{aligned} \tag{81}$$

5. Non-CF spacetimes

In this section we deal with non-CF 2 + 2 reducible spacetimes. We refer the reader to the list of eleven cases which appears at the end of section 2 for the various possibilities regarding the symmetries of $(M, \hat{g})$.

The forms of the metrics h_1 and h_2 (or equivalently their associated line elements $d\sigma_1^2$ and $d\sigma_2^2$) and those of the symmetries (HV and KV) that the 2-spaces (V_1, h_1) and (V_2, h_2) admit can be gathered from the results in theorems 2, 3 and 4 (see also section 4 for issues regarding non-standard coordinate choices). We list them for the sake of clarity and summarize the various possibilities for $(M, \hat{g})$ in a table later in this section.

The possibilities for the Lorentz 2-space (V_1, h_1) are as follows.

- (V_1, h_1) admits 3 KV $\vec{\xi}_A$ $(A = 1, 2, 3)$ and one HV $\vec{\eta}$ with homothetic constant $\psi \neq 0$. (i.e., is of constant zero curvature).

$$d\sigma_1^2 = -dt^2 + dx^2, \qquad \vec{\xi}_1 = \partial_t, \qquad \vec{\xi}_2 = \partial_x, \qquad \vec{\xi}_3 = x\partial_t + t\partial_x, \tag{82}$$

$$\vec{\eta} = \psi(t\partial_t + x\partial_x). \tag{83}$$

Other useful coordinate gauges are

$$d\sigma_1^2 = -2\,du\,dv, \qquad \vec{\xi}_1 = \partial_u, \qquad \vec{\xi}_2 = \partial_v, \qquad \vec{\xi}_3 = u\partial_v - v\partial_u, \tag{84}$$

$$\vec{\eta} = \psi(u\partial_u + v\partial_v), \tag{85}$$

$$\begin{gathered} d\sigma_1^2 = -[a(t) + x]^2\,dt^2 + dx^2, \\ \vec{\xi}_A = [-(a+x)^{-1}(\epsilon_1\,e^t - \epsilon_2\,e^{-t} + \dot{h}) + \omega_1]\partial_t + [\epsilon_1\,e^t + \epsilon_2\,e^{-t} + h]\partial_x, \end{gathered} \tag{86}$$

$$\vec{\eta} = (-(a+x)^{-1}\dot{f})\partial_t + (\psi x + f)\partial_x, \tag{87}$$

where $\epsilon_1, \epsilon_2, \omega_1$ are constants, and $h(t)$ and $f(t)$ satisfy

$$\ddot{h} - h = \dot{a}\omega_1, \qquad \ddot{f} - f = -a\psi \tag{88}$$

and $h = 0$ if $\dot{a}\omega_1 = 0$.

- (V_1, h_1) admits 3 KV $\vec{\xi}_A$ $(A = 1, 2, 3)$ and no HV (i.e. non-zero constant curvature $k = \pm 1$).

$$\begin{gathered} d\sigma_1^2 = -\eta(x,k)^2\,dt^2 + dx^2, \qquad \eta(x,k) = \sin x, \sinh x \quad \text{for} \quad k = +1, -1. \\ \vec{\xi}_1 = \cosh t\,\partial_x - \eta'\eta^{-1}\sinh t\,\partial_t, \qquad \vec{\xi}_2 = \partial_t, \qquad \vec{\xi}_3 = -\sinh t\,\partial_x + \eta'\eta^{-1}\cosh t\,\partial_t. \end{gathered} \tag{89}$$

Other useful coordinate gauges are

$$d\sigma_1^2 = -4\frac{du\,dv}{k(u+v)^2}, \qquad \vec{\xi}_1 = u^2\partial_u - v^2\partial_v, \qquad \vec{\xi}_2 = u\partial_u + v\partial_v, \qquad \vec{\xi}_3 = \partial_u - \partial_v, \tag{90}$$

$$\begin{gathered} d\sigma_1^2 = -[a(t)r^{-\lambda} + r^{\lambda}]^2\,dt^2 + r^{-2}\,dr^2, \qquad \text{constant curvature} \quad -\lambda^2(\neq -1), \\ \vec{\xi} = \left[\lambda D + \frac{\ddot{D}}{2\lambda}(a(t) + r^{2\lambda})^{-1}\right]\partial_t - \dot{D}r\partial_r, \end{gathered} \tag{91}$$

where the function $D(t)$ satisfies the third-order equation

$$\dddot{D} + 4\lambda^2 a\dot{D} + 2\lambda^2\dot{a}D = 0, \tag{92}$$

the three constants of integration giving then three independent KV.

$$\begin{gathered} d\sigma_1^2 = -[a(t)\cos\lambda x + \sin\lambda x]^2\,dt^2 + dx^2 \qquad \text{constant curvature} \quad +\lambda^2(\neq 1). \\ \vec{\xi} = [F - \lambda^{-2}(a^{-1}\dot{F})^{\cdot}\cos\lambda x(a\cos\lambda x + \sin\lambda x)^{-1}]\partial_t + \lambda^{-1}a^{-1}\dot{F}\partial_x \end{gathered} \tag{93}$$

where $F(t)$ satisfies the third-order equation

$$\lambda^{-2}(a^{-1}\dot{F})^{\cdot\cdot} - (a + a^{-1})\dot{F} - \dot{a}F = 0, \tag{94}$$

and again, the three constants of integration give rise to three independent KV.

