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RADIATING, SLOWLY ROTATING BODIES IN GENERAL RELATIVITY

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ABSTRACT

We propose a method to obtain axisymmetric, dynamic solutions to the Einstein equations that can represent a radiating collapsing body in slow differential rotation. The method is a generalization of the seminumeric approach developed by Herrera, Jiménez, & Ruggeri, in 1980, for the spherically symmetric case. Solutions are properly matched to the exterior Kerr-Vaidya metric, and the values of the physically relevant variables (density, pressure, fluid velocity, and energy flux) are obtained inside the matter distribution. As an example of the method, a model based on Schwarzschild interior homogeneous static solution is presented.

Subject headings: relativity — stars: neutron — stars: rotation

1. INTRODUCTION

One of the most important pending problems in general relativity is the construction of a "physically reasonable" source for the Kerr metric (Kerr 1963). Indeed, in spite of a considerable effort, the question of finding a source for the Kerr metric describing a finite body, with a physically relevant stress tensor and matched to a Kerr solution, remains unsolved (see Krasiński 1979; Herlt 1988; Wahlquist 1992, and references therein).

On the other hand, a steady increase in the number of observed rotating neutron stars with rotational periods within the millisecond range (Backer & Kulkarni 1990) has renewed the interest in the influence and the implications of rotation on the properties of compact objects (Weber, Glendenning, & Weigel 1991).

Fortunately, if in the adiabatic case "slow rotation" is assumed, then the field equations reduce, up to the first order in angular momentum, to the equations with spherical symmetry plus an additional equation to be solved for the angular velocity of the inertial frame along rotational axis. This assumption is very sensible because it considers that the tangential velocity of every fluid element is much less than the speed of light and the centrifugal forces are little compared with the gravitational ones. This important simplification, and the fact that all known pulsars satisfy these conditions, justify the intense activity developed around this subject. Thus, equations governing equilibrium configurations in slow uniform rotation have been derived for the first time by Hartle (1967) (up to the second order in the angular velocity) and Cohen (1970) (up to first order). An analytic theory of slowly and uniformly rotating relativistic bodies was proposed by Abramowicz & Wagoner (1978). Applications of these methods leading to slowly rotating neutron star models or to analytic solutions of the Einstein equations for rotating sources (always in the slow rotation approximation) may be found in Hartle & Thorne (1968); Adams et al. (1973, 1974); Chandrasekhar & Miller (1974); Adams & Cohen (1975); Whitman & Pizzo (1979); Whitman (1982, 1985); Stewarts (1983); Ibañez (1983); Friedman, Ipser, & Parker (1986); Datta (1988); Weber et al. (1991); Weber & Glendenning (1992) and references therein. It is particularly interesting for the present work to mention some previous effort building sources to the Kerr-Vaidya metric in the slow rotation approximation (Murenbeeld & Trollope 1970; Bayin 1981, 1983).

It is the purpose of this paper to present a general method to obtain nonstationary (time dependent), radiating, and slowly rotating sources. At the outside of the matter distribution a Kerr-Vaidya metric (Carmeli & Kaye 1977) is assumed. This method is an extension of an approach introduced some years ago to model the collapse of general relativistic radiating spheres (Herrera, Jiménez, & Ruggeri 1980) and which has been successfully applied to a variety of astrophysical scenarios (Herrera & Núñez 1990, and references therein). Models of axisymmetric radiating bodies based on the slow rotation approximation may be obtained from static "seed" solutions. For simplicity, we shall restrain ourselves in this paper, to the first order in the slow rotation limit (i.e., we shall need only linear terms in the angular velocity of the local inertial frame). Therefore all effects related to deviations from spherical symmetry are purely relativistic. In fact, in the slow rotation formalism, up to first order in the angular velocity of local inertial frames, all matter variables are angular independent, and effects of rotation manifest through the dragging of local inertial frames (a purely relativistic effect). This is understandable if we recall that in the Newtonian theory, where the parameter measuring the "strength" of rotation (the ratio of centrifugal acceleration to gravity at the equator) is not linear in the angular velocity, but proportional to the square of it.

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The paper is organized as follows. In §§ 2 and 3 the energy momentum tensor and the field equations are introduced. Junction conditions are presented in § 4. Section 5 displays the method and in § 6 we work out a model starting with the homogeneous Schwarzschild static "seed" solution. Some comments and conclusions are included in § 7. Finally, spin coefficients for the interior and exterior metrics are given in the Appendix.

2. ENERGY-MOMENUM TENSOR

Let us consider a nonstatic and axisymmetric distribution of matter formed by a fluid and radiation. The exterior metric, in radiation coordinates (Bondi 1964), takes the Kerr-Vaidya form (Carmeli & Kaye 1977):

$$ds^2 = \left(1 - \frac{2mr}{r^2 + \alpha^2 \cos^2 \theta}\right)du^2 + 2\,du\,dr - 2\alpha \sin^2 \theta\, dr\, d\phi + 4\alpha \sin \theta^2 \frac{mr}{r^2 + \alpha^2 \cos^2 \theta}\, du\, d\phi - (r^2 + \alpha^2 \cos^2 \theta)d\theta^2$$
$$- \sin^2 \theta \left[r^2 + \alpha^2 + \frac{2mr\alpha^2 \sin^2 \theta}{r^2 + \alpha^2 \cos^2 \theta}\right]d\phi^2 . \tag{1}$$

Here α is the Kerr parameter, representing angular momentum per unit mass in the weak field limit, and m is the total mass.

The interior metric is written as (Herrera & Jiménez 1982)

$$ds^2 = e^{2\beta}\left\{\frac{V}{r}\, du^2 + 2\, du\, dr\right\} - (r^2 + \tilde{\alpha}^2 \cos^2 \theta)d\theta^2 + 2\tilde{\alpha}e^{2\beta} \sin^2 \theta \left\{1 - \frac{V}{r}\right\}du\, d\phi - 2e^{2\beta}\tilde{\alpha} \sin^2 \theta\, dr\, d\phi$$
$$- \sin^2 \theta \left\{r^2 + \tilde{\alpha}^2 + 2\tilde{\alpha}^2 \sin^2 \theta \frac{V}{r}\right\}d\phi^2 . \tag{2}$$

In the above equations (1) and (2), $u = x^0$ is a timelike coordinate, $r = x^1$ is the null coordinate, and $\theta = x^2$ and $\phi = x^3$ are the usual angle coordinates. The u-coordinate is the retarded time in flat space time and, therefore, u-constant surfaces are null cones open to the future.

The Kerr parameter for the interior spacetime (2) is denoted $\tilde{\alpha}$ and, for the present work, will be considered constant and only (as well as α in eq. [1]) up to the *first order.* Notice that in these coordinates the r = constant = r_0 surfaces are not spheres but *oblate spheroids*, whose eccentricity depends upon the Kerr parameter $\tilde{\alpha}$ and is given by

$$e^2 = 1 - \frac{r_0^2}{r_0^2 + \tilde{\alpha}^2} . \tag{3}$$

Metric elements β and V in equation (2), are functions of u, r, and θ. A function $\tilde{m}(u, r, \theta)$ can be defined by

$$V = e^{2\beta}\left[r - \frac{2\tilde{m}(u, r, \theta)r^2}{r^2 + \tilde{\alpha}^2 \cos^2 \theta}\right], \tag{4}$$

which is the generalization, inside the distribution, of the "mass aspect" defined by Bondi and collaborators (Bondi, Van der Burg, & Metzner 1962) and in the static limit coincides with the Schwarzschild mass.

