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CENTER MANIFOLD AND EXPONENTIALLY BOUNDED SOLUTIONS OF A FORCED NEWTONIAN SYSTEM WITH DISSIPATION

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Abstract

In this note, we study the following second order system of ordinary differential equations with dissipation

$$u'' + cu' + dAu + kH(u) = P(t), \quad u \in I\!R^n, \quad t \in I\!R,$$

where c, d and k are positive constants, $H : I\!R^n \to I\!R^n$ is a globally Lipschitz function and $P : R \to I\!R^n$ is a continuous and bounded function. A is a $n \times n$ matrix whose first eigenvalue λ_1 is equal to zero and the others are positive($0 = \lambda_1 < \lambda_2 < \cdots \lambda_l$). Under these conditions, we prove that for some values of c and k there exist a positive number η depending on c and a continuous manifold $\mathcal{M} = \mathcal{M}(c, k, P(\cdot))$ such that any solution of this system starting in $\mathcal{M}$ is exponentially bounded. i.e., $\sup_{t \in I\!R} e^{-\eta|t|} \left\{ \|u'(t)\|^2 + \|u(t)\|^2 \right\}^{1/2} < \infty$. These results are applied to the spatial discretization of very well known second order partial differential equations with Neumann boundary conditions.

Key words. differential equation, center manifold, exponentially bounded solutions.

AMS(MOS) subject classifications. primary: 34; secondary: 45.

1 Introduction

In this note, we study the existence of exponentially bounded solutions of the following second order system of ordinary differential equations with a dampping force and dissipation in $I\!R^n$

$$u'' + cu' + dAu + kH(u) = P(t), \quad u \in I\!R^n, \quad t \in I\!R, \tag{1.1}$$

where c, d and k are positive constants, $H : I\!R^n \to I\!R^n$ is a globally Lipschitz function and $P \in C_b(I\!R; I\!R^n)$, the space of continuous and bounded functions. A is a $n \times n$ matrix whose first eigenvalue λ_1 is equal to zero and the others are positive

$$0 = \lambda_1 < \lambda_2 < \cdots \lambda_l$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

The equation (1.1) has been studied in [6] for the case that the first eigenvalue λ_1 of the matrix A is positive ($\lambda_1 > 0$); under these conditions they prove that for some values of c and k the

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equation (1.1) has a bounded solution which is exponentially stable and, if $P(t)$ is almost periodic, this bounded solution is also almost periodic.

The following second order system of differential equations in $I\!R^n$ has been studied by Alonso and Ortega in [2]

$$u'' + cu' + Au + \nabla G(u) = P(t), \quad u \in I\!R^n, \quad t \in I\!R, \tag{1.2}$$

where $c > 0$ is a constant, A is a $n \times n$ symetric semidefinite positive constant mätrix, P is a continuous function and bounded, and $G \in C^2(I\!R^n)$. They were interested in the existence of a bounded solution of (1.2) which is exponentially asymptotically stable. In fact, they prove that: If $\lambda_1(A) \geq 0$ and there exist non-negative constants a and b such that

$$aI_n \leq D^2G(\xi) \leq bI_n, \quad \forall \xi \in I\!R^n, \tag{1.3}$$

with $a + \lambda_1(A) > 0$ and

$$b < a + c^2 + 2c\sqrt{a + \lambda_1(A)}.$$

Then (1.2) has a unique bounded solution which is exponential asymptotically stable.

Moreover; if $P(t)$ is τ-periodic, then such a solution is also τ-periodic.

The fact that, the first eigenvalue λ_1 of the matrix A is equal to zero in the equation (1.1), does not allow us to prove the existence of bounded solutions of (1.1) in general.