- (V_1, h_1) admits one KV $\vec{\xi}$ and one HV $\vec{\eta}$ (with homothetic constant ψ):

$$d\sigma_1^2 = x^{2\psi}(-dt^2 + dx^2), \qquad \vec{\xi} = \partial_t, \tag{95}$$

$$\vec{\eta} = t\partial_t + x\partial_x, \tag{96}$$

or

$$d\sigma_1^2 = t^{2\psi}(-dt^2 + dx^2), \qquad \vec{\xi} = \partial_x, \tag{97}$$

$$\vec{\eta} = t\partial_t + x\partial_x. \tag{98}$$

- (V_1, h_1) admits one HV $\vec{\eta}$ and no KV:

$$d\sigma_1^2 = \exp(2\psi t)\exp(2g(x))(-dt^2 + dx^2), \qquad \vec{\eta} = \partial_t,$$

or

$$d\sigma_1^2 = \exp(2\psi x)\exp(2g(t))(-dt^2 + dx^2), \qquad \vec{\eta} = \partial_x. \tag{99}$$

- (V_1, h_1) admits one KV $\vec{\xi}$ and no HV:

$$d\sigma_1^2 = \exp(2g(x))(-dt^2 + dx^2), \qquad \vec{\xi} = \partial_t,$$

or

$$d\sigma_1^2 = \exp(2g(t))(-dt^2 + dx^2), \qquad \vec{\xi} = \partial_x. \tag{100}$$

- (V_1, h_1) admits no KV and no HV:

$$d\sigma_1^2 = \Omega^2(t, x)(-dt^2 + dx^2), \qquad \text{equivalently} \quad d\sigma_1^2 = -A^2(t, x)\,dt^2 + B^2(t, x)\,dx^2, \tag{101}$$

where Ω, A, B are arbitrary functions of their arguments.

The possibilities for (V_2, h_2), the Riemannian 2-space, are

- (V_2, h_2) admits 3 KV $\vec{\xi}_A$ $(A = 1, 2, 3)$ and 1 HV $\vec{\eta}$ with homothetic constant $\psi \neq 0$ (i.e. it is of constant zero curvature).

$$d\sigma_2^2 = dy^2 + dz^2, \qquad \vec{\xi}_1 = \partial_y, \qquad \vec{\xi}_2 = \partial_z, \qquad \vec{\xi}_3 = z\partial_y - y\partial_z, \tag{102}$$

$$\vec{\eta} = \psi(y\partial_y + z\partial_z). \tag{103}$$

- (V_2, h_2) admits 3 KV $\vec{\xi}_A$ $(A = 1, 2, 3)$ and no HV (i.e. non-zero constant curvature $k = \pm 1$).

$$\begin{aligned} &d\sigma_2^2 = d\theta^2 + \eta(\theta, k)^2\,d\phi^2, \qquad \eta(\theta, k) = \sin\theta, \sinh\theta \qquad \text{for} \quad k = +1, -1. \\ &\vec{\xi}_1 = \cos\phi\,\partial_\theta - \eta'\eta^{-1}\sin\phi\,\partial_\phi, \qquad \vec{\xi}_2 = \partial_\phi, \qquad \vec{\xi}_3 = -\sin\phi\,\partial_\theta + \eta'\eta^{-1}\cos\phi\,\partial_\phi. \end{aligned} \tag{104}$$

- (V_2, h_2) admits one KV $\vec{\xi}$ and one HV $\vec{\eta}$ (with homothetic constant ψ):

$$d\sigma_2^2 = z^{2\psi}(dy^2 + dz^2), \qquad \vec{\xi} = \partial_y, \tag{105}$$

$$\vec{\eta} = y\partial_y + z\partial_z. \tag{106}$$

- (V_2, h_2) admits one HV $\vec{\eta}$ and no KV:

$$d\sigma_2^2 = e^{2\psi z}\,e^{2g(y)}(dy^2 + dz^2), \qquad \vec{\eta} = \partial_z. \tag{107}$$

Table 1. The numbers in the table refer to the the equations in the text. Whenever different coordinate gauges exist, this is indicated in square brackets. Further, the entries in the column labelled KV are to be understood as the KV appearing in the corresponding equations in the text (i.e. the KV of (V_1, h_1) and those of (V_2, h_2), see section 2 for details); whereas the entries in the column HV (whenever non-void) mean that the HV in $(M, \hat{g})$ is the sum of those existing in (V_1, h_1) and (V_2, h_2) their homothetic constants being *the same* (again, see section 2 for details).

Case	$d\sigma_1^2$	$d\sigma_2^2$	KV	HV
6 KV	(89) [or (90), (91)]	(104)	(89) [or (90), (91)] and (104)	–
	(82) [or (84), (86)]	(104)	(82) [or (84), (86)] and (104)	–
	(89) [or (90), (91)]	(102)	(89) [or (90), (91)] and (102)	–
4 KV	(82) [or (84), (86)]	(105)	(82) [or (84), (86)] and (105)	(82) [or (84), (86)] + (105)
1 HV	(95)	(102)	(95) and (102)	(95) + (102)
4 KV	(89) [or (90), (91)]	(108)	(89) [or (90), (91)] and (108)	–
	(100)	(104)	(100) and (104)	–
	(82) [or (84), (86)]	(108)	(82) [or (84), (86)] and (108)	–
	(89) [or (90), (91)]	(105)	(89) [or (90), (91)] and (105)	–
	(95) [or (97)]	(104)	(95) [or (97)] and (104)	–
	(100)	(102)	(100) and (102)	–
3 KV	(82) [or (84), (86)]	(107)	(82) [or (84), (86)] and (107)	(82) [or (84), (86)] + (107)
1 HV	(99)	(102)	(99) and (102)	(99) + (102)
3 KV	(89) [or (90), (91)]	(109)	(89) [or (90), (91)] and (109)	–
	(101)	(104)	(101) and (104)	–
	(82) [or (84), (86)]	(109)	(82) [or (84), (86)] and (109)	–
	(101)	(102)	(101) and (102)	–
2 KV	(95)	(105)	(95) and (105)	(95) + (105)
1 HV				
2 KV	(100)	(108)	(100) and (108)	–
	(95) [or (97)]	(108)	(95) [or (97)] and (108)	–
	(100)	(105)	(100) and (105)	–
1 KV	(95)	(107)	(95) and (107)	(95) + (107)
1 HV	(99)	(105)	(99) and (105)	(99) + (105)
1 KV	(100)	(109)	(100) and (109)	–
	(99)	(107)	(99) and (107)	–
1 HV	(99)	(107)	–	(99) + (107)
	(95) [or (97)]	(109)	(95) [or (97)] and (109)	–
	(101)	(105)	(101) and (105)	–
0 KV	(101)	(109)	–	–
0 HV				