In order to give a clear physical significance to the above formulae, we now introduce local Minkowski coordinates (t, x, y, z):

$$dt = e^{\beta}\left(\sqrt{\frac{V}{r}}\, du + \sqrt{\frac{r}{V}}\, dr\right) + \tilde{\alpha} \sin^2 \theta e^{\beta}\left(\sqrt{\frac{r}{V}} - \sqrt{\frac{V}{r}}\right)d\phi , \tag{5}$$

$$dx = e^{\beta} \sqrt{\frac{r}{V}}\, (dr + \tilde{\alpha} \sin^2 \theta\, d\phi) , \tag{6}$$

$$dy = (r^2 + \tilde{\alpha}^2 \cos^2 \theta)^{1/2}\, d\theta , \tag{7}$$

$$dz = (r^2 + \tilde{\alpha}^2 \cos^2 \theta)^{1/2} \sin \theta\, d\phi . \tag{8}$$

It is assumed that, for a local Minkowskian observer comoving with the fluid, the space-time contains:

1. an isotropic fluid of density ρ and pressure P and,
2. a radiation field of specific intensity $I(x, t; \boldsymbol{n}, \nu)$.

The specific intensity of the radiation field, $I(x, t; \boldsymbol{n}, \nu)$, is measured at the position x and time t, traveling in the direction $\boldsymbol{n}$ with a frequency ν. As in classical radiative transfer theory, the moments of the *specific intensity of radiation* for a planar geometry can be written as (Mihalas & Mihalas 1984):

$$\rho_R = \tfrac{1}{2} \int_0^{\infty} d\nu \int_{-1}^{1} d\mu I(x, t; \boldsymbol{n}, \nu) , \tag{9}$$

$$\mathscr{F} = \tfrac{1}{2} \int_0^{\infty} d\nu \int_{-1}^{1} d\mu \mu I(x, t; \boldsymbol{n}, \nu) , \tag{10}$$

and

$$\mathscr{P} = \frac{1}{2} \int_0^\infty d\nu \int_{-1}^{1} d\mu \mu^2 I(x, t; \boldsymbol{n}, \nu) , \tag{11}$$

where $\mu = \cos\theta$. Physically, ρ_R, $\mathscr{F}$, and $\mathscr{P}$ represent the radiation contribution to energy density, energy flux density, and pressure, respectively.

For a moving observer the covariant energy momentum tensor can be written as

$$\hat{T}_{\mu\nu} = \hat{T}^M_{\mu\nu} + \hat{T}^R_{\mu\nu} , \tag{12}$$

where the material part, $\hat{T}^M_{\mu\nu}$, is

$$\hat{T}^M_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} . \tag{13}$$

To construct the tensor for the radiation field, we start with the energy momentum tensor for the radiation field, $\hat{T}^R_{\mu\nu}$, as seen by a local Minkowskian *nonrotating* observer (Lindquist 1966; Mihalas & Mihalas 1984):

$$\hat{T}^R_{\mu\nu} = \begin{pmatrix} \rho_R & -\mathscr{F} & 0 & 0 \\ -\mathscr{F} & \mathscr{P} & 0 & 0 \\ 0 & 0 & \frac{1}{2}(\rho_R - \mathscr{P}) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(\rho_R - \mathscr{P}) \end{pmatrix} . \tag{14}$$

With a local infinitesimal rotation, we find the tensor as seen by an observer comoving with the fluid:

$$\hat{T}^R_{\mu\nu} = \begin{pmatrix} \rho_R & -\mathscr{F} & 0 & \mathscr{D}(3\rho_R - \mathscr{P})/2 \\ -\mathscr{F} & \mathscr{P} & 0 & -\mathscr{D}\mathscr{F} \\ 0 & 0 & (\rho_R - \mathscr{P})/2 & 0 \\ \mathscr{D}(3\rho_R - \mathscr{P})/2 & -\mathscr{D}\mathscr{F} & 0 & (\rho_R - \mathscr{P})/2 \end{pmatrix} , \tag{15}$$

where $\mathscr{D}$ is an unknown function of u, r, and θ, associated with the local "dragging of inertial frames" effect. In the slow rotation limit, $\mathscr{D}$ is also taken up to the first order.

Once the Minkowskian comoving energy momentum tensor is built in terms of physical observables on a local frame (ρ, P, ρ_R, $\mathscr{F}$, $\mathscr{P}$, and $\mathscr{D}$), it can be transformed as follows:

$$T_{\alpha\beta} = \frac{\partial \hat{x}^\gamma}{\partial x^\alpha} \frac{\partial \hat{x}^\lambda}{\partial x^\beta} L^\mu_\gamma(\boldsymbol{\omega}) L^\nu_\lambda(\boldsymbol{\omega}) \hat{T}_{\mu\nu} , \tag{16}$$

where $L^\nu_\lambda(\boldsymbol{\omega})$ is a Lorentz boost. In the rotating case, the boost velocity, $\boldsymbol{\omega}$, has components ω_x in the radial direction and ω_z in the ϕ-direction that represent the radial and orbital velocities of the fluid as measured by a local Minkowskian observer. The coordinate transformations, $\partial\hat{x}^\gamma/\partial x^\alpha$, connecting Minkowskian coordinates (t, x, y, z) to Bondi coordinates (u, r, θ, ϕ) emerge from equations (5) through (8) (see Herrera & Núñez 1990 for details). We thus obtain the stress-energy tensor as seen by a general observer, written in terms of local (Minkowskian) variables.

In radiation coordinates the radial velocity of matter is given by

$$\frac{dr}{du} = \frac{V}{r} \frac{\omega_x}{1 - \omega_x} , \tag{17}$$

and the orbital velocity by

$$\frac{d\phi}{du} = \frac{\omega_z}{1 - \omega_x} \frac{1}{r \sin\theta} e^\beta \sqrt{\frac{V}{r}} . \tag{18}$$

3. LIMITS FOR THE RADIATION FIELD AND EINSTEIN EQUATIONS

Herrera and collaborators have considered collapsing spherical radiating configurations in the two limits for the radiation field: *free streaming out* (Herrera et al. 1980) and *diffusion* (Herrera, Jiménez, & Esculpi 1987). Barreto & Núñez (1991) have also studied general relativistic spheres where diffusion and free streaming processes coexist. It is clear that in order to deal with realistic physical scenarios, a relativistic Boltzmann transport-equation should be considered to describe the evolution of the radiation through the matter configuration (Lindquist 1966). In this way the above radiation momenta (9), (10), and (11) are related to the physical properties of the medium (absorption and/or emission). Despite this, it is possible to consider several physically interesting situations in the above mentioned limits (Mihalas & Mihalas 1984). The *free streaming out* limit assumes that the radiation (neutrinos and/or photons) mean free path is of the order of the dimensions of the spheroid. With this assumption it is obtained that

$$\rho_R = \mathscr{F} = \mathscr{P} = \epsilon \, . \tag{19}$$

On the other hand, in the *diffusion limit approximation* radiation is considered to have a mean free path much smaller than the characteristic length of the system. Within this limit, radiation is locally isotropic and we have

$$\rho_R = 3\mathscr{P} \quad \text{and} \quad \mathscr{F} = \hat{q} \, . \tag{20}$$

