Static Anisotropic Solutions to Einstein Equations with a Nonlocal Equation of State

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Abstract

We present a general method to obtain static anisotropic spherically symmetric solutions, satisfying a nonlocal equation of state, from known density profiles. This equation of state describes, at a given point, the components of the corresponding energy-momentum tensor not only as a function at that point, but as a functional throughout the enclosed configuration. In order to establish the physical aceptability of the proposed static family of solutions satisfying nonlocal equation of state, we study the consequences imposed by the junction and energy conditions for anisotropic fluids in bounded matter distribution. It is shown that a general relativistic spherically symmetric bounded distributions of matter, at least for certain regions, could satisfy a nonlocal equation of state.

1 Introduction

The structure of a relativistic star is believed to be rather complicated (solid crust, superfluid interior, different exotic phases transitions). Despite that its bulk properties seem to be computed with reasonable accuracy making several simplifying assumptions, the true equation of state that describes the properties of matter at densities higher than nuclear ($\approx 10^{14}\ gr/cm.^3$) is essentially unknown. This is mainly due to our inability to verify experimentally the theories that should describe the microphysics of nuclear matter at such high densities [1, 2, 3, 4]. Presently, what is known comes from the experimental insight and extrapolations from the ultra high energy accelerators and experimental cosmic physics (see [4, 5, 6] and references therein). Having this uncertainty in mind, it seems be reasonable to explore what is allowed by the laws of physics, in particular, within the framework of the theory of General Relativity and considering spherical and axial symmetries.

Classical continuum theories are based on the assumption that the state of a body is determined entirely by the behavior of an arbitrary infinitesimal neighborhood centered at any of its material points. Furthermore, there is also a premise that any small piece of the material can serve as a representative of the entire body in its behavior and, hence the governing balance laws are assumed to be valid for every part of the body, no matter how small. Clearly, the influence of the neighborhood on motions of the material points, emerging as a result of the interatomic interaction of the rest of the body, is neglected. Moreover, the isolation of an arbitrary small part of the body to represent the entire one clearly ignores the effects of the action of the applied load at distance. These applied loads are important because their transmissions from one part of the body to another, through their common boundaries, affect the motions and hence the state of the body at every point.

The relevance of long-range or nonlocal outcomes on the mechanical properties of materials are well known. The main ideas of non local a continuum were introduced during the 1960s and are based on considering the stress to be a function of the mean of the strain from a certain representative volume of the material centered at that point. From that time there have been many situations of common occurrence wherein nonlocal effects seem to dominate the macroscopic behavior of matter. Interesting problems coming from a wide variety of areas such as damage and cracking analysis of materials, surface phenomena between two liquids or two phases, mechanics of liquid crystals, blood flow, dynamics of colloidal suspensions seem to demand this type of nonlocal approach which has made this area very active concerning recent developments in material and fluid science and engineering (see [7] and references therein).

In a recent work [8], it is shown that under particular circumstances a general relativistic spherically symmetric anisotropic distribution of matter could satisfy a *Nonlocal Equation of State* (*NLES* from now on). Some types of dynamic bounded matter configurations having a *NLES*, with constant gravitational potentials at the surface, admit a Conformal Killing Vector and fulfill the energy conditions for anisotropic imperfect fluids. We also developed several analytical and numerical models for collapsing radiating anisotropic spheres in general relativity.

The present paper is focussed in a more general family of *NLES* and in determining the conditions under it could represent reasonable bounded matter distribution in General Relativity.

The static limit of the particular *NLES* considered in reference [8] can be written as

$$P_r(r) = \rho(r) - \frac{2}{r^3}\int_0^r \bar{r}^2\rho(\bar{r})\ \mathrm{d}\bar{r}\ + \frac{\mathcal{C}}{2\pi r^3}\ ; \tag{1}$$

where $\mathcal{C}$ is an arbitrary integration constant. It is clear in equation (1) a collective behavior on the physical variables $\rho(r)$ and $P_r(r)$ is present. The pressure $P_r(r)$ is not only a function of the energy density, $\rho(r)$, at that point but also its functional throughout the rest of the configuration. Any change in the pressure takes into account the effects of the variations of the energy density within an entire volume.