However; we prove that, for some values of c and k there exist a positive number η depending on c and a continuous manifold $\mathcal{M} = \mathcal{M}(c, k, P(\cdot))$ such that any solution of the system (1.1) starting in $\mathcal{M}$ is exponentially bounded. i.e.,

$$\sup_{t \in I\!R} e^{-\eta|t|} \left\{ \|u'(t)\|^2 + \|u(t)\|^2 \right\}^{1/2} < \infty.$$

Our method is similar to the one used in [6], we just rewrite the equation (1.1) as a first orden system of ordinary differential equations and prove that the linear part of this system has an exponential trichotomy with trivial unstable space. Next, we use the variation constant formula and some ideas from [8] [9] to find a formula for the exponentially bounded solutions of (1.1). From this formula we can prove the existence of such manifold $\mathcal{M} = \mathcal{M}(c, k, P(\cdot))$. These results are applied to the spatial discretization of very well known second order partial differential equations with Neumann boundary conditions:

Example 1.1 *The Sine-Gordon Equation with Neumman boundary conditions is given by:*

$$\begin{cases} U_{tt} + cU_t - U_{xx} + k \sin U = p(t,x), & 0 < x < L, \quad t \in I\!R, \\ U_x(t,0) = U_x(t,L) = 0, & t \in I\!R, \end{cases} \tag{1.4}$$

where c and k are positive constants, $p : I\!R \times [0, L] \to I\!R$ is continuous and bounded.

This equation is physically interesting, the average value of the function U is not expected to remain bounded and actually leads to notrivial dynamics. From mathematical point of view this case is also interesting.

For each $N \in I\!N$ the spatial discretization of this equation is given by

$$\begin{cases} u_i'' + cu_i' + \delta^{-2}(2u_i - u_{i+1} - u_{i-1}) + k \sin u_i = p_i(t), & 1 \leq i \leq N, \quad t \in I\!R, \\ u_0 = u_1, \quad u_N = u_{N+1} = 0. \end{cases} \tag{1.5}$$

The equation (1.5) can be written in the form of (1.1) with the matrix A been as follows

$$\delta^{-2}\begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix} \tag{1.6}$$

where $\delta = L/(N+1)$. The eigenvalues of this matrix are simple, the first of it is zero and the others are positive

Example 1.2 *A telegraph equation with Neumman boundary conditions*

$$\begin{cases} U_{tt} + cU_t - U_{xx} + arctan U = & p(t,x), \quad 0 < x < L, \quad t \in I\!R, \\ U_x(t,0) = U_x(t,L) = 0, & t \in I\!R, \end{cases} \tag{1.7}$$

where c is a positive constant, $p : I\!R \times [0,L] \to I\!R$ is continuous and bounded.
For each $N \in I\!N$ the spatial discretization of this equation is given by

$$\begin{cases} u_i'' + cu_i' + \delta^{-2}(2u_i - u_{i+1} - u_{i-1}) + arctan u_i = & p_i(t), \quad 1 \le i \le N, \quad t \in I\!R, \\ u_0 = u_1, \quad u_N = u_{N+1} = 0. \end{cases} \tag{1.8}$$

The equation (1.8) can be written in the form of (1.1) with the the same matrix A given by (1.6).

2 Preliminaries

Most of the ideas present in this section can be found in [6]. So, we shall prove only the new results. The equation (1.1) can be written as a first order system of ordinary differential equations in the space $W = I\!R^n \times I\!R^n$ as follow:

$$w' + \mathcal{A}w + k\mathcal{H}(w) = \mathcal{P}(t), \quad w \in W, \quad t \in I\!R, \tag{2.1}$$

where $v = u'$ and

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} 0 \\ H(u) \end{pmatrix}, \quad \mathcal{P} = \begin{pmatrix} 0 \\ P(t) \end{pmatrix} \quad \text{and} \quad \mathcal{A} = \begin{pmatrix} 0 & -I \\ A & cI \end{pmatrix}. \tag{2.2}$$