In this work, we have considered only the *free streaming out* limit; therefore the stress-energy tensor in the local Minkowskian comoving frame takes the form

$$\hat{T}_{\mu\nu} = \begin{pmatrix} \rho + \epsilon & -\epsilon & 0 & \epsilon\mathscr{D} \\ -\epsilon & P + \epsilon & 0 & -\epsilon\mathscr{D} \\ 0 & 0 & P & 0 \\ \epsilon\mathscr{D} & -e\mathscr{D} & 0 & P \end{pmatrix} . \tag{21}$$

Give the metric (2), using the transformations of equation (16), and considering the slow-rotation limit (i.e., to first order in the orbital velocity ω_z, the dragging function $\mathscr{D}$, and the Kerr parameter α), we can finally write the Einstein field equations as

$$8\pi r^2 e^{-4\beta} T_{uu} = 8\pi r^2 \left(1 - \frac{2\tilde{m}}{r}\right)\left[\frac{\rho + P\omega^2}{1 - \omega^2} + \epsilon\left(\frac{1 + \omega_x}{1 - \omega_x}\right)\right]$$

$$= 2\left(1 - \frac{2\tilde{m}}{r}\right)\tilde{m}_1 - \frac{\tilde{m}_{22}}{r} - \frac{\tilde{m}_2 \cot\theta}{r} - 2e^{-2\beta}\tilde{m}_0 \, , \tag{22}$$

$$8\pi r^2 e^{-2\beta} T_{ur} = 8\pi r^2 \left(\frac{\rho - P\omega_x}{1 + \omega_x}\right) = 2\tilde{m}_1 - \beta_{22} - \beta_2 \cot\theta \, , \tag{23}$$

$$8\pi r^2 T_{rr} = \frac{8\pi r^2}{(1 - 2\tilde{m}/r)}\left(\frac{1 - \omega_x}{1 + \omega_x}\right)(\rho + P) = 4r\beta_1 \, , \tag{24}$$

$$8\pi r^2 T_{\theta\theta} = 8\pi r^3 P = 2\beta_2 \cot\theta - \tilde{m}_{11} r - 2e^{-2\beta} r^2 \beta_{10} - 6\beta_1 \tilde{m}_1 r + 3\beta_1 r + \left(1 - \frac{2\tilde{m}}{r}\right)(4\beta_1^2 r + 2\beta_{11} r - \beta_1) r \, , \tag{25}$$

$$\frac{8\pi r^2}{\sin^2\theta} T_{r\phi} = 8\pi r^2 \left\{ -e^{2\beta}\tilde{\alpha}\left(\frac{\rho - P\omega_x}{1 + \omega_x}\right) + \left(\frac{1 - \omega_x}{1 + \omega_x}\right)\tilde{\alpha}\,\frac{\rho + P}{(1 - 2\tilde{m}/r)} - \frac{r\omega_z}{\sin\theta}\sqrt{\left(1 - \frac{2\tilde{m}}{r}\right)^{-1}}\,\frac{\rho + P}{1 + \omega_x} \right\}$$

$$= \tilde{\alpha}\{2r(\beta_1 + \beta_0) - 1 + r^2(\beta_{11} - \beta_{01}) + e^{2\beta}(1 - 2\tilde{m}_1)\} \, , \tag{26}$$

$$e^{-2\beta}\frac{8\pi r^3}{\sin^2\theta} T_{u\phi} = 8\pi r^3 \left\{ -e^{2\beta}\left(1 - \frac{2\tilde{m}}{r}\right)\tilde{\alpha}\left[\frac{\rho + P\omega^2}{1 - \omega^2} + \epsilon\left(\frac{1 + \omega_x}{1 - \omega_x}\right)\right] + \frac{\tilde{\alpha}}{1 + \omega_x}(\rho - P\omega_x) \right.$$

$$\left. + \mathscr{D}\epsilon\,\frac{r}{\sin\theta}\sqrt{\left(1 - \frac{2\tilde{m}}{r}\right)}\sqrt{\left(\frac{1 + \omega_x}{1 - \omega_x}\right)} - \frac{r\omega_z}{\omega_x \sin\theta}\sqrt{\left(1 - \frac{2\tilde{m}}{r}\right)}\left[\frac{\omega_x(\rho + P)}{1 - \omega^2} + \epsilon\left(\frac{1 + \omega_x}{1 - \omega_x} + \sqrt{\frac{1 + \omega_x}{1 - \omega_x}}\right)\right]\right\}$$

$$= \tilde{\alpha} r \left\{ \left(1 - \frac{2\tilde{m}}{r}\right)[r^2(4\beta_1\beta_0 - 4\beta_1^2 + \beta_{10}) - (\beta_0 - 3\beta_1) r + 1] - r[(\beta_0 - 3\beta_1)(1 - 2\tilde{m}_1) \right.$$

$$\left. - \tilde{m}_{10} + \tilde{m}_{11} - \tilde{m}_0/r(4\beta_1 r - 3)] + e^{2\beta}\left(1 - \frac{2\tilde{m}}{r}\right)(1 - 2\tilde{m}_1) + e^{-2\beta} r^2 [\beta_{10} - \beta_{00}] \right\} , \tag{27}$$

$$T_{r\theta} = 0 = \beta_{12} - 2\frac{\beta_2}{r} \, , \tag{28}$$

$$T_{\phi\phi} - \sin^2\theta\, T_{\theta\theta} = 0 = \beta_{22} - \beta_2 \cot\theta \, , \tag{29}$$

$$T_{u\theta} = 0 = \left(1 - \frac{2\tilde{m}}{r}\right)(r\beta_{12} + 4r\beta_1\beta_2 - \beta_2) + 2\tilde{m}_{12} + \beta_2 - 2\beta_2\tilde{m}_1 - 4\beta_1\tilde{m}_2 + \frac{\tilde{m}_2}{r} + e^{-2\beta}\beta_{02} \, . \tag{30}$$

Differentiation with respect to u, r, and θ are denoted by subscripts 0, 1, and 2, respectively.

With $\beta(u, r)$, $\tilde{m}(u, r)$ and their derivatives as known functions, equations (22) through (30) can be considered as a system of six algebraic equations in the physical variables ω_x, ω_z, ρ, P, ρ_R, $\mathscr{F}$, $\mathscr{P}$, and $\mathscr{D}$. Therefore, only six (of the eight) of these variables can be algebraically obtained. Accordingly, more information has to be provided to this system.