An additional physical insight of the meaning of the nonlocality for this particular equation of state can be gained by considering equation (1) re-written as

$$P_r(r) = \rho(r) - \frac{2}{3}\left\langle\rho\right\rangle_r + \frac{\mathcal{C}}{r^3}\ , \qquad \text{with} \quad \left\langle\rho\right\rangle_r = \frac{\int_0^r 4\pi\bar{r}^2\rho(\bar{r})\ \mathrm{d}\bar{r}}{\frac{4\pi}{3}r^3} = \frac{M(r)}{V(r)} \tag{2}$$

Clearly the nonlocal term represents an average over the function $\rho(r)$ within the volume enclosed by the radius r. Moreover, equation (2) can be easily rearranged as

$$P(r) = \frac{1}{3}\rho(r) + \frac{2}{3}\ \left(\rho(r) - \langle\rho(r)\rangle\right)\ + \frac{\mathcal{C}}{r^3} = \frac{1}{3}\rho(r) + \frac{2}{3}\ \sigma_\rho + \frac{\mathcal{C}}{r^3}\ , \tag{3}$$

where we have used the concept of statistical standard deviation σ_ρ from the local value of energy density. Furthermore, we may write:

$$P_r(r) = \mathcal{P}(r) + 2\sigma_{\mathcal{P}(r)} + \frac{\mathcal{C}}{r^3} \quad \text{where} \quad \begin{cases} \mathcal{P}(r) = \frac{1}{3}\rho(r) \\ \\ \sigma_{\mathcal{P}(r)} = \left(\frac{1}{3}\rho(r) - \frac{1}{3}\left\langle\rho\right\rangle_r\right) = \left(\mathcal{P}(r) - \bar{\mathcal{P}}(r)\right) \end{cases} \tag{4}$$

Therefore, if at a particular point within the distribution the value of the density, $\rho(r)$, gets very close to its average $\langle\rho(r)\rangle$ the equation of state of the material becomes similar to the typical radiation dominated environment, $P_r(r) \approx \mathcal{P}(r) \equiv \frac{1}{3}\rho(r)$.

The structure of this contribution is the following: first we give the general conventions and the field equations; secondly in Section III the nonlocal equation of state for anisotropic static models is presented; following is Section IV where we consider the consequences imposed by the junction and energy conditions; and finally the new solutions are shown in the Section V.

2 The Einstein Field Equations

To explore the feasibility of nonlocal equations of state for bounded configurations in General Relativity, we shall consider a static spherically symmetric anisotropic distribution of matter with an energy-momentum represented by $\mathbf{T}^\mu_\nu = diag\ (\rho, -P_r, -P_\perp, -P_\perp)$. Here ρ is the energy density, P_r the radial pressure and $P_\perp$ the tangential pressure. Although the perfect pascalian fluid assumption (i.e. $P_r = P_\perp$) is supported by solid observational and theoretical grounds, an increasing amount of theoretical evidence strongly suggests that, for certain density ranges, a variety of very interesting physical phenomena may take place giving rise to local anisotropy (see [9] and references therein).

We adopt standard Schwarzschild coordinates (t, r, θ, ϕ) where the line element can be written as

$$\mathrm{d}s^2 = e^{2\nu(r)}\mathrm{d}t^2 - e^{2\lambda(r)}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \tag{5}$$

with $d\Omega^2 \equiv d\theta^2 + \sin^2\theta d\phi^2$, the solid angle.