- (V_2, h_2) admits one KV $\vec{\xi}$ and no HV:

$$d\sigma_2^2 = e^{2g(y)}(dy^2 + dz^2), \qquad \vec{\xi} = \partial_z. \tag{108}$$

- (V_2, h_2) admits no KV and no HV:

$$d\sigma_2^2 = \Omega^2(y, z)(dy^2 + dz^2), \qquad \text{equivalently} \quad d\sigma_2^2 = C^2(y, z)\, dy^2 + D^2(y, z)\, dz^2), \tag{109}$$

where Ω, C, D are arbitrary functions of their arguments.

Bearing all this in mind, we summarize all the possibilities for the 2 + 2 reducible spacetime $(M, \hat{g})$ in table 1.

Table 2. The columns list the curvatures of each 2-space as $\pm\lambda^2, 0$ for $d\sigma_1^2$ and $\pm\omega^2, 0$ for $d\sigma_2^2$, where $\lambda\omega \neq 0$, for the cases when each 2-space is of constant curvature but the spacetime is not CF, corresponding to the six KV case of table 1.

Case	Curvature $d\sigma_1^2$	Curvature $d\sigma_2^2$
(i)	$-\lambda^2$	$\omega^2 \neq \lambda^2$
(ii)	0	ω^2
(iii)	λ^2	ω^2
(iv)	$-\lambda^2$	0
(v)	λ^2	0
(vi)	$-\lambda^2$	$-\omega^2$
(vii)	0	$-\omega^2$
(viii)	λ^2	$-\omega^2 \neq -\lambda^2$

The cases, in which both 2-spaces are of constant curvature but the spacetime is not CF, correspond to the first entry in table 1, that is, $(M, \hat{g})$ admits six KV, three from each 2-space. Now, if $(M, \hat{g})$ is not conformally flat, the sum of the curvatures of the two 2-spaces cannot be zero (see section 2 and also [6] for details). Denoting these constant curvatures (whenever non-zero) as $\pm\lambda^2$ and $\pm\omega^2$, the possible cases are listed in table 2.

Note that, without loss of generality we can take one of λ, ω to be unity (but, in general, not both). We choose $\omega = 1$.

6. Examples

In this section we shall provide examples of physically significant spacetimes which are 2 + 2 conformally reducible. We shall concentrate specifically in vacuum and perfect fluid solutions and will follow the conventions established in section 2; thus, $\hat{g}$ and g will designate the metrics of the reducible and the conformally reducible spacetimes, respectively (associated line elements $d\Sigma^2$ and ds^2), and the coordinates will be labelled $x^a = t, x, y, z$, $(a, b, \ldots = 0, 1, 2, 3)$, $x^\alpha = \{t, x\}$ $(\alpha, \beta, \ldots = 0, 1)$ and $x^A = \{y, z\}$ $(A, B, \ldots = 2, 3)$.

We shall look for perfect fluid spacetimes which are conformally 2 + 2 reducible according to the definition given in section 3. In so doing, vacuum will be regarded as a special (limit) case of those, whenever $\rho = p = 0$.

The line element of (M, g) will be written, for convenience, as

$$ds^2 = \omega^{-2}(x^\alpha, x^A)\left[-L_0^2(x^\alpha)\,dt^2 + L_1^2(x^\alpha)\,dx^2 + e^{2P(x^A)}(dy^2 + dz^2)\right] \equiv \omega^{-2}\,d\Sigma^2, \qquad (110)$$

and we shall restrict ourselves to the case in which the four-velocity of the fluid, $\vec{u}$, is tangent to the t, x plane.

Noting now the Einstein tensors associated with the metrics g and $\hat{g}$ as G and $\hat{G}$, respectively, we will have, in an arbitrary coordinate chart [1],

$$G_{ab} = \hat{G}_{ab} + 2\omega^{-1}\omega_{a/b} - S\hat{g}_{ab}, \qquad S \equiv 2\omega^{-1}\left(\omega^c_{/c} - \tfrac{3}{2}\omega^{-1}\omega^c\omega_c\right), \qquad (111)$$

where a slash denotes covariant derivative relative to the metric $\hat{g}$ and all the contractions appearing in the above equation have been performed with $\hat{g}$.

We next choose a null tetrad in $(M, \hat{g})$, say $\{\hat{l}^a, \hat{n}^a, \hat{y}^a, \hat{z}^a\}$ (with $\hat{l}^a\hat{n}_a = \hat{y}^a\hat{y}_a = \hat{z}^a\hat{z}_a = 1$ and all the other inner products zero), adapted to the 2 + 2 structure; i.e., $\hat{l}^a, \hat{n}^a$ are defined on the t, x 2-space and $\hat{y}^a, \hat{z}^a$ on the two-dimensional submanifold coordinated by y, z; we thus have

$$\hat{G}_{ab} = -\tfrac{1}{2}R_2(x^A)(\hat{l}_a\hat{n}_b + \hat{n}_a\hat{l}_b) - \tfrac{1}{2}R_1(x^\alpha)(\hat{y}_a\hat{y}_b + \hat{z}_a\hat{z}_b). \qquad (112)$$