4. JUNCTION CONDITIONS

If one wishes the solutions to the Einstein equations to represent the interior of a fluid spheroid, they must be matched to an exterior solution, represented by the Kerr-Vaidya metric (eq. [1]). To accomplish this, we have required the continuity across the boundary surface of the *tetrad components* and the *spin coefficients* for the metrics (1) and (2). These requirements have been shown to be equivalent to demand the continuity of both *first* and *second fundamental forms* (g_{ij} and K_{ij}) across the boundary (Herrera & Jiménez 1983).

The Newman & Penrose (1962) null tetrad components of the exterior metric can be written as

$$l^\mu = \delta_r^\mu , \tag{31}$$

$$n^\mu = \delta_u^\mu - \frac{1}{2}\left[1 - \frac{2mr}{(r^2 + \alpha^2 \cos^2 \theta)}\right]\delta_r^\mu , \tag{32}$$

$$m^\mu = \frac{1}{\sqrt{2}(r + i\alpha \cos \theta)} [i\alpha \sin \theta(\delta_u^\mu - \delta_r^\mu) + \delta_\theta^\mu + i \csc \theta\, \delta_\phi^\mu] . \tag{33}$$

And for the interior metric we have

$$l^\mu = e^{-2\beta} \delta_r^\mu , \tag{34}$$

$$n^\mu = \delta_u^\mu - \frac{1}{2} e^{-2\beta}\left[1 - \frac{2\tilde{m}r}{(r^2 + \tilde{\alpha}^2 \cos^2 \theta)}\right]\delta_r^\mu , \tag{35}$$

$$m^\mu = \frac{1}{\sqrt{2}(r + i\tilde{\alpha} \cos \theta)} [i\tilde{\alpha} \sin \theta(\delta_u^\mu - \delta_r^\mu) + \delta_\theta^\mu + i \csc \theta\, \delta_\phi^\mu] . \tag{36}$$

Continuity of these tetrad components across the surface $r = a(u)$ implies

$$\beta_a = 0 , \tag{37}$$

$$\tilde{m}_a = m , \tag{38}$$

and

$$\tilde{\alpha}_a = \alpha , \tag{39}$$

where the subscript a means evaluation at the corresponding surface.

The continuity of the spin coefficients τ, γ, and ν gives

$$\beta_{1a}\left(1 - \frac{2m}{a}\right) - \beta_{0a} = \frac{\tilde{m}_{1a}}{2a} , \tag{40}$$

$$\beta_{2a} = \tilde{m}_{2a} = 0 , \tag{41}$$

and

$$\alpha(\beta_{1a} - \beta_{0a}) = \alpha(m_{0a} - \tilde{m}_{0a} + \tilde{m}_{1a}) = 0 . \tag{42}$$

It is to say, $(\beta_{1a} - \beta_{0a})$ and $(m_{0a} - \tilde{m}_{0a} + \tilde{m}_{1a})$ are of order α.

In order to find the consequences of the junction condition upon the physical variables at the surface, all this results are introduced in the Einstein field equations evaluated at boundary surface $r = a(u)$. The first outcome emerges from equation (28) that can be integrated to give

$$\beta_2 = \mathscr{C}(u, \theta)r^2 . \tag{43}$$

Now, using condition (41), we find $\mathscr{C}(u, \theta) = 0$. Therefore β *does not depend on the angular variable.* A similar result can be found using this fact and integrating field equation (30):

$$\tilde{m}_2 = \frac{e^{4\beta}}{r} \mathscr{D}(u, \theta) . \tag{44}$$

Again, equation (41) implies that $\tilde{m}$ is a *function of r and u only.* From field equations (22) through (30), it is easily seen that the independence of both metric variables on the angular coordinate will lead to the independence of the physical variables ρ, P, ω_x, and ϵ on this coordinate. This conclusion, that ρ and P do not depend on θ, to first order in α and in these coordinates (where $r =$ const. defines an oblate spheroid), was found by Hartle (1967) for the stationary case. Here, it is obtained as a consequence of the junction conditions.

The second important consequence comes from equation (40). Using that β is continuous and $\beta = 0$ for the Kerr-Vaidya metric, we may expand it near the boundary $r = a(u)$

$$\beta_{0a} + \dot{a}\beta_{1a} = 0 , \tag{45}$$

where $\dot{a} = da/du$. Substituting equation (45) into equation (40) and using field equations (23) and (24)

$$\dot{a} = \left(1 - \frac{2m}{a}\right)\left[\frac{\rho_a \omega_{xa} - P_a}{(\rho_a + P_a)(1 - \omega_x)}\right] . \tag{46}$$

On the other hand, it follows from equation (17) that

$$\dot{a} = \frac{\omega_{xa}}{1 - \omega_{xa}} \left(1 - \frac{2m}{a}\right) . \tag{47}$$

Comparing equations (46) and (47) we obtain

$$P_a = 0 . \tag{48}$$

As in the spherically symmetric case, the pressure vanishes at the surface in the free streaming out limit (Herrera et al. 1980; Herrera & Núñez 1990).

Finally, we can use equation (45) and a similar expansion for $\tilde{m}$ to write conditions (42) as

$$\alpha \beta_{1a}(1 + \dot{a}) = 0 , \tag{49}$$

and

$$\alpha \tilde{m}_{1a}(1 + \dot{a}) = 0 . \tag{50}$$

5. SURFACE EQUATIONS AND THE HJR METHOD

To obtain the physical variables for the matter configuration, we will use the so-called HJR method. It can be considered a method to find *dynamic, spherically symmetric* solutions to the Einstein field equations starting from a known *static* solution. We will follow it closely and find that it can be extended to the axially symmetric slowly rotating case. For details of the method and some of its applications to astrophysical scenarios, see Herrera & Núñez (1990).

We start by defining two auxiliary functions (*the effective variables*):

$$\tilde{\rho} = \frac{\rho - P\omega_x}{1 + \omega_x} , \tag{51}$$

and

$$\tilde{P} = \frac{P - \rho\omega_x}{1 + \omega_x} . \tag{52}$$

These are the *effective density* and the *effective pressure*, respectively. It is clear that these variables coincide with the corresponding "physical" density and pressure in the static case. Now field equations (23) and (24) can be integrated yielding

$$\tilde{m} = \int_0^r d\bar{r} 4\pi \bar{r}^2 \tilde{\rho} , \tag{53}$$

and

$$\beta = \int_r^a \frac{2\pi \bar{r}^2 \, d\bar{r}}{\bar{r} - 2\tilde{m}} (\tilde{\rho} + \tilde{P}) . \tag{54}$$

The crucial point of the HJR method is the system of ordinary differential equations for quantities evaluated at the surface which is called *the System of Surface Equations* (SSE). The first of these *surface equations* is equation (47). Scaling the radius a, the total mass $\tilde{m}_a = m$, and the timelike coordinate u by the total initial mass, $m(u = 0) = m(0)$, i.e.,

$$A = \frac{a}{m(0)} , \qquad M = \frac{m}{m(0)} , \qquad u = \frac{u}{m(0)} \tag{55}$$

and defining

$$F = 1 - \frac{2M}{A} , \qquad \Omega = \frac{1}{1 - \omega_{xa}} . \tag{56}$$

Equation (47) can be written as

$$\dot{A} = F(\Omega - 1) . \tag{57}$$

The second *surface equation* emerges from the evaluation of field equation (22) at $r = a + 0$. It takes the form of

$$\dot{M} = -FL , \tag{58}$$

where L can be written as

$$L = 4\pi A^2 \epsilon_a (2\Omega - 1) . \tag{59}$$

Notice that L is related to the total luminosity.