The resulting Einstein equations are:

$$8\pi\rho = \frac{1}{r^2} + \frac{e^{-2\lambda}}{r}\left[2\lambda' - \frac{1}{r}\right]\,, \tag{6}$$

$$-8\pi P_r = \frac{1}{r^2} - \frac{e^{-2\lambda}}{r}\left[2\nu' + \frac{1}{r}\right] \qquad \text{and} \tag{7}$$

$$-8\pi P_\perp = e^{-2\lambda}\left[\frac{\lambda'}{r} - \frac{\nu'}{r} - \nu'' + \nu'\lambda' - (\nu')^2\right]\,, \tag{8}$$

where primes denote differentiation with respect to r.

Using equations (7) and (8), or equivalently the conservation law $\mathbf{T}^\mu_{\nu;\mu} = 0$, we obtain the hydrostatic equilibrium equation for anisotropic fluids

$$P_r' = -\left(\rho + P_r\right)\nu' + \frac{2}{r}\left(P_\perp - P_r\right)\,. \tag{9}$$

It can be formally integrated to give

$$e^{-2\lambda} = 1 - 2\frac{m(r)}{r}\,, \tag{10}$$

where the mass function $m(r)$ defined by

$$m(r) = 4\pi\int_0^r \rho\,\bar{r}^2\,\mathrm{d}\bar{r}\,, \tag{11}$$

is the mass inside a sphere of radius r as seen by a distant observer. Finally, from(9), (6) and (7) the anisotropic Tolman-Oppenheimer-Volkov (TOV) equation [10] can be written as

$$\frac{\mathrm{d}\, P_r}{\mathrm{d}\, r} = -\left(\rho + P_r\right)\left(\frac{m + 4\pi r^3 P_r}{r\left(r - 2m\right)}\right) + \frac{2}{r}\left(P_\perp - P_r\right) . \tag{12}$$

Obviously, in the isotropic case ($P_\perp = P_r$) it becomes the usual TOV equation.

3 A Family of Solutions with a *NLES*

In this section we are going to present a family of static solution of the Einstein Equation satisfying a *NLES* Defining the new variables:

$$e^{2\nu(r)} = h\left(r\right)\, e^{4\beta(r)}, \quad \text{and} \qquad e^{2\lambda(r)} = \frac{1}{h\left(r\right)}; \qquad \text{with} \qquad h(r) \equiv 1 - 2\frac{m\left(r\right)}{r} , \tag{13}$$

the above metric (5) can be re-written as

$$\mathrm{d}s^2 = h\left(r\right)\, e^{4\beta(r)}\mathrm{d}t^2 - \frac{1}{h(r)}dr^2 - r^2\mathrm{d}\Omega^2 , \tag{14}$$

and the resulting Einstein Equations are:

$$8\pi\rho = \frac{1 - h - h'r}{r^2} , \tag{15}$$

$$8\pi P_r = -\frac{1 - h - h'r}{r^2} + \frac{4\, h\, \beta'}{r} \qquad \text{and} \tag{16}$$

$$8\pi P_\perp = \frac{h' + 2\, h\, \beta'}{r} + \frac{1}{2}\left[\, h'' + 4\, h\, \beta'' + 6\, h'\beta' + 8\, h\, \left(\beta'\right)^2\right] . \tag{17}$$

Now, if equation (1) is re-stated as

$$\rho - 3P_r + r\left(\rho' - P_r'\right) = 0 , \tag{18}$$

we have, from (18) and by using (15) - (16),

$$\frac{2}{r}\left(h' + 2h\beta'\right) + h'' + 2\beta' h' + 2h\beta'' = 0 . \tag{19}$$

It can be formally integrated yielding

$$\beta(r) = \frac{1}{2}\ln\left(\frac{\mathrm{C}}{h}\right) + \int \frac{\mathcal{C}}{r^2\, h}\, \mathrm{d}\, r + C_1 \tag{20}$$

where C and C_1 are arbitrary integration constants.