Now, we are ready to study the linear part of the equation (2.1):

$$w' + \mathcal{A}w = 0, \quad w \in W, \quad t \in I\!R. \tag{2.3}$$

From now on, we shall suppose that each eigenvalue of the matrix A has multiplicity γ_j equal to the dimension of the corresponding eigenspace and the first one is equal to zero and the others are positive. Therefore, if $0 = \lambda_1 < \lambda_2 < \cdots \lambda_l$ are the eigenvalues of A, we have the following:
a) there exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvector of A in $I\!R^n$.

b) for all $x \in I\!R^n$ we have

$$Ax = \sum_{j=1}^{l} \lambda_j \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{l} \lambda_j E_j x, \tag{2.4}$$

where $< \cdot, \cdot >$ is the inner product in $I\!R^n$ and

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}. \tag{2.5}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in R^n and $x = \sum_{j=1}^{l} E_j x, \quad x \in I\!R^n$.

c) the exponential matrix e^{-At} is given by

$$e^{-At} = \sum_{j=1}^{l} e^{-\lambda_j t} E_j. \tag{2.6}$$

Theorem 2.1 *Suppose that* $c \neq 2\sqrt{\lambda_j}, \quad j = 1, 2, \ldots, l$. *Then the exponential matrix* $e^{-\mathcal{A}t}$ *of the matrix* $-\mathcal{A}$ *given by (2.2) can be written as follow*

$$e^{-\mathcal{A}t} w = \sum_{j=1}^{l} \left\{ e^{\rho_1(j)t} Q_1(j) w + e^{\rho_2(j)t} Q_2(j) w \right\}, \quad w \in W, \quad t \in I\!R, \tag{2.7}$$

where

$$\rho(j) = \frac{-c \pm \sqrt{c^2 - 4\lambda_j}}{2}, \quad j = 1, 2, \ldots, l \tag{2.8}$$

and $\{Q_i(j) : i = 1, 2\}_{j=1}^{l}$ *is a complete orthogonal system of projections in* W.

Corollary 2.1 *The spectrum* $\sigma(-\mathcal{A})$ *of the matrix* $-\mathcal{A}$ *is given by*

$$\sigma(-\mathcal{A}) = \{ \frac{-c \pm \sqrt{c^2 - 4\lambda_j}}{2}, \quad j = 1, 2, \ldots, l \}.$$

Corollary 2.2 *Under the hypothesis of Theorem 2.1, there exist two orthogonal projectors* $\pi_0, \pi_s : W \to W$ *and a constant* $M > 0$ *such that*

$$\begin{aligned} \|e^{-\mathcal{A}t} \pi_0\| &\leq M, \quad t \in I\!R, \\ \|e^{-\mathcal{A}t} \pi_s\| &\leq e^{-\beta t}, \quad t \geq 0, \\ I_W = \pi_0 + \pi_s, \quad & W = W_0 \oplus W_s \end{aligned}$$

where $W_0 = Ran(\pi_0), \quad W_s = Ran(\pi_0)$ *and* $\beta = \beta(c)$ *is given by*

$$0 > -\beta = max \left\{ -c, \quad Re(\rho_j) = Re \left(\frac{-c \pm \sqrt{c^2 - 4\lambda_j}}{2} \right) \quad : j = 2, \ldots, l. \;\; i = 1, 2. \right\}.$$

Proof The hypothesis $c \neq 2\sqrt{\lambda_j}, \ \ j = 1, 2, \ldots, l$ implies that $\mathrm{Re}(\rho_i(j) < 0, \ \ j = 2, \ldots, l$. Therefore $\beta > 0$. Since $\lambda_1 = 0$, the formula (2.7) can be written as follows

$$e^{-\mathcal{A}t}w = Q_1(1)w + e^{-ct}Q_2(1) + \sum_{j=2}^{l}\left\{e^{\rho_1(j)t}Q_1(j)w + e^{\rho_2(j)t}Q_2(j)w\right\}, \quad w \in W, \quad t \in I\!R.$$