Now, using above definitions (55) and (56), and equation (57), we can restate equation (58) as

$$\frac{\dot{F}}{F} = \frac{2L + (1 - F)(\Omega - 1)}{A} . \tag{60}$$

The third *surface equation* comes from field equations (23), (24), and (25). After some straightforward manipulations, it is obtained

$$e^{2\beta}\left(\frac{\tilde{\rho} + \tilde{P}}{1 - 2\tilde{m}/r}\right)_{,0} - \frac{\partial \tilde{P}}{\partial r} - \frac{\tilde{\rho} + \tilde{P}}{1 - 2\tilde{m}/r}\left(4\pi r\tilde{P} + \frac{\tilde{m}}{r^2}\right) = \frac{-2}{r}(P - \tilde{P}) , \tag{61}$$

which is the generalization of Tolman-Oppenheimer-Volkov (TOV) equation for a radiative situation. Now, evaluating equation (61) at $r = a + 0$ the third *surface equation* takes the form of

$$(\Omega - 1)\left(4\pi A\tilde{\rho}_a \frac{3\Omega - 1}{\Omega} - \frac{3 + F}{2A} + \tilde{\rho}_{1a}\frac{\Omega F}{\rho_a}\right) + \frac{\dot{F}}{F} + \frac{\dot{\Omega}}{\Omega} - \frac{\dot{\tilde{\rho}}_a}{\tilde{\rho}_a} + \frac{\Omega^2 F\tilde{R}}{\tilde{\rho}_a} = 0 , \tag{62}$$

with

$$\tilde{R} = \left[\frac{\partial \tilde{P}}{\partial r} + \frac{\tilde{\rho} + \tilde{P}}{1 - 2\tilde{m}/r}\left(4\pi r\tilde{P} + \frac{\tilde{m}}{r^2}\right)\right]_a . \tag{63}$$

Finally, equations (49) and (50) can be written as a boundary condition on the *effective density* and the evolution of the boundary:

$$[\tilde{\rho}_a a^2(1 + \dot{a})]\alpha = 0 . \tag{64}$$

This equation implies a severe restriction on the value of $\tilde{\rho}$ at the surface, i.e., it must be of order α. This is to be considered a restriction imposed on the equation of state by the junction conditions.

Equations (57), (60), and (62) conform the SSE. This system may be integrated (numerically) for any given radial dependence of the *effective variables*, provided that equation (64) is satisfied.

These equations alone suffice to completely determine metric functions β and $\tilde{m}$ and are the same equations found in the spherically symmetric case, with the radial coordinate having a different meaning. This is due to the fact that the metric functions have been found to be independent of the angular variable θ to this order in α.

The remaining two equations ([27] and [26]), give the θ-dependent variables $\mathscr{D}$ and ω_z as functions of β and $\tilde{m}$. It can be seen from these equations how simple is their dependence on θ, i.e.:

$$\omega_z = \tilde{\alpha} \sin\theta\, \mathscr{F}[\beta(u, r), \tilde{m}(u, r), \text{and derivatives with respect } u \text{ and } r\, ; r] \tag{65}$$

and

$$\mathscr{D} = \tilde{\alpha} \sin\theta\, \mathscr{G}[\beta(u, r), \tilde{m}(u, r), \text{and derivatives with respect } u \text{ and } r\, ; r] . \tag{66}$$

Thus, we are lead to the following scheme, that allows one to generate a *radiating, slowly rotating model* from a known *static* solution to the Einstein equations, and may be considered a generalization of the HJR method to the axially symmetric case;

1. Take a static interior solution of the Einstein equations for a fluid with spherical symmetry, $\rho_{\text{static}} = \rho(r)$ and $P_{\text{static}} = P(r)$.
2. Assume that the r dependence of $\tilde{P}$ and $\tilde{\rho}$ are the same as that of P_{static} and ρ_{static}, respectively. Be aware of the boundary conditions (see § 7 below):

$$\tilde{P}_a = -\omega_{xa}\tilde{\rho}_a , \tag{67}$$

and

$$[\tilde{\rho}_a A^2(1 + \dot{A})] \qquad \text{of order } \alpha . \tag{68}$$

3. With the r dependence of $\tilde{P}$ and $\tilde{\rho}$ and using equations (53) and (54), we have the metric elements $\tilde{m}$ and β up to some functions of u.
4. In order to obtain these unknown functions of u, we integrate SSE: equations (57), (60), and (62). The first two, equations (57) and (60), are model independent. The third one, equation (62), and the condition (64) depend of the particular choice of the equation of state.
5. One has four unknown functions of u for the SSE. These functions are boundary radius A, the velocity of the boundary surface (related to Ω), the total mass M (related to F), and the "total luminosity" L. Providing one of these functions, the SSE can be integrated for any particular set of initial data.
6. By substituting the result of the integration in the expressions for $\tilde{m}$ and β, these metric functions become completely determined.
7. The complete set of matter variables can be algebraically found for any part of the spheroid by using the field equations (22)–(26).

6. AN EXAMPLE MODEL

We shall work out, as an example, a model that has been studied in the nonrotating case, the homogeneous *Schwarzschild-like* solution (Herrera et al. 1980). In the static limit, it represents an incompressible fluid of constant density. Despite the simplicity of this model, which provides the clearest illustration of the effects of slow rotation in a radiating general relativistic sphere, this model has some physical interest. The homogeneity of the mass energy density in this model enables us to study the effects of rotation on radiating spheres in general relativity under conditions that are "more extreme" than any that one encounters with "more normal" equations of state. The model obtained represents a source of the exterior vacuum Kerr metric, as was imposed by the junction conditions in § 4.

The *effective density* and the *effective pressure* are assumed to be

$$\tilde{\rho} = f(u) = \frac{3}{8\pi}\frac{1-F}{A^2}, \qquad (69)$$

and

$$\tilde{P} = \tilde{\rho}\left\{\frac{3g(u)[1-(8\pi/3)f(u)r^2]^{1/2}-1}{3-3g(u)[1-(8\pi/3)f(u)r^2]^{1/2}}\right\} \qquad (70)$$

Following equations (67), the functions $f(u)$ and $g(u)$ are related through boundary conditions

$$g(u) = \frac{3-2\Omega}{[1-(8\pi/3)f(u)a^2]^{1/2}}. \qquad (71)$$

The third SE can be written as

$$\dot{\Omega} = \frac{-\Omega}{1-F}\left[\frac{3(1-F)^2(2\Omega-1)(\Omega-1)}{2A\Omega} + \frac{\dot{F}}{F}\right]. \qquad (72)$$

Therefore, the SSE is formed by equations (57), (60), and (72) above. The boundary condition (64) now reads

$$(1-F)(1+\dot{A})\alpha = 0\,. \qquad (73)$$

Given the "total luminosity" L the SSE can be integrated. The function L has been assumed to be a Gaussian such that the total radiated energy is a fraction of the initial total mass.