The corresponding Einstein equations in terms of the metric elements (13) are:

$$8\pi\rho = \frac{2m'}{r^2}, \tag{21}$$

$$8\pi P = 8\pi\rho - \frac{4\left(m - \mathcal{C}\right)}{r^3}, \qquad \text{and} \tag{22}$$

$$8\pi P_\perp = \frac{m''r^2\left(r - 2m\right) + 2r^2 m'\left(m' - 1\right) - 2m\left(m - r\right) - 2\mathcal{C}\left(r + 2m - 3rm'\right) + 4\mathcal{C}^2}{r^3\left(r - 2m\right)} \tag{23}$$

At this point, equation (20) deservers several comments.

- Firstly, if we set $\mathcal{C} = C_1 = 0$, a particular solution is found, i.e.,

$$\beta(r) = \frac{1}{2}\ln\left(\frac{\mathrm{C}}{h}\right), \tag{24}$$

  with C a constant parameter. We have considered this family of solutions in a previous work [8] and some of their geometric properties and several collapsing models of radiating anisotropic spheres were presented.

- The second comment concerning equation (20) is the approach we have followed in order to obtain static anisotropic solutions having a *NLES*. It is clear that if the profile of the energy density, $\rho(r)$, is provided, the metric elements $h(r)$ and $\beta(r)$ can be calculated through (11), (13) and (20). Therefore we can device a consistent method to obtain static solutions having *NLES* from a known static solutions.

It is clear that the metric elements describing bounded matter distribution should fulfill the junctions conditions and the physical variables coming from the energy momentum tensor are only restricted by some elementary criteria of physical acceptability and the hydrostatic equilibrium equation (12). The next section is devoted to list those conditions for anisotropic fluids.

4 Energy and Junctions Conditions

Most of the spherically symmetric exact solutions found in the literature do not represent physically "realistic" fluids (see for examples, two interesting and complementaries reviews on this subject [11] and [12]). The condition of what is a realistic fluid is subjective and depends from author to author on which of the energy conditions are considered valid. In order to establish the physical aceptability of the proposed static family of solutions (20) satisfying a *NLES, (*1) we shall study consequences imposed by the junction and energy conditions for anisotropic fluids on bounded matter distribution.

In order to be matched to the exterior Schwarzschild solution,

$$\mathrm{d}s^2 = \left(1 - 2\frac{M}{r}\right)\mathrm{d}t^2 - \left(1 - 2\frac{M}{r}\right)^{-1}\mathrm{d}r^2 - r^2\mathrm{d}\Omega^2\,, \qquad (25)$$

the interior metric (14) should satisfy the following conditions at the surface of the sphere $r = a$:

$$\beta\,(a) = \beta_a = 0, \qquad m(a) = M \quad \text{and} \quad P_r(a) = 0 \qquad (26)$$

thus equation (22) leads to

$$\mathcal{C} = M - 2\pi a^3 \rho\,(a) \qquad (27)$$

The elementary criteria we use to characterize a physically meaningful anisotropic fluid are:

1. The pressure must be positive:

$$P_r \geq 0 \quad \text{and} \quad P_\perp \geq 0 \qquad (28)$$

2. The pressure gradient must be negative:

$$\frac{\partial P_r}{\partial r} \leq 0 \qquad (29)$$

3. The speed of sound should be subluminal:

$$v_s \equiv \frac{\partial P_r}{\partial \rho} \leq 1 \qquad (30)$$

4. The trace of the energy-momentum tensor must be positive (*Strong Energy Condition*):

$$\rho > P_r + 2\,P_\perp \qquad (31)$$

5. The density must be larger that the pressure (*Dominant Energy Condition*):

$$\rho > P_r \quad \text{and} \quad \rho > P_\perp \qquad (32)$$

5 The Method and *NLES* Static Solutions

In the present section we state a general method to obtain *NLES* static anisotropic spherically symmetric solution from density profiles.