Hence, if we define $\pi_0 = Q_1(1)$ and

$$\pi_s = I - \pi_0 = Q_2(1) + \sum_{j=2}^{l}\{Q_1(j) + Q_2(j)\},$$

we obain the require projections and

$$\|e^{-\mathcal{A}t}\pi_0 w\| = \|\pi_0 e^{-\mathcal{A}t}w\| = \|Q_1(1)w\| \leq \|Q_1(1)\|\|w\|, \quad t \in I\!R.$$

Therefore, $\|e^{-\mathcal{A}t}\pi_0\| \leq \|Q_1(1)\| = M$. In the same way we get

$$\begin{aligned}
\|e^{-\mathcal{A}t}\pi_s w\|^2 &= \|\pi_s e^{-\mathcal{A}t}w\|^2 = \|e^{-ct}Q_2(1)w + \sum_{j=2}^{l}\left\{e^{\rho_1(j)t}Q_1(j)w + e^{\rho_2(j)t}Q_2(j)w\right\}\|^2 \\
&= e^{-ct}\|Q_2(1)w\|^2 + \sum_{j=2}^{l}\left\{e^{2\mathrm{Re}\rho_1(j)t}\|Q_1(j)w\|^2 + e^{2\mathrm{Re}\rho_2(j)t}\|Q_2(j)w\|^2\right\} \\
&\leq e^{-2\beta t}\left\{\|Q_2(1)w\|^2 + \sum_{j=2}^{l}\|Q_1(j)w\|^2 + \|Q_2(j)w\|^2\right\} \\
&\leq e^{-2\beta t}\|w\|^2, \quad t \geq 0.
\end{aligned}$$

So, $\qquad \|e^{-\mathcal{A}t}\pi_s\| \leq e^{-\beta t}, \quad t \geq 0.$ □

Corollary 2.3 *For each $\epsilon \in [0, \beta)$ there exists some $M(\epsilon) > 0$ such that*

$$\begin{aligned}
\|e^{-\mathcal{A}t}\pi_0\| &\leq M(\epsilon)e^{\epsilon|t|}, \quad t \in I\!R, \\
\|e^{-\mathcal{A}t}\pi_s\| &\leq M(\epsilon)e^{-(\beta-\epsilon)t}, \quad t \geq 0.
\end{aligned}$$

3 Main Result

In this section we shall prove the main Theorem of this paper, under the hypothesis of Theorem 2.1($c \neq 2\sqrt{\lambda_j}, \ \ j = 1, 2, \ldots, l$).

The solution of (2.1) passing through the point w_0 at time $t = t_0$ is given by the variation constant formula

$$w(t) = e^{-\mathcal{A}(t-t_0)}w_0 + \int_{t_0}^{t} e^{-\mathcal{A}(t-s)}\left\{-k\mathcal{H}(w(s)) + \mathcal{P}(s)\right\}ds, \quad t \in I\!R. \tag{3.1}$$

We shall use the following notation: For each $\eta \geq 0$ we denote by Z_η the Banach space

$$Z_\eta = \left\{ z \in C(I\!R; W) : \quad \|z\|_\eta = \sup_{t \in I\!R} e^{-\eta|t|} \|z(t)\| < \infty \right\}. \tag{3.2}$$

In particular, $Z_0 = C_b(I\!R, W)$ the space of bounded and continuous functions defined in $I\!R$ taking values in $W = I\!R^n \times I\!R^n$.

Theorem 3.1 (**Main Theorem**)*Suppose that H is a bounded function or $H(0) = 0$. Then for some c and k positive there exist $\eta = \eta(c) \in (0, \beta)$ and a continuous manifold $\mathcal{M} = \mathcal{M}(c, k, P)$ such that any solution $u(t)$ of (1.1) with $(u(0), u'(0)) \in \mathcal{M}$ satisfies:*

$$\sup_{t \in I\!R} e^{-\eta|t|} \left\{ \|u(t)\|^2 + \|u'(t)\|^2 \right\}^{1/2} < \infty. \tag{3.3}$$