Our *Schwarzschild-like* solution has been integrated using either

$$A(0) = 2000.0 \quad \text{or} \quad A(0) = 3000.0\,,$$

as an initial radius of the configuration and

$$\Omega(0) = 0.999 \quad \text{and} \quad F(0) = 0.8\,,$$

as a set of initial conditions, with the following parameters:

$$\dot{M} = 0.001 \quad \text{and} \quad \alpha = 0.001\,.$$

The assumption of slow rotation implies that the physical variables ρ, P, ω_z, and ϵ do not depend on the angular coordinate θ (Hartle 1967). The dependence on θ of the quantities ω_z (orbital velocity) and $\epsilon\mathscr{D}$ (dragged energy flux) has been found algebraically from the field equations. Figure 1 shows the evolution of this θ-dependent variables at the equator for two different initial radii $A(0)$. Because the evolution of the θ-independent variables are qualitatively the same as in the spherical case given by Herrera et al. (1980) they are not displayed in the present work. The evolution of the surface variables: luminosity, radius, eccentricity, and mass are presented in Figure 2.

7. DISCUSSION OF THE RESULTS AND CONCLUSIONS

We have developed a seminumeric method to integrate the Einstein field equations for a radiating body in the slow-rotation approximation. It allows one to construct a dynamic, slowly rotating solution from a static "seed" solution to the Einstein equations that satisfies the appropriate boundary conditions with the exterior spacetime.

Three main features can be noticed in this method:

1. To construct the energy-momentum tensor in terms of physical variables *as measured by a local Minkowskian comoving observer.*
2. To employ Bondi radiation coordinates which lead to metric functions having the same radial dependence as in the static case whereas physical variables contains corrections of order ω.
3. To assume that the r dependence of $\tilde{P}$ and $\tilde{\rho}$, equations (51) and (52), is the same as that of P_{static} and ρ_{static}, respectively. This last assumption leads to the *system of surface equations*, one of the crucial points of the present scheme.

The comoving Minkowskian observer coincides with the *Lagrangian frame* (the *proper frame*) which is the frame where the interaction between radiation and matter are most easily handled (Mihalas, Kunasz, & Hummer 1976; Mihalas & Mihalas 1984). Thus, the physical variables are obtained as measured by this observer. The effects of gravitation and those corresponding to the dragging of inertial frames are clearly obtained through the appropriate transformation of coordinates.

It is worth noticing the fact that in the nonrotating case, Bondi coordinates lead to components T_{ur} and T_{rr} of the energy momentum tensor, which enter in the field equations for $\tilde{m}_1$ and β_1 (eqs. [23] and [24]), containing terms of the order of ω_x. This is in contrast with the situation in Schwarzschild coordinates where the lowest terms are of the order ω_x^2. Therefore, for the slow

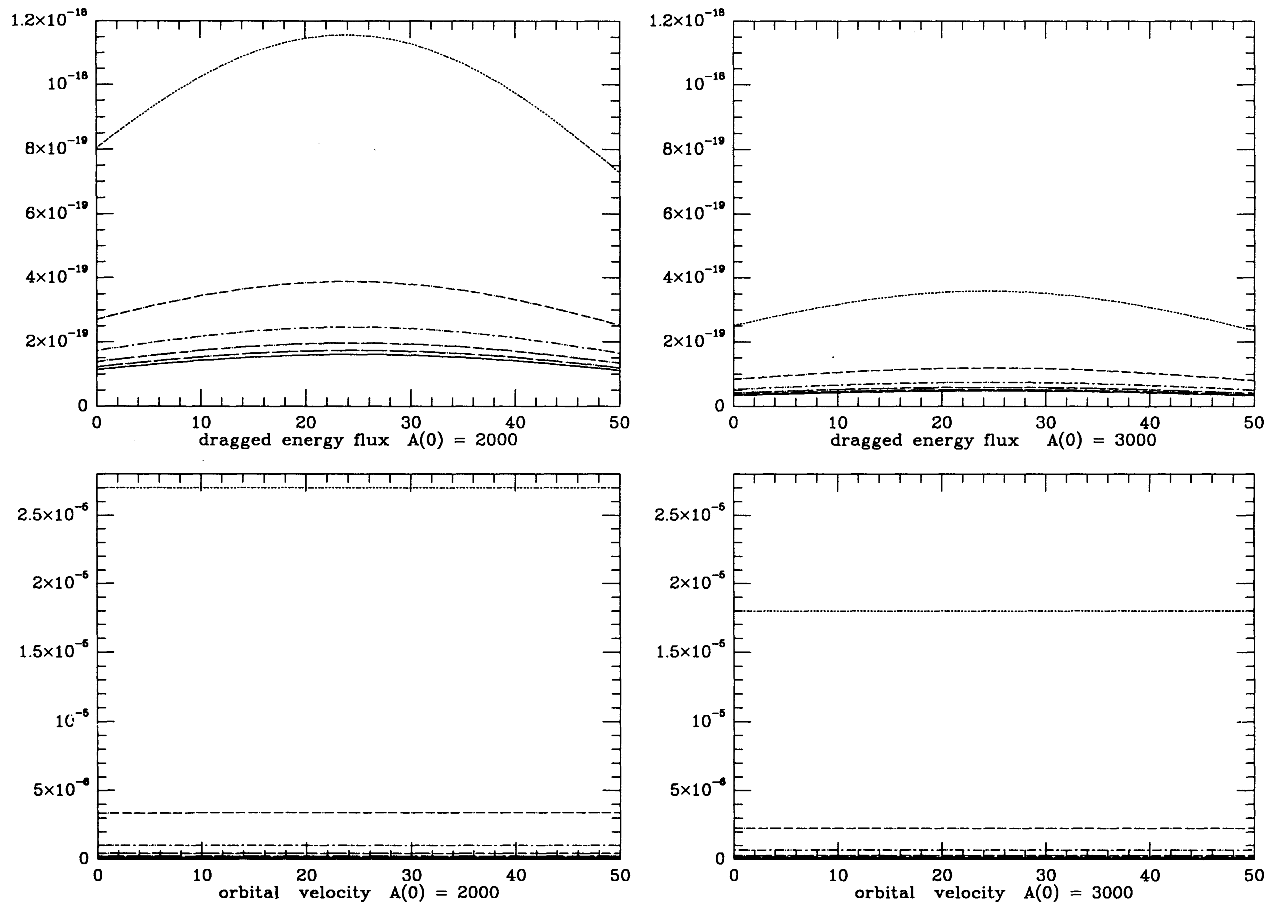

FIG. 1.—These figures show the evolution of this θ-dependent variables at the equator for two different initial radii $A(0)$. All these variables are dimensionless. The horizontal axes represent the timelike coordinate u while vertical ones denote the value of the variable labeled under the corresponding horizontal axis. Curves sketching the evolution of the respective variables, from inner to outermost shell (surface), are represented by dotted, dotted-dashed, dashed and continuous curves, respectively. Notice that inner shells rotate faster and drag more energy flux than the outer ones. It is also clear that the more compact the sources the greater is the orbital velocity and the dragged energy distribution.

contraction approximation in Schwarzschild type coordinates both the metric functions and the physical variables share the same r-dependence as in the corresponding static variables. However, in Bondi radiation coordinates only metric functions have the same radial functionality as in the static case, whereas physical variables contain corrections of order ω_x. In addition to this, Bondi radiation coordinates are adapted to radiation problems allowing the physical variables to be solved, algebraically in tems of the metric elements and their derivatives.