1. Select a static density profile $\rho(r)$, from a known static solution. Thus, the mass function can be obtained through equation (11) and by the junction condition (26) fulfills the continuity of $h(a)$ giving the expression for the total mass $m(a) = M$.
2. Following, the metric coefficient $\beta(r)$ can be found by using equation (20).
3. Finally, Einstein equations (22) through (23) provide the expressions for the radial and tangential pressures, P_r and $P_\perp$, respectively. The integration constants are obtained as consequences of the junction conditions at the boundary surface, $r = a$, i.e. $\beta(a) = 0$, $m(a) = M$ and $P_r(a) = 0$.
4. Once we have a candidate for a bounded matter distribution having a *NLES* we should explore where, within the configuration, all the above conditions (equations (28) through (32)) are valid.

In order to illustrate the above procedure we shall work out seven static solutions: *Tolman VI*, and *Tolman-V* [13], an static solution proposed by B. W. Stewart [14], three solutions proposed by M. C. Durgapal [15] and the M. K. Gokhroo and A. L. Mehra solution [16], (see Appendix). Tolman and Durgapal solutions are borrowed from isotropic static distribution of matter but Stewart and Gokhroo & Mehra solutions describe static bounded anisotropic ones. Stewart solution outlines a conformally flat anisotropic distribution which is more stable than its isotropic counterpart.

The Gokhroo and Mehra solution corresponds to an anisotropic fluid with variable density and under some circumstances [17], represents densities and pressures given rise to an equation of state similar to the Bethe-Börner-Sato newtonian equation for nuclear matter [1, 2].

The expressions for the starting energy density can be summarized as follows:

Singular models:

Equation of State	Density ρ
Tolman V	$\frac{1}{8\pi}\left[\frac{3}{7r^2}+\frac{10}{3R^2}\left(\frac{r}{R}\right)^{\frac{1}{3}}\right]$
Tolman VI	$\frac{3\sigma}{56\,\pi\,r^2}$
Stewart	$\frac{1}{8\,\pi r^2}\frac{\left(e^{2Kr}-1\right)\left(e^{4Kr}+8Kre^{2Kr}-1\right)}{\left(e^{2Kr}+1\right)^3}$

No singular models:

Equation of State	Density ρ
Durgapal (n=1)	$\frac{C}{8\,\pi}\left[\frac{1-3K-3Kx}{(1+2\,x)}+\frac{2(1+Kx)}{(1+2\,x)^2}\right]$
Durgapal (n=2)	$-\frac{C}{8\,\pi}\frac{K(3+5\,x)}{(1+3\,x)^{\frac{5}{3}}}$
Durgapal (n=3)	$\frac{C}{8\,\pi(1+x)^2}\left[\frac{3}{2}\,(3+x)-\frac{3K(1+3\,x)}{(1+4\,x)^{\frac{3}{2}}}\right]$
Gokhroo & Mehra	$\sigma\left[1-K\frac{r^2}{a^2}\right]$

The parameters that characterize bounded configurations: mass M, M/a, boundary redshift z_a, surface density ρ_a and central density ρ_c the are the shown in the following table. Let us consider stars with radius of 10 Km.

Equation of State	M/a	M $(M_\odot)$	z_a	$\rho_a\times 10^{14}\,(gr.cm^{-3})$
TolmanV	0.25	1.69	0.4	3.57
TolmanVI	0.25	1.70	0.4	2.68
Stewart	0.32	2.15	0.6	6.80

and for the models with central density:

Equation of State	M/a	$M\ (M_\odot)$	z_a	$\rho_a\ \times 10^{14}\,(gr.cm^{-3})$	$\rho_c\ \times 10^{15}\ (gr.cm^{-3})$
Durgapal (n=1)	0.25	1.69	0.4	5.36	2.00
Durgapal (n=2)	0.33	2.26	0.7	7.15	2.70
Durgapal (n=3)	0.38	2.54	1.0	8.04	2.41
Gokhroo & Mehra	0.20	1.35	0.3	4.29	0.96