The Field equations specialized to a perfect fluid imply

$$G_{ab} = (\rho + p)u_a u_b + p g_{ab}, \tag{113}$$

where

$$u_a = \frac{1}{\sqrt{2}}\omega^{-1}(\hat{l}_a - \hat{n}_a), \tag{114}$$

on account of the previous hypothesis regarding the fluid velocity. Also notice that the freedom we still have in choosing the coordinates t, x preserving the form (110) of the metric (and hence the freedom in choosing the null vectors $\hat{l}^a, \hat{n}^a$) allows us to write, without loss of generality,

$$\vec{u} = \frac{\omega}{L_0}\partial_t. \tag{115}$$

From (111)–(114) we get

$$2\omega^{-1}\omega_{a/b} = \left[\tfrac{1}{2}(p-\rho)\omega^{-2} + S + \tfrac{1}{2}R_2(x^A)\right](\hat{l}_a\hat{n}_b + \hat{n}_a\hat{l}_b) + \tfrac{1}{2}(p+\rho)\omega^{-2}(\hat{l}_a\hat{l}_b + \hat{n}_a\hat{n}_b)$$
$$+ \left[p\omega^{-2} + S + \tfrac{1}{2}R_1(x^\alpha)\right](\hat{y}_a\hat{y}_b + \hat{z}_a\hat{z}_b); \tag{116}$$

from where it follows, on account of the block-diagonal form of $\hat{G}_{ab}$, that

$$\omega(x^\alpha, x^A) = \omega_1(x^\alpha) + \omega_2(x^A); \tag{117}$$

substituting this back into (111) and taking into account (112–115) we get, in the coordinates in which (110) is written,

$$\omega_{1t/x} = \omega_{2y/z} = 0, \tag{118}$$

$$\tfrac{1}{2}R_2 + 2\omega^{-1}L_0^{-2}\omega_{1t/t} + S = \rho\omega^{-2}, \tag{119}$$

$$-\tfrac{1}{2}R_2 + 2\omega^{-1}L_1^{-2}\omega_{1x/x} - S = p\omega^{-2}, \tag{120}$$

$$-\tfrac{1}{2}R_1 + 2\omega^{-1}\,\mathrm{e}^{-2P}\omega_{2y/y} - S = p\omega^{-2}, \tag{121}$$

$$-\tfrac{1}{2}R_1 + 2\omega^{-1}\,\mathrm{e}^{-2P}\omega_{2z/z} - S = p\omega^{-2}; \tag{122}$$

from (120, 121) it follows

$$-\tfrac{1}{2}R_2 + 2\omega^{-1}L_1^{-2}\omega_{1x/x} = -\tfrac{1}{2}R_1 + 2\omega^{-1}\,\mathrm{e}^{-2P}\omega_{2y/y}. \tag{123}$$

There are now various different cases to be considered:

(1) neither R_1 nor R_2 is constant; i.e., none of the two 2-spaces is of constant curvature;
(2) R_2 is constant;
(3) R_1 is constant;
(4) both R_1 and R_2 are constant.

Let us deal with each case separately.

(1) Both R_1 and R_2 are non-constant. Consider (123) and take derivatives first with respect to x^α and then with respect to x^A; we get

$$(\partial_A R_2)(\partial_\alpha\omega_1) - (\partial_A\omega_2)(\partial_\alpha R_1) = 0,$$

that is,

$$\omega_1 = kR_1 + k_1 \quad \text{and} \quad \omega_2 = kR_1 + k_2,$$

where $k \neq 0$, k_1 and k_2 are constants. Note that k_1 or k_2 (but not both) can be set equal to zero without loss of generality, thus we shall write, from now on,

$$\omega = k(R_1 + R_2) + k', \tag{124}$$

with k, k' constants ($k \neq 0$). Substituting this back into (123), dividing through by k and setting $c \equiv k'/k$ we get

$$R_1^2 + cR_1 + \frac{4}{L_1^2} R_{1x/x} = R_2^2 + cR_2 + 4\,\mathrm{e}^{-2P} R_{2y/y} = N, \qquad N = \text{constant}. \tag{125}$$

Now, the above equations together with

$$R_{1t/x} = R_{2y/z} = 0 \quad \text{and} \quad R_{2y/y} = R_{2z/z}, \tag{126}$$

which come from (118), (121) and (122), yield a system of differential equations for R_1 and R_2, which once integrated provides, on account of (124), an expression for ω and then, through (119) and (120), give expressions for the density ρ and the pressure p of the fluid.

(2) R_2 is constant. Putting now $R_2 = k_2$ in (123) and differentiating as in the previous case with respect to x^α and x^A we easily get

$$R_{1,\alpha}\omega_{2,A} = 0, \tag{127}$$

that is, either $R_1 =$ constant (thus falling into case (4) above) or else $\omega_2 =$ constant. We shall assume the latter (the possibility $R_1 =$ constant will be left until case (4) is dealt with). If $\omega_2 =$ constant it follows that the conformal factor depends only on the co-ordinates of M_1 and the resulting spacetime is then warped (class B) [4]; further, (M_2, h_2) is of constant curvature, thus all of the spherically symmetric, plane symmetric and hyperbolic symmetric perfect fluid solutions existing in the literature fall into this class. Other examples of class B warped perfect fluid spacetimes can be found in [4] and also in [17].

(3) R_1 is constant. Now put $R_1 = k_1$ in (123). Proceeding as before, we get

$$R_{2,A}\omega_{1,\alpha} = 0. \tag{128}$$

Thus again either $R_2 =$ constant (and again, we shall deal with this possibility when analyzing case (4)) or else $\omega_1 =$ constant and the same comments as above apply here; namely, the spacetime is class B warped but, as it can be shown, the energy-momentum tensor of this particular class of type B warped spacetimes (type B_S in the classification given in [17]) can only be $\{(1, 1)11\}$ (or any degeneracy thereof), see [17] for details, and therefore perfect fluids (with $\rho + p \neq 0$) are excluded in this case.