The rationale behind the assumptions on the r dependence of the *effective variables* $\tilde{P}$ and $\tilde{\rho}$ can be grasped in terms of the characteristic times for different processes involved in a collapse scenario. If the hydrostatic timescale $\mathscr{T}_{\rm HYDR}$, which is of the order $\sim 1/(G\rho)^{1/2}$ (where G is the gravitational constant and ρ denotes the mean density) is much smaller than the Kelvin-Helmholtz timescale ($\mathscr{T}_{\rm KH}$), then in a first approximation the inertial terms in the equation of motion (61) can be ignored (Kippenhahn & Weigert 1990). Therefore in this first approximation the r dependence of P and ρ are the same as in the static solution. Then the assumption that the *effective variables* (51) and (52) have the same r dependence as the physical variables of the static situation represents a correction to that approximation, and is expected to yield good results whenever $\mathscr{T}_{\rm KH} \gg \mathscr{T}_{\rm HYDR}$. Fortunately enough, $\mathscr{T}_{\rm KH} \gg \mathscr{T}_{\rm HYDR}$, for almost all kind of stellar objects. Thus for example for the Sun we get $\mathscr{T}_{\rm KH} \sim 107$ yr, whereas $\mathscr{T}_{\rm HYDR} \sim 27$ minutes. Also, the Kelvin-Helmholtz phase of the birth of a neutron star last for about tens of seconds (Burrows & Lattimer 1986), whereas for a neutron star of one solar mass and a 10 km radius, we obtain $\mathscr{T}_{\rm HYDR} \sim 8.61 \times 10^{-11}$ s.

Other important point which deserves some comments is that our solutions have differential fluid rotation, since we demand only the angular momentum per unit mass to be a constant inside the distribution. The orbital velocity is strongly differential and a decreasing function of the radial coordinate (Fig. 1). That is to say, inner shells rotate faster than the outer ones. This effect can be also appreciated from the dragged energy flux evolution. This behavior of the dragging was already reported in rigid-rotation models (Hartle 1967; Hartle and Thorne 1968). It can be also noticed from this figure that more compact configurations (smaller initial radius) have greater orbital velocity and dragged energy flux distribution. The assumption of rigid rotation is common in the studies of rotating self-gravitating objects, whether in Newtonian or Einstein theory of gravity. In stellar interiors rigid rotation is usually justified through the argument that differential rotation with depth in the radiative interiors of stars would amplify even a small internal magnetic field over a short timescale, generating thereby toroidal magnetic fields strong enough to remove the angular velocity gradients (Spruit 1987; Mestel, Moss, & Taylor 1988). This hypothesis however has been questioned by others

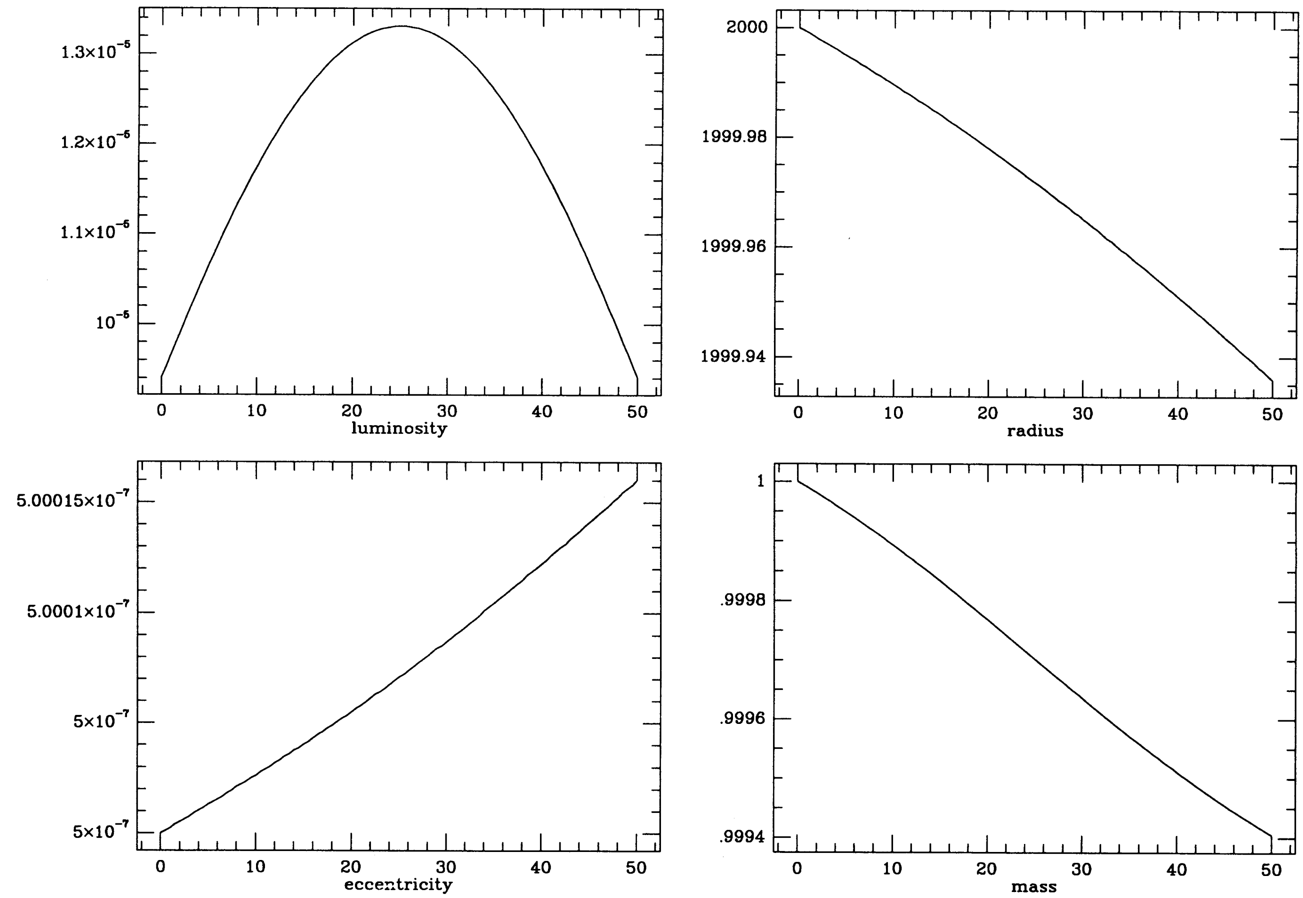

FIG. 2.—In these figures the evolution of the dimensionless variables: luminosity profile (*upper left*), exterior radius (*upper right*), surface eccentricity (*lower left*), and the total mass (*lower right*) are displayed. Horizontal axes represent the timelike coordinate u while vertical ones denote the value of the variable labeled under the corresponding horizontal axis. As it can be appreciated from evolution of the surface eccentricity (*lower left*), it increases with the timelike coordinate.