Within the above configurations the range of validity for the *NLES* are

Equation of State	$r\ (Km)$
Tolman V	$7.8 \leq r \leq 10.0$
Tolman VI	$7.5 \leq r \leq 10.0$
Stewart	$0.7 \leq r \leq 10.0$
Durgapal(n=1)	$0.0 \leq r \leq 10.0$
Durgapal(n=2)	$0.0 \leq r \leq 10.0$
Durgapal(n=3)	$0.0 \leq r \leq 10.0$
Gokhroo & Mehra	$1.1 \leq r \leq 8.2$

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7 Appendix

7.1 Tolman V-like solution

We shall now take the solution V of Tolman as providing the first of the examples of a sphere of fluid surrounded by empty space. The density is given by:

$$\rho = \frac{1}{8\pi}\left[\frac{3}{7r^2} + \frac{10}{3R^2}\left(\frac{r}{R}\right)^{\frac{1}{3}}\right] \tag{33}$$

¿From equations (11) and (1) we obtain the expressions for the mass and radial pressure,

$$m = \frac{r^3}{2}\left[\frac{3}{7r^2} + \frac{1}{R^2}\left(\frac{r}{R}\right)^{\frac{1}{3}}\right] \qquad \text{and} \qquad P_r = \rho - \frac{1}{4\pi}\left[\frac{3}{7r^2} + \frac{1}{R^2}\left(\frac{r}{R}\right)^{\frac{1}{3}}\right] + \frac{\mathcal{C}}{2\pi r^3} \tag{34}$$

respectively.

¿From the boundary conditions, C_1 and R are found to be

$$\mathcal{C} = \frac{a}{84}\left[9 - 28\left(\frac{a}{R}\right)^{\frac{7}{3}}\right] \qquad \text{and} \qquad R = \frac{7^{\frac{3}{7}} a^{\frac{10}{7}}}{(14M - 3a)^{\frac{3}{7}}} \tag{35}$$

The following table contains the ranks of validity for the different criteria indicated in the section IV

	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	(0, 10.0]	(0, 10.0]	(0, 0.3] $\cup$ [7.8, 10.0]	(0, 10.0]	[5.8, 10.0]	[3.4, 10.0]	[3.6, 10.0]	[2.0, 10.0]

7.2 Tolman VI-like solution

In this case, the model to be studied is the solution VI of Tolman. We recall that the equation of state of this model, for large ρ, approaches that for a highly compressed Fermi gas. The density matter is

$$\rho = \frac{3\,\sigma}{56\,\pi\,r^2} \tag{36}$$

Respectively, the mass and pressure are found to be

$$m = \frac{3}{14}\sigma r \qquad \text{and} \qquad P_r = -\rho + \frac{\mathcal{C}}{2\,\pi r^3}\ . \tag{37}$$

The constant C_1 and the boundary conditions lead to

$$\mathcal{C} = \frac{3}{28}\,\sigma\,a \qquad \text{and} \qquad \sigma = \frac{14\,M}{3\,a} \tag{38}$$

The energy conditions yield to

$r \leq 10.0$	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	(0, 10.0]	(0, 10.0]	(0, 5.0]	(0, 10.0]	[7.5, 10.0]	[5.0, 10.0]	[5.0, 10.0]	[3.0, 10.0]

7.3 Stewart Solution

B.W. Stewart examines several anisotropic, conformally flat, internal solutions. He presents four examples of mass distributions, we select the example 2 from that paper [14]

$$\rho = \frac{1}{8\,\pi r^2}\frac{\left(e^{2Kr}-1\right)\left(e^{4Kr}+8Kre^{2Kr}-1\right)}{\left(e^{2Kr}+1\right)^3} \tag{39}$$

The expressions for the mass distribution and pressure are

$$m = \frac{r}{2}\left(\frac{e^{2Kr}-1}{e^{2Kr}+1}\right)^2 \qquad \text{and} \qquad P_r = \rho - \frac{1}{4\pi r^2}\left(\frac{e^{2Kr}-1}{e^{2Kr}+1}\right)^2 + \frac{\mathcal{C}}{2\,\pi r^3} \tag{40}$$