(4) R_1, R_2 are constant. Suppose now that both R_1 and R_2 are constant. Since there are normal forms for the two 2-metrics it is, at this point, best to deal with them dropping the assumption of co-moving velocity for the fluid (but retaining the condition that it is tangent to (V_1, h_1)). Thus we shall write

$$\mathrm{d}s^2 = (\omega_1(t, x) + \omega_2(y, z))^{-2}[m^2(-\Sigma^2(x, \epsilon)\,\mathrm{d}t^2 + \mathrm{d}x^2) + \mathrm{d}y^2 + \eta^2(y, k)\,\mathrm{d}z^2], \tag{129}$$

where $R_1 = -2\epsilon/m^2, \epsilon = -1, 0, 1$ with $\Sigma(x, \epsilon) = \sin x$, 1, $\sinh x$, respectively and $R_2 = 2k$, $k = -1, 0, 1$ and $\eta(y, k) = \sin y$, 1, $\sinh y$, respectively. Notice that, from the comments at the end of the previous section, it follows that ϵ and k cannot be zero simultaneously if we want to avoid the CF case.

From the field equations (111) and (113), we get $\omega_{2y/z} = 0$ and $\eta^2\omega_{2y/y} = \omega_{2z/z}$ which can be easily integrated to give

$$\omega_2 = B_1 + B_0 \int \eta(y, k)\,\mathrm{d}y, \qquad \text{whenever} \quad k \neq 0 \quad \text{or} \quad \omega_2 = B_1 + B_0 y + az \quad \text{for} \quad k = 0 \tag{130}$$

where B_0, B_1 and a are constants.

Now turning our attention back to (111) and (113) we get, on account of (129),

$$(\rho + p)u_t^2 - p\omega^{-2}m^2\Sigma^2 = G_{tt}, \tag{131}$$

$$(\rho + p)u_t u_x = G_{tx}, \tag{132}$$

$$(\rho + p)u_x^2 + p\omega^{-2}m^2\Sigma^2 = G_{xx}, \tag{133}$$

$$p\omega^{-2} = G_{yy}. \tag{134}$$

Further, we also have

$$\Sigma^{-2}u_t^2 = u_x^2 + m^2\omega^{-2}. \tag{135}$$

Now solving for $\rho + p$, u_t and u_x from the above equations and substituting those results back into (132), we get

$$\rho + p = \omega^2[2G_{yy} + m^{-2}(\Sigma^{-2}G_{tt} - G_{xx})], \tag{136}$$

$$u_x^2 = \omega^{-2}[G_{xx} - m^2G_{yy}][2G_{yy} + m^{-2}(\Sigma^{-2}G_{tt} - G_{xx})]^{-1}, \tag{137}$$

$$\Sigma^{-2}u_t^2 = \omega^{-2}[G_{tt}\Sigma^{-2} + m^2G_{yy}][2G_{yy} + m^{-2}(\Sigma^{-2}G_{tt} - G_{xx})]^{-1}, \tag{138}$$

$$\Sigma[G_{xx} - m^2G_{yy}]^{-1/2}[G_{tt}\Sigma^{-2} + m^2G_{yy}]^{-1/2} = G_{tx}. \tag{139}$$

Now, equations (136)–(138) provide expressions for $\rho + p$, u_t and u_x in terms of the metric functions Σ, η and ω_1, whereas (139) provides a differential equation for $\omega_1(t, x)$ which is the only unknown function; namely,

$$\begin{aligned}(\omega_1 + \omega_2)\Sigma[(\epsilon - km^2) + 2(\omega_1 + \omega_2)^{-1}(\omega_{1xx} - B_0m^2\eta')]^{1/2}[-(\epsilon - km^2)\\ + 2(\omega_1 + \omega_2)^{-1}(\Sigma^{-2}\omega_{1tt} - \Sigma^{-1}\Sigma'\omega_{1x} + B_0m^2\eta')]^{1/2} = 2[\omega_{1tx} - \Sigma^{-1}\Sigma'\omega_{1t}],\end{aligned} \tag{140}$$

where a prime denotes differentiation with respect to its argument. Notice that if $k = 0$ then $\eta' = 0$ and similarly, if $\epsilon = 0$ then $\Sigma' = 0$.

Now, it is immediately seen that the right-hand side of the above equation has no dependence on y or z and therefore one can differentiate with respect to these variables in order to obtain equations for $\omega_1(t, x)$. After some straightforward calculations, one gets:

- if $\epsilon = 0$ or $k = 0$ then, either $\omega_2(y, z) =$ constant, in which case the spacetime is warped thus falling into one of the previously considered cases, or else both k and ϵ are zero and the spacetime is CF (both 2-spaces being of curvature zero), all perfect fluid solutions then being known (see [1]);
- if $k \neq 0$, then necessarily $\epsilon = -k$ and $m^2 = 1$ in which case the resulting spacetime is CF (since the underlying 2 + 2 reducible spacetime $(M, \hat{g})$ is flat) and, again, all perfect fluid solutions are known [1].

So far, we have presented only the general case, discussing the various possibilities regarding the curvature of each 2-space and setting up the basic equations in every case which, upon integration, would yield actual perfect fluid exact solutions, but no attempt has been made at finding those solutions. In what follows, we shall present some selected examples.