researchers (Fox & Bernstein 1987; Tassoul & Tassoul 1982). On the other hand, in the case of isotropic viscosity, a self-gravitating sphere which initially starts out with differential rotation approaches rigid rotation after viscous timescale. However, nonisotropic turbulent viscosity (which is the case in convective regions) causes differential rotation (Bierman 1951). Also, from the requirement that the fluid energy momentum-tensor of the source be that of a perfect fluid it has been argued (Tauber & Weinberg 1961) that the source should rotate rigidly. However, as it has been recently pointed out (Chinea & González-Romero 1992), the necessary and sufficient condition for the energy momentum tensor to reduce to the perfect fluid type is less stringent than what is required by Tauber & Weinberg. Also it is worth mentioning that rigid rotation on the giant branch has been found to be ruled out by the observations (Pinsonneault et al. 1991), once the effects of mass loss on the giant branch are taken into account. In the context of general relativity attempts have been made to exclude rigidly rotating perfecting fluids as sources of the Kerr metric (see Krasiński 1979 and references therein; and Herrera & Manko 1993).

Notice also that, because of the coordinate system used, the surfaces ρ = const. are spheroids of increasing eccentricity as $r \to 0$. This is opposite to the behavior found by Hartle (1967) for the adiabatic case. As it was stressed by this author, the eccentricity in his models varies from 1 (spherical) at the center to a value which describes the shape of the model at the surface. It can be appreciated from Figure 2 that our models radiate 0.06% of their initial mass. It also emerges from this figure that the eccentricity of the surface increases with the timelike coordinate.

As a further consequence of the boundary conditions, a restriction on the equations of state allowed has been found, namely, that the *effective density* at the surface must be of order α. This condition must be taken into account on choosing the "seed" static solution. Equation (64) restrains the value of $\tilde{\rho}$ at the surface, to be of order α. In a homogeneous model, this condition obviously implies a low global density. It is straightforward to find from equations (73) and (56), that for $M = 1$ models, we must have A of order α^{-1}. Because of the above mentioned restriction on the surface density, such model is not highly relativistic, having radius of the order 10^3 km and one solar mass. The θ-independent variables show the usual behavior reported for spherical configurations, but now the density at each shell *decreases*, because the mass loss is not compensated by an appropriated decrease in radius. The static limit is not reached, although it was found that the radial velocity tends to a constant value at each shell when the configuration ceases to radiate.

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APPENDIX

SPIN COEFFICIENTS

We give here the spin coefficients for the interior and exterior metrics as given by the null tetrad components (31)–(36). For details of the calculations, see, for example, Carmeli (1977).

A1. EXTERIOR METRIC

$$\lambda = \sigma = \kappa = \epsilon = \tau = 0 , \qquad \rho = -\frac{1}{(r - i\alpha\cos\theta)} , \qquad \pi = -\bar{\tau} , \qquad \alpha = \frac{\bar{\tau}}{2} - \frac{\cot\theta}{4\sqrt{2}(r + i\alpha\cos\theta)} ,$$

$$\beta = \bar{\alpha} - \frac{\cos\theta}{\sin\theta\sqrt{2}(r + i\alpha\cos\theta)} , \qquad \mu = \frac{1}{(r^2 + \alpha^2\cos^2\theta)}\left[-(r + i\alpha\cos\theta)\frac{\mathscr{Y}^2}{2} + i\alpha\cos\theta\right] ,$$

$$\gamma = \frac{\mathscr{Y}\mathscr{Y}_1}{4}\frac{i}{(r^2 + \alpha^2\cos^2\theta)}\left[\frac{\mathscr{Y}^2}{2}\alpha + i\alpha\cos\theta\left(\frac{\mathscr{Y}}{X} - \frac{\mathscr{Y}^2}{2}\right)\right] , \qquad \nu = \frac{\mathscr{Y}}{\sqrt{2}(r - i\alpha\cos\theta)}[i\alpha\sin\theta(\mathscr{Y}_1/\mathscr{Y}_0)] ,$$

where

$$\mathscr{Y} = \sqrt{1 - \frac{2mr}{r^2 + \alpha^2\cos^2\theta}} . \tag{74}$$

A2. INTERIOR METRIC

$$\lambda = \sigma = \kappa = 0 , \qquad \rho = -\frac{X/Y}{(r^2 + \tilde{\alpha}^2\cos^2\theta)}\left[r + i\tilde{\alpha}\cos\theta\frac{Y}{X}\right] ,$$

$$\tau = -\frac{Y/X}{2\sqrt{2}(r + i\tilde{\alpha}\cos\theta)}Y^{-2}i\tilde{\alpha}\sin\theta[XY_1 - X_1Y + YX_0 - X_0Y + YX_2 - XY_2] , \qquad \pi = -\bar{\tau} ,$$

$$\alpha = \frac{\bar{\tau}}{2} - \frac{\cot\theta}{4\sqrt{2}(r + i\tilde{\alpha}\cos\theta)} , \qquad \beta = \bar{\alpha} - \frac{\cos\theta}{\sin\theta\sqrt{2}(r + i\tilde{\alpha}\cos\theta)} , \qquad \epsilon = \frac{i\cos\theta}{2(r^2 + \tilde{\alpha}^2\cos^2\theta)}\tilde{\alpha}\left(\frac{X}{Y} - 1\right) ,$$

$$\mu = \frac{1}{(r^2 + \tilde{\alpha}^2\cos^2\theta)}\left[i\tilde{\alpha}\cos\theta\left(\frac{Y}{X} - \frac{Y^2}{2}\right) - \frac{XY}{2}r\right] ,$$

$$\gamma = \frac{XY_1}{2} - \frac{X}{2Y}\left(\frac{Y}{X}\right)_0 + \frac{i}{2(r^2 + \mathscr{A}^2\cos^2\theta)}\left[\frac{\tilde{\alpha}}{2}XY + i\mathscr{A}\cos\theta\left(\frac{Y}{X} - \frac{Y^2}{2}\right) - i\tilde{\alpha}\sin\theta\frac{Y^2}{2}\left(\frac{X_2}{Y} - \frac{Y_2}{X}\right)\right] ,$$

$$\nu = \frac{Y/X}{2\sqrt{2}(r - i\mathscr{A}\cos\theta)}[i\tilde{\alpha}\sin\theta(XY_1 + YX_1 - YX_0 - XY_0) + YX_2 + XY_2] ,$$

where

$$Y = e^{2\beta}\sqrt{1 - \frac{2\tilde{m}r}{r^2 + \tilde{\alpha}^2\cos^2\theta}} , \tag{75}$$

and

$$X = e^{-2\beta}Y . \tag{76}$$

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