Again, the constant C_1 is obtained to make $P_r(a) = 0$, i.e.

$$\mathcal{C} = \frac{a}{4}\frac{e^{2Ka}\left(e^{3Ka}-8Kae^{2Ka}-e^{2Ka}+8Ka-1\right)+1}{e^{2Ka}\left(e^{3Ka}+3e^{2Ka}+3\right)+1} \tag{41}$$

while the constant K is found from the boundary conditions as

$$K = \frac{1}{2a}\ln\left[\frac{1+\left(\frac{2M}{a}\right)^{\frac{1}{2}}}{1-\left(\frac{2M}{a}\right)^{\frac{1}{2}}}\right] \tag{42}$$

The energy conditions lead

$r \leq 10.0$	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0.5,\ 10)$	$(0,\ 10.0]$	$[1.9,\ 10.0]$	$[0.6,\ 10.0]$	$(0.5,\ 10.0)$	$(0,\ 10.0]$

7.4 Gokhroo and Mehra solution

Now, the Gokhroo and Mehra [16] solution is presented. The mass density is

$$\rho = \sigma\left[1 - K\frac{r^2}{a^2}\right] \tag{43}$$

the mass and the radial pressure, can be written as

$$m = \frac{\sigma\, r^3}{6}\left[1\; - \frac{3\,K}{5}\frac{r^2}{a^2}\right] \qquad \text{and} \qquad P_r = \frac{3}{5}\rho - \frac{1}{2\pi}\left[\frac{\sigma}{15} - \frac{\mathcal{C}}{r^3}\right] \tag{44}$$

and the boundary conditions lead to

$$\mathcal{C} = \frac{\sigma a^3}{60}\left[9K - 5\right] \qquad \text{and} \qquad \sigma = \frac{30\,M}{a^3\,(5-3K)} \tag{45}$$

The physical variables are valid in the following range

$r \leq 10.0$	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 8.2]$	$(0,\ 10.0]$	$[1.1, 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$

7.5 Durgapal Solution

The paper of M. C. Durgapal [15] report several solutions. The autor assume that the value of the metric coefficient is given by: $e^{\nu} = A\left(1 + Cr^2\right)^n$, where A and C are constant and n a parameter. Durgapal shows that Einstein's equations can be solved explicitly for different values from n.

- For $n = 1$

  The first solution are identical to Tolman IV solution. The expression for the density is given by

$$\rho = \frac{C}{8\,\pi}\left[\frac{1-3K-3Kx}{(1+2\,x)} + \frac{2\,(1+Kx)}{(1+2\,x)^2}\right] \tag{46}$$

  Again, the expressions for mass and radial pressure are obtained from the equation (1):

$$m = -\frac{x^{\frac{3}{2}}}{2C^{\frac{1}{2}}}\frac{K\,(1+x)-1}{(1+2x)} \qquad \text{and} \qquad P_r = \rho + \frac{C}{4\,\pi}\frac{K\,(1+x)-1}{(1+2x)} + \frac{\mathcal{C}}{2\,\pi\left(\frac{x}{C}\right)^{\frac{3}{2}}} \tag{47}$$

  where

$$x = C\,r; \qquad x_1 = C\,a^2 = \frac{M}{a-3M}; \qquad \text{and} \qquad K = \frac{\sqrt{C}\,[ax_1 - 2M\,(1+2x_1)]}{x_1^{\frac{3}{2}}\,(1+x_1)} \tag{48}$$

  The constant K is obtained for boundary conditions. The pressure in the surface ($P_r(a) = 0$) leads to that:

$$\mathcal{C} = \frac{ax_1}{4}\frac{K\left(1+x_1+2x_1^2\right)+2x_1-1}{1+4x_1\,(1+x_1)} \tag{49}$$