Example 1. First we consider the CF spacetimes of section 4. The general line element for CF perfect fluid spacetimes can be written in the form [1]

$$ds^2 = V^{-2}(-F^2\,dt^2 + dx^2 + dy^2 + dz^2) \tag{141}$$

where for the expanding ($\Theta = \Theta(t) \neq 0$) generalized FRW models V and F are given by

$$V = H(r^2 - 2x_0x - 2y_0y - 2z_0z) + V_0 + Hr_0^2, \qquad F = 3\Theta^{-1}\frac{dV}{dt}, \tag{142}$$

where H, x_0, y_0, z_0, V_0 are arbitrary functions of t and $r^2 = x^2 + y^2 + z^2$, $r_0^2 = x_0^2 + y_0^2 + z_0^2$.

For the non-expanding generalized Schwarzschild models V and F are given by

$$V = \tfrac{1}{2}C(1+r^2), \qquad F = -\tfrac{1}{2}(Cf_4+1)r^2 + f_1x + f_2y + f_3z + \tfrac{1}{2}(Cf_4-1), \tag{143}$$

where $f_1, \ldots, f_4$ are the arbitrary functions of t. From this it follows that the general CF spherically symmetric spacetime has a metric which can be put into the form

$$\mathrm{d}s^2 = V^{-2}[-(a(t)+r^2)^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)], \tag{144}$$

where V is given by (143) for the non-expanding case and by

$$V = M(t) + N(t)r^2 \tag{145}$$

for the expanding case, M, N being arbitrary functions of t satisfying

$$M_t = a(t)N_t. \tag{146}$$

For the case of plane symmetry, the general metric takes the form

$$\mathrm{d}s^2 = [M(t)+N(t)x]^{-2}[-(a(t)+x)^2\,\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2] \tag{147}$$

where M, N again satisfy (146).

Note that the underlying $2+2$ spacetimes for the metrics (144) and (147) are precisely those given by equations (52) and (63), respectively. Thus, the CKV for these perfect fluid spacetimes are precisely those given by equations (55) and by equations (64), respectively, with the conformal scalar, ψ, recalculated for the metrics (144) and (147).

As a simple illustration of this consider the static Schwarzschild interior solution which, in isotropic coordinates, takes the form

$$\mathrm{d}s^2 = 4C^{-2}(1+r^2)^{-2}[-(a+r^2)^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)], \tag{148}$$

i.e., metric (52) multiplied by the factor $4C^{-2}(1+r^2)^{-2}r^2$ and with a, C being non-zero constants. We will assume a to be positive, i.e. $a = k^2$. Equations (57) and (58) then yield

$$m_i = \mu_i\cos 2kt + \nu_i\sin 2kt + \kappa_i, \tag{149}$$

$$n_i = k^2\left(\mu_i\cos 2kt + \nu_i\sin 2kt - \kappa_i\right), \tag{150}$$

where μ_i, ν_i, κ_i ($i = 1, 2, 3$) are a set of nine arbitrary constants. Equation (59) yields

$$D = \delta\cos 2kt + \epsilon\sin 2kt + \omega, \tag{151}$$

where δ, ϵ, ω are three arbitrary constants. From the first two equations in (55) we find that, apart from the three KV of spherical symmetry given by the constants a_1, a_2, a_3 in the first equation in (55) and the timelike hypersurface orthogonal KV $\vec{\xi} = \partial_t$ corresponding to the constant ω in (151), there are eleven proper CKV given by

$$\begin{aligned}
\xi^0 &= (k^2+r^2)^{-1}\{2kr[(-\mu_1\sin 2kt + \nu_1\cos 2kt)\sin\theta\sin\phi \\
&\quad + (\mu_2\sin 2kt - \nu_2\cos 2kt)\sin\theta\cos\phi \\
&\quad + (\mu_3\sin 2kt - \nu_3\cos 2kt)\cos\theta] + (r^2-k^2)(\delta\cos 2kt + \epsilon\sin 2kt)\}, \\
\xi^1 &= (k^2-r^2)\{(\mu_1\cos 2kt + \nu_1\sin 2kt)\sin\theta\sin\phi - (\mu_2\cos 2kt + \nu_2\sin 2kt)\sin\theta\cos\phi \\
&\quad - (\mu_3\cos 2kt + \nu_3\sin 2kt)\cos\theta\} - (r^2+k^2)(\kappa_1\sin\theta\sin\phi \\
&\quad - \kappa_2\sin\theta\cos\phi - \kappa_3\cos\theta) + 2kr(\delta\sin 2kt - \epsilon\cos 2kt), \\
\xi^2 &= r^{-1}(k^2+r^2)\{(\mu_1\cos 2kt + \nu_1\sin 2kt)\cos\theta\sin\phi - (\mu_2\cos 2kt + \nu_2\sin 2kt)\cos\theta\cos\phi \\
&\quad + (\mu_3\cos 2kt + \nu_3\sin 2kt)\sin\theta\} - r^{-1}(k^2-r^2)(\kappa_1\cos\theta\sin\phi \\
&\quad - \kappa_2\cos\theta\cos\phi + \kappa_3\sin\theta), \\
\xi^3 &= r^{-1}(k^2+r^2)\operatorname{cosec}\theta\,\{(\mu_1\cos 2kt + \nu_1\sin 2kt)\cos\phi + (\mu_2\cos 2kt + \nu_2\sin 2kt)\sin\phi\} \\
&\quad - r^{-1}(k^2-r^2)\operatorname{cosec}\theta(\kappa_1\cos\phi + \kappa_2\sin\phi).
\end{aligned} \tag{152}$$

The corresponding conformal scalar, calculated for the metric (148) is given by

$$\begin{aligned}\psi = 2r^{-1}(1+r^2)^{-1}\{(k^2+r^4)[-(\mu_1\cos 2kt+\nu_1\sin 2kt)\sin\theta\sin\phi\\ +(\mu_2\cos 2kt+\nu_2\sin 2kt)\sin\theta\cos\phi+(\mu_3\cos 2kt+\nu_3\sin 2kt)\cos\theta]\\ +(k^2-r^4)(\kappa_1\sin\theta\sin\phi-\kappa_2\sin\theta\cos\phi-\kappa_3\cos\theta)\\ +kr(r^2-1)(\delta\sin 2kt-\epsilon\cos 2kt)\}.\end{aligned} \tag{153}$$

Example 2. These examples illustrate the case in which both 2-spaces are of constant curvature but the spacetime is not CF, i.e., the first entry in table 1, or in the foregoing discussion, the case of a perfect fluid spacetime with both R_1 and R_2 constant. As follows from our previous analysis of this case, the spacetime (M, g) has to be warped, that is, the conformal factor can only depend on the coordinates of one of the 2-spaces, namely (V_1, h_1). The underlying 2 + 2 spacetime has (V_1, h_1) given by (82) and (V_2, h_2) by (104) with $k = +1$ and so admits the three KV of spherical symmetry together with the three KV given by (82).