Moreover,
(a) there exist a globally Lipshitz function $\psi : W_0 \to W_s$ such that

$$\mathcal{M} = \{ w_0 + \psi(w_0) \quad : w_0 \in W_0 \}. \tag{3.4}$$

Moreover, there exist $M \geq 1$ and $0 < \Gamma < 1$ such that

$$\|\psi(w_1, P_1) - \psi(w_2, P_2)\| \leq \frac{kLM(1-\Gamma)^{-1}}{\beta - \eta} \|w_1 - w_2\| + \frac{1}{\beta} \|P_1 - P_2\|, \tag{3.5}$$

for $w_1, w_2 \in W$, $\quad P_1, P_2 \in C_b(I\!R, I\!R^n)$.
(b) if H is bounded, then ψ is also bounded.
(c) if $P = 0$ and $H(0) = 0$, then $\mathcal{M}$ is unique and invariant under the equation $w' + \mathcal{A}w + k\mathcal{H}(w) = 0$. In this case $\mathcal{M}$ is called center manifold and it is tangent to the space W_0 at $w_0 = 0$.

Before we prove the main theorem, we shall need some previous results.

Lemma 3.1 *Let $z \in Z_0 = C_b(I\!R, W)$. Then, z is a solution of (2.1) if and only if there exists some $w_0 \in W_0$ such that*

$$\begin{aligned} z(t) &= e^{-\mathcal{A}t} w_0 + \int_0^t e^{-\mathcal{A}(t-\tau)} \pi_0 \{ -k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau) \} \, d\tau \\ &+ \int_{-\infty}^t e^{-\mathcal{A}(t-\tau)} \pi_s \{ -k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau) \} \, d\tau, \quad t \in I\!R. \end{aligned} \tag{3.6}$$

Proof Suppose that z is a solution of (2.1). Then, from corollary 2.2 we get $z(t) = \pi_0 z(t) + \pi_s z(t)$ and from the variation constant formula (3.1) we obtain

$$\pi_0 z(t) = e^{-\mathcal{A}t} \pi_0 z(0) + \int_0^t e^{-\mathcal{A}(t-\tau)} \pi_0 \{ -k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau) \} \, d\tau, \quad t \in I\!R. \tag{3.7}$$

From the uniqueness of the solutions of (2.1) we get that

$$\pi_s z(t) = e^{-\mathcal{A}(t-t_0)} \pi_s z(t_0) + \int_{t_0}^t e^{-\mathcal{A}(t-\tau)} \pi_s \{ -k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau) \} \, d\tau, \quad t \in I\!R. \tag{3.8}$$

Since $z(t)$ is bounded, there exists $R > 0$ such that $\|z(t)\| \leq R$, for all $t \in \mathbb{R}$. then , from corollary 2.2 we obtain that

$$\|e^{-\mathcal{A}(t-t_0)}\pi_s z(t_0)\| \leq Re^{-\beta(t-t_0)} \to 0 \quad \text{as} \quad t_0 \to -\infty.$$

Now, if we put

$$l = \sup_{\tau\in\mathbb{R}} \|\mathcal{H}(z(\tau))\| \quad \text{and} \quad L_p = \sup_{\tau\in\mathbb{R}} \|\mathcal{P}(\tau)\|,$$

then

$$\begin{aligned} \|\int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)}\pi_s \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau\| &\leq \|\int_{-\infty}^{t} e^{-\beta(t-\tau)}\{kl + L_p\}\,d\tau\| \\ &= \frac{kl + L_p}{\beta}. \end{aligned}$$

Hence, passing to the limit in (3.8) when t_0 goes to $-\infty$, we obtain

$$\pi_s z(t) = \int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)}\pi_s \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau, \quad t \in \mathbb{R}. \tag{3.9}$$

Therefore, putting $w_0 = \pi_0 z(0)$ we get (3.6).