  The validity range are:

$r \leq 10.0$	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]

- For $n = 2$

  This solution is identical to that obtained by Kuchowicz, Adler and Adams and Cohen, according to the author. The density is given by

$$\rho = -\frac{C}{8\,\pi}\frac{K\,(3+5\,x)}{(1+3\,x)^{\frac{5}{3}}} \tag{50}$$

  The mass and radial pressure, are :

$$m = -\frac{1}{2C^{\frac{1}{2}}}\frac{Kx^{\frac{3}{2}}}{(1+3x)^{\frac{2}{3}}} \qquad \text{and} \qquad P_r = \rho + \frac{1}{4\,\pi}\frac{CK}{(1+3x)^{\frac{2}{3}}} + \frac{\mathcal{C}}{2\,\pi\left(\frac{x}{C}\right)^{\frac{3}{2}}} \tag{51}$$

  where

$$x = C\,r; \qquad x_1 = C\,a^2 = \frac{M}{2a-5M}; \qquad \text{and} \qquad K = -\frac{2\,M}{a\,x_1}\,(1+3\,x_1)^{\frac{2}{3}} \tag{52}$$

  The constant K is obtained for boundary conditions and $\mathcal{C}$ by ($P_r(a) = 0$)

$$\mathcal{C} = \frac{CKa^3}{4}\frac{1-x_1}{(1+3x_1)^{\frac{5}{3}}} \tag{53}$$

  The regions where the physical variables are valid

$r \leq 10.0$	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]	(0, 10.0]

- For $n = 3$

  In the third solution studied by Durgapal the density has the following form:

$$\rho = \frac{C}{8\,\pi\,(1+\,x)^2}\left[\frac{3}{2}\,(3+x) - \frac{3K\,(1+3\,x)}{(1+4\,x)^{\frac{3}{2}}}\right] \tag{54}$$

This expression for the density is exactly the same expression for the densited reported in Durgapal and Bannerji for the case with $K = 0$

Again, the mass and the radial pressure are

$$m = -\frac{x^{\frac{3}{2}}}{4C^{\frac{1}{2}}}\frac{2K - 3\,(1+4x)^{\frac{1}{2}}}{(1+x)\,(1+4x)^{\frac{1}{2}}}; \qquad \text{and} \qquad P_r = \rho + \frac{C}{8\,\pi}\frac{2K - 3\,(1+4x)^{\frac{1}{2}}}{(1+x)\,(1+4x)^{\frac{1}{2}}} + \frac{\mathcal{C}}{2\,\pi\left(\frac{x}{C}\right)^{\frac{3}{2}}} \tag{55}$$

where

$$x = C\,r; \qquad x_1 = C\,a^2 = \frac{M}{3a - 7M}; \qquad \text{and} \qquad K = -\frac{(1+4\,x_1)^{\frac{1}{2}}}{2a\,x_1}\left[4M\,(1+x_1) - 3\,a\,x_1\right] \tag{56}$$

Again, te constant K is obtained for boundary conditions and $\mathcal{C}$ by: $(P_r(a) = 0)$

$$\mathcal{C} = \frac{x_1^{\frac{3}{2}}}{8C^{\frac{1}{2}}}\frac{2K\left(1 - x_1 - 8x_1^2\right) - 3\,(1+4x_1)^{\frac{1}{2}}\left(1 + 3x_1 - 4x_1^2\right)}{(1+4x_1)^{\frac{3}{2}}\,(1 + 2x_1 + x_1^2)} \tag{57}$$

The validity ranks are

$r \leq 10.0$	$\rho \geq 0$	$P_r \geq 0$	$P_\perp \geq 0$	$\frac{\partial P_r}{\partial r} < 0$	$\frac{\partial P_r}{\partial \rho} \leq 1$	$\rho > P_r + 2\,P_\perp$	$\rho > P_r$	$\rho > P_\perp$
$r \in$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$	$(0,\ 10.0]$

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