(1) Consider the spacetime with metric

$$\mathrm{d}s^2 = \cosh^{-2}(r/\sqrt{2})(-\mathrm{d}t^2+\mathrm{d}r^2+\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \tag{154}$$

which represents a perfect fluid, with co-moving velocity, density and pressure given by

$$\mu = \tfrac{1}{2}(\cosh^2(r/\sqrt{2})+3), \qquad p = \tfrac{1}{2}(\cosh^2(r/\sqrt{2})-3),$$

thus satisfying the dominant energy condition everywhere. The three KV of spherical symmetry together with $\vec{\xi}_1$ of (82) remain as KV while $\vec{\xi}_2$ and $\vec{\xi}_3$ become proper CKV with $\psi_2 = -(\sqrt{2})^{-1}\tanh(r/\sqrt{2})$ and $\psi_3 = -(\sqrt{2})^{-1}t\tanh(r/\sqrt{2})$.

(2) The spacetime

$$\mathrm{d}s^2 = \exp\left[\tfrac{1}{2}(t^2-r^2)\right](-\mathrm{d}t^2+\mathrm{d}r^2+\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2), \tag{155}$$

represents a perfect fluid with tilting velocity given by

$$\vec{u} = (t^2-r^2)^{-1/2}\exp\left[-\tfrac{1}{4}(t^2-r^2)\right](t\partial_t+r\partial_r)$$

and density and pressure

$$\mu = \tfrac{1}{4}\exp\left[-\tfrac{1}{2}(t^2-r^2)\right][3(t^2-r^2)+8],$$
$$p = -\tfrac{1}{4}\exp\left[-\tfrac{1}{2}(t^2-r^2)\right][(t^2-r^2)+8].$$

The dominant energy condition is satisfied everywhere in the region $t > r$ although p is always negative there. In this example $\vec{\xi}_3$ of (82) remains a KV while $\vec{\xi}_1$ and $\vec{\xi}_2$ become proper CKV with $\psi_1 = -(1/2)t$ and $\psi_2 = (1/2)r$.

Example 3. Spacetimes admitting four KV and one HV are easily found (see table 1). First consider the plane symmetric 2 + 2 spacetime given by

$$\mathrm{d}\Sigma^2 = x^{2(\psi-1)}(-\mathrm{d}t^2+\mathrm{d}x^2)+\mathrm{d}y^2+\mathrm{d}z^2. \tag{156}$$

Redefining the coordinate x, rescaling t, and taking $\psi = 1/4$ for this example, results in the metric

$$\mathrm{d}\Sigma^2 = -x^{-6}\,\mathrm{d}t^2+\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2, \tag{157}$$

which admits the four KV

$$\vec{\xi}_1 = \partial_t, \qquad \vec{\xi}_2 = \partial_y, \qquad \vec{\xi}_3 = \partial_z, \qquad \vec{\xi}_4 = z\partial_y - y\partial_z \tag{158}$$

and the HV

$$\vec{\eta} = 4t\partial_t + x\partial_x + y\partial_y + z\partial_z \tag{159}$$

with homothetic constant $\psi = 1$.

(1) The spacetime with metric

$$ds^2 = x^4\,d\Sigma^2, \tag{160}$$

is a vacuum solution. The four KV and HV above remain as such but now $\psi = 3$ for (159).

(2) A (co-moving) perfect fluid spacetime with (157) as its underlying 2+2 spacetime has a metric

$$ds^2 = (a^2x^3 + b^2x^{-2})^{-2}\,d\Sigma^2, \tag{161}$$

its density and pressure being given by

$$\mu = 15a^2x^{-1}(4b^2 - a^2x^5), \qquad p = 15a^2x^{-1}(3a^2x^5 - 2b^2).$$

The dominant energy condition is satisfied for $x^5 < 3b^2/(2a^2)$. The four KV in (158) remain as KV, but the HV $\vec{\eta}$ becomes a CKV with $\psi = (3b^2 - 2a^2x^5)(a^2x^5 + b^2)^{-1}$.

Non-static examples of this type can be found by considering the plane-symmetric 2+2 spacetime

$$d\Sigma^2 = -t^{-1}\,dt^2 + t^{-1}\,dx^2 + dy^2 + dz^2 \tag{162}$$

which admits 4 KV, $\vec{\xi}_2, \vec{\xi}_3, \vec{\xi}_4$ as before (see (158)) but now $\vec{\xi}_1 = \partial_x$, the HV being $\vec{\eta} = 2t\partial_t + 2x\partial_x + y\partial_y + z\partial_z$ with $\psi = 1$.

(1) The spacetime

$$ds^2 = t\,d\Sigma^2 \tag{163}$$

is a Kasner-type perfect fluid solution with $\mu = p = (1/4)t^{-2}$. In this case $\vec{\eta}$ remains a HV but with $\psi = 2$.