Conversely, suppose that z is a solution of (3.6). Then

$$\begin{aligned} z(t) &= e^{-\mathcal{A}t}w_0 + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau \\ &+ \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_s\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau \\ &+ \int_{-\infty}^0 e^{-\mathcal{A}(t-\tau)}\pi_s\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau \\ &= e^{-\mathcal{A}t}\left\{w_0 + \int_{-\infty}^0 e^{\mathcal{A}\tau}\pi_s\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau\right\} \\ &+ \int_0^t e^{-\mathcal{A}(t-\tau)}\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau \\ &= e^{-\mathcal{A}t}z(0) + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau, \end{aligned}$$

where

$$z(0) = w_0 + \int_{-\infty}^0 e^{\mathcal{A}\tau}\pi_s\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau. \tag{3.10}$$

This concludes the proof of the lemma. □

Lemma 3.2 *Suppose that $H(0) = 0$ and $z \in Z_\eta$ for $\eta \in [0, \beta)$. Then, z is a solution of (2.1) if and only if there exists some $w_0 \in W_0$ such that z satisfies (3.6).*

Proof Suppose that z is a solution of (2.1). Then, in the same way as the proof of lemma 3.1, we consider:

$$\pi_0 z(t) = e^{-\mathcal{A}t}z(0) + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\}\,d\tau, \quad t \in \mathbb{R}.$$

and

$$\pi_s z(t) = e^{-\mathcal{A}(t-t_0)} z(t_0) + \int_{t_0}^{t} e^{-\mathcal{A}(t-\tau)} \pi_s \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\} d\tau, \quad t \in I\!R.$$

Since z belong to Z_η, there exists $R > 0$ such that $\|z(t)\| \le Re^{\eta|t|}$, for all $t \in I\!R$. Fix some $t \in I\!R$ and let $t_0 \le \min\{t, 0\}$; then we have:

$$\|e^{-\mathcal{A}(t-t_0)} \pi_s z(t_0)\| \le Re^{-\beta(t-t_0)} e^{-\eta t_0} = Re^{-\beta t} e^{(\beta-\eta)t_0} \to 0 \text{ as } t_0 \to -\infty.$$

On the other hand, we obtain the following estimate

$$\| \int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)} \pi_s \{-k\mathcal{H}(z(\tau)) + \mathcal{P}(\tau)\} d\tau \| \le e^{\eta|t|} \left\{ \frac{kLR}{\beta+\eta} + \frac{kLR}{\beta-\eta} + \frac{L_p}{\beta} \right\}. \tag{3.11}$$

Where L is the Lischitz contant of H. Hence, putting $w_0 = \pi_0 z(0)$ and passing to the limit when t_0 goes to $-\infty$ we get (3.6).

The converse follows in the same way as the foregoing lemma. □

Lemma 3.3 *Suppose that H is bounded and $z \in Z_\eta$ for $\eta \in [0, \beta)$. Then, z is a solution of (2.1) if and only if there exists some $w_0 \in W_0$ such that z satisfies (3.6).*

Now, from (3.6) we only have to prove that the following set

$$\mathcal{M} = \mathcal{M}(c, k, P) = \{z(0) \ : z \in Z_\eta, \ z \text{ satisfying } (3.6)\} \tag{3.12}$$

is a continuous manifold for some values of c, k and $\eta \in (0, \beta(c))$. From (3.10) we get that

$$\mathcal{M} = \{w_0 + \pi_s z(0) \ : (w_0, z) \in W_0 \times Z_\eta, \ (w_0, z) \text{ satisfying } (3.6)\} \tag{3.13}$$

We shall need the following definition and notations:

Definition 3.1 *(a) for each $w_0 \in W_0$ we define the function $Sw_0 : I\!R \to W$ by:*

$$(Sw_0)(t) = e^{-\mathcal{A}t} w_0, \quad t \in I\!R;$$

(b) for each function $z : I\!R \to W$ we define the non-autonomous Nemytski operator $G(z) : I\!R \to W$ by

$$G(z)(t) = -k\mathcal{H}(z(t)) + \mathcal{P}(t), \quad t \in I\!R;$$

(c) for those functions $z : I\!R \to W$ for which the integrals make sense we define $Kz : I\!R \to W$ by

$$Kz(t) = \int_0^t e^{-\mathcal{A}(t-\tau)} \pi_0 z(\tau) d\tau + \int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)} \pi_s z(\tau) d\tau, \quad t \in I\!R.$$

With these notations, the equation (3.6) can be written in the following equivalent form in Z_η

$$z = Sw_0 + K \circ G(z). \tag{3.14}$$

Lemma 3.4 *(A) S is a bounded operator from W_0 into Z_η for each $\eta \geq 0$.*
(B) if $\mathcal{H}(0) = 0$ or $\mathcal{H}$ is bounded, then G maps Z_η into itself for $\eta \geq 0$ and

$$\|G(z_1) - G(z_2)\|_\eta \leq kL\|z_1 - z_2\|_\eta, \quad z_1, z_2 \in Z_\eta;$$

(C) for $\eta \in (0, \beta)$ the linear operator K is bounded from Z_η into itself and

$$\|K\|_\eta \leq R(c) = M(\epsilon)\left\{\frac{1}{\eta - \epsilon} + \frac{1}{\beta(c) - \eta}\right\}, \tag{3.15}$$

where $0 < \epsilon < \eta < \beta$ and $M(\epsilon)$ is given by corollary 2.3.

Lemma 3.5 *Let $c > 0$, $k > 0$ and $\eta \in (0, \beta)$ such that*

$$\Gamma = \|K\|_\eta kL < 1. \tag{3.16}$$

Then $(I - K \circ G) : Z_\eta \to Z_\eta$ is a homeomorphism with inverse $\Psi : Z_\eta \to Z_\eta$ and the manifold $\mathcal{M} = \mathcal{M}(c, k, P)$ is given by

$$\mathcal{M} = \{w_0 + \pi_s \Psi(Sw_0)(0) \quad : w_0 \in W_0\}. \tag{3.17}$$

Proof It follows from Lemma 3.4 that $K \circ G$ maps Z_η into itself for $\eta \in (0, \beta)$ and is globally Lipschitzian with Lipschitz constant Γ. Then, under the condition (3.16) the map $(I - K \circ G)$: $Z_\eta \to Z_\eta$ is invertible, with inverse $\Psi : Z_\eta \to Z_\eta$ which is also globally Lipschitzian with Lipschitz constant $(1 - \Gamma)^{-1}$. In particular Ψ is a continuous function. Therefore, the equation (3.14) has a unique solution given by

$$z(t) = (I - K \circ G)^{-1}(Sw_0)(t) = \Psi(Sw_0)(t), \quad t \in \mathbb{R}. \tag{3.18}$$

Hence, from (3.13) we get (3.17). □

Proof of Theorem 3.1. If we take for example $\eta = \beta/2$, then $R(c)$ given by (3.15) can be written as follow

$$R(c) = M(\epsilon)\left\{\frac{2}{\beta - 2\epsilon} + \frac{2}{\beta(c)}\right\},$$

with $0 < \epsilon < \frac{\beta}{2}$. Hence, $\lim_{c \to \infty} R(c) = 0$. So, we can choose c big enough such that

$$\Gamma = \|K\|_\eta kL \leq R(c)kL < 1.$$

Then, using Lemma 3.5 we get (3.3) and define $\psi : W_0 \to W_s$ by

$$\psi(w_0) = \pi_s \Psi(Sw_0)(0), \quad w_0 \in W_0,$$

we obtain (3.3). Clearly, the function ψ is globally Lipschitzian.