(2) The spacetime

$$ds^2 = [a(x^2 - 2t^2) + b]^{-2}t\,d\Sigma^2, \tag{164}$$

where a, b are non-zero constants, is also a perfect fluid solution. In this case $\vec{\xi}_1$ and $\vec{\eta}$ become CKVs with conformal factors $\psi = -2ax[a(x^2 - 2t^2) + b]^{-1}$ and $\psi = -2 + 4b[a(x^2 - 2t^2) + b]^{-1}$, respectively.

(3) A further perfect fluid solution is found by considering the spacetime

$$ds^2 = t\sec^2 kx\,d\Sigma^2, \tag{165}$$

where k is an arbitrary non-zero constant. As before, $\vec{\xi}_1$ and $\vec{\eta}$ become CKVs with conformal factors $\psi = k\tan kx$ and $\psi = 2 + 2kx\tan kx$, respectively.

Example 4. We note briefly that there are many spacetimes admitting four KV only. These include all the static spherically and plane symmetric spacetimes that admit no HV or CKV such as the Schwarzschild solution, the Reissner–Nordström solution and their plane symmetric counterparts (all of them are instances of warped spacetimes).

Example 5. We now consider spacetimes admitting two KV and one HV.

(1) From table 1, we take as the starting point for the line element of the underlying 2+2 reducible spacetime

$$d\Sigma^2 = x^{2(\psi-1)}(-dt^2 + dx^2) + y^{2(\psi-1)}(dy^2 + dz^2), \tag{166}$$

and next make a coordinate transformation similar to that between (156) and (157) to obtain

$$d\Sigma^2 = -x^{-6}\,dt^2 + dx^2 + dy^2 + y^{-6}\,dz^2 \tag{167}$$

which admits the two KV $\vec{\xi}_1 = \partial_t$ and $\vec{\xi}_2 = \partial_z$ and the HV $\vec{\eta} = 4t\partial_t + x\partial_x + y\partial_y + 4z\partial_z$ with $\psi = 1$. Now, the spacetime with metric

$$\mathrm{d}s^2 = (x^{-2} + y^{-2})^{-2}\,\mathrm{d}\Sigma^2 \tag{168}$$

is a vacuum spacetime. The KV and HV remain as such but now for $\vec{\eta}$ we have $\psi = 3$.

(2) A second such spacetime, again vacuum, has the underlying 2 + 2 metric

$$\mathrm{d}\Sigma^2 = -x^{-6}\,\mathrm{d}t^2 + t^{-6}\,\mathrm{d}x^2 + z^{-6}\,\mathrm{d}y^2 + y^{-6}\,\mathrm{d}z^2 \tag{169}$$

which admits the two KV

$$\vec{\xi}_1 = (t^9 + 3x^8t)\partial_t + (x^9 + 3t^8x)\partial_x, \qquad \vec{\xi}_2 = (y^9 - 3z^8y)\partial_y + (-z^9 + 3y^8z)\partial_z$$

and the HV $\vec{\eta} = t\partial_t + x\partial_x + y\partial_y + z\partial_z$ with homothetic constant $\psi = -2$. The vacuum spacetime has metric

$$\mathrm{d}s^2 = (-x^6t^{-2} + t^6x^{-2} + z^6y^{-2} + y^6z^{-2})^{-2}\,\mathrm{d}\Sigma^2 \tag{170}$$

and the KV and HV remain as such with $\psi = -6$ for $\vec{\eta}$.

7. Conclusion

We have investigated spacetimes which are conformally related to 2 + 2 reducible spacetimes and have given an invariant characterization for them (see theorem 1). Also, we have collected and proved a number of results regarding KV and HV in two-dimensional spaces with metrics of arbitrary signature; most of them were known previously but a few others, as far as we are aware of, were not. A distinction has been made between the situation in which the underlying 2 + 2 reducible spacetime is conformally flat and those cases in which it is not, showing that, in the latter, no CKV are admitted and also that their KV and/or HV are simply the ones admitted by the two 2-spaces whose product gives rise to the reducible spacetime (plus a certain condition regarding the homothetic constants in the case in which a HV exists) (see [6]). Taking all this into account, a classification of conformally reducible spacetimes in terms of their conformal algebra (or equivalently, in terms of the conformal and/or homothetic algebra of the underlying reducible spacetime) is given. All this is done in section 2.

Section 3 deals with the issue of isometries and homotheties with fixed points in two-dimensional spaces of arbitrary signature; again, most of the results were essentially known [9, 8, 11] but were scattered in the literature. Further, we provide normal forms for the metrics of those 2-spaces holding in a coordinate neighbourhood containing the fixed point. Related to this, an example is presented of a spacetime which admits a group G_4 of isometries acting everywhere on three-dimensional orbits except for one given two-dimensional orbit, a fact that does not contradict Fubini's theorem as it would appear at first sight.

Section 4 is devoted to the study of conformally flat spacetimes; normal forms for the metric of the 2 + 2 reducible spacetime are given in various coordinate gauges, and expressions for the 15 generators of the conformal algebra are also provided.

In section 5 we study in detail non-conformally flat spacetimes from the point of view of their classification in terms of the homothetic algebra of the underlying reducible spacetime (conformal algebra of the conformally reducible one), such as it is put forward in section 2. Again, normal forms for both the line elements and generators of the algebras are given (sometimes in more than one coordinate gauge). All the relevant information is presented in two tables.

Finally, in section 6, some examples are presented and some others that already exist in the literature are referred to; in so doing, we have concentrated on just the case of perfect fluid spacetimes (although some vacuum solutions are also given). Besides the explicit examples, a thorough discussion of the general case (with the only limiting assumption being that the velocity of the fluid is tangent to one of the 2-spaces) is also provided.

The results of this study enable us to classify this important class of spacetimes in terms of their conformal algebras and, conversely, allows us to find physically meaningful spacetimes by assuming the existence of one of the specific sets of symmetries associated with such spacetimes.

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