On the other hand, from (3.10) we get that

$$\begin{aligned} z(0) &= w_0 + \pi_s z(0) = w_0 + \psi(w_0) \\ &= w_0 + \int_{-\infty}^{0} e^{\mathcal{A}\tau} \pi_s \{-k\mathcal{H}(\Psi(Sw_0)(\tau)) + \mathcal{P}(\tau)\} \, d\tau. \end{aligned}$$

Therefore,

$$\psi(w_0, P) = \int_{-\infty}^{0} e^{\mathcal{A}\tau}\pi_s\{-k\mathcal{H}(\Psi(Sw_0)(\tau)) + \mathcal{P}(\tau)\}\,d\tau. \tag{3.19}$$

To complete the proof of part (a), let us consider $w_1, w_2 \in W$, $P_1, P_2 \in C_b(I\!R, I\!R^n)$ and

$$\begin{aligned}\psi(w_1, P) - \psi(w_2, P) &= \int_{-\infty}^{0} -ke^{\mathcal{A}\tau}\pi_s\{\mathcal{H}(\Psi(Sw_1)(\tau)) - \mathcal{H}(\Psi(Sw_2)(\tau))\}\,d\tau \\ &+ \int_{-\infty}^{0} e^{\mathcal{A}\tau}\pi_s\{\mathcal{P}_1(\tau) - \mathcal{P}_2(\tau)\}\,d\tau.\end{aligned}$$

From Corollary 2.2 we get that

$$\begin{aligned}\|Sw_1 - Sw_2\|_\eta &= \sup_{\tau\in I\!R} e^{-\eta|\tau|}\|e^{\mathcal{A}\tau}(Sw_1 - Sw_2)\| \\ &\le M\|w_1 - w_2\|,\end{aligned}$$

and from Lemma 3.5 we get that

$$\begin{aligned}\|\Psi(Sw_1) - \Psi(Sw_2)\|_\eta &\le (1-\Gamma)^{-1}\|Sw_1 - Sw_2\|_\eta \\ &\le M(1-\Gamma)^{-1}\|w_1 - w_2\|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\psi(w_1, P) - \psi(w_2, P)\| &\le \int_{-\infty}^{0} kLM(1-\Gamma)^{-1}\|w_1 - w_2\|e^{(\beta-\eta)\tau}\,d\tau \\ &+ \int_{-\infty}^{0} \|P_1 - P_2\|e^{\beta\tau}\,d\tau.\end{aligned}$$

Hence,

$$\|\psi(w_1, P_1) - \psi(w_2, P_2)\| \le \frac{kLM(1-\Gamma)^{-1}}{\beta - \eta}\|w_1 - w_2\| + \frac{1}{\beta}\|P_1 - P_2\|.$$

To prove part (b), let us suppose that: $\|H(u)\| \le l$, $u \in I\!R^n$ and $L_p = \sup_{\tau\in I\!R}\|P(\tau)\|$. Then, from (3.19) we get that

$$\|\psi(w_0)\| \le \int_{-\infty}^{0} e^{\beta\tau}\{kl + L_P\}\,d\tau \le \frac{kl + L_P}{\beta}, \quad w_0 \in W_0.$$

Part (c) follows from Theorem 2.1 of [8]. □

Remark 3.1 *The equation (3.6) may not have bounded solutions in $I\!R$. However, if $H = 0$ and $\mathcal{P}$ satisfies the condition*

$$\sup\left\{\left|\int_0^t \|\pi_0\mathcal{P}(\tau)\|d\tau\right| : \ t \in I\!R\right\} < \infty,$$

then for each $w_0 \in W_0$ the equation (3.6) has a bounded solution which is given by

$$z(t) = e^{-\mathcal{A}t}w_0 + \int_0^t e^{-\mathcal{A}(t-\tau)}\pi_0\mathcal{P}(\tau)d\tau + \int_{-\infty}^{t} e^{-\mathcal{A}(t-\tau)}\pi_s\mathcal{P}(\tau)d\tau, \quad t \in I\!R.$$

An open question, is the following:

What conditions do we have to impose to the functions H and P to insure the existence of bounded solutions of the equation (3.6) ?.

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