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"AN INEQUALITY FOR GENERALIZED MODULI OF CONTINUITY"

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Abstract. It is shown that $\omega_X(f^*,t) \leq 7\omega_X(f,t)$, where ω_X is a Moduli of continuity associated with a rearrangement invariant space X.

1. INTRODUCTION. Let $X(0,1)$ be a rearrangement invariant space of Lebesgue measurable functions on $(0,1)$ (cf. [7]). To this space we associate a modulus of continuity $\omega_X(f,t)$ defined as follows: $\omega_X(f,t) = \sup_{0\leq h\leq t} \| \Delta_h f \|_X$, where $(\Delta_h f)(x) = |f(x+h) - f(x)| \chi_{(0,1-h)}(x)$

In [8] we derived estimates for f^* ("the" non-increasing rearrangement of f) of the following type

$$(1.1)\quad f^*(t)-f^*(1/2) \leq \text{const} \int_t^1 \frac{\omega_X(f^*,s)}{\phi_X(s)} \frac{ds}{s}, \quad 0<t\leq 1/2, \quad \text{where}$$

$\Phi_X(s) = \|\chi_{(0,s)}\|_X$, i.e. Φ_X is the "fundamental function" of the space X.

The purpose of this paper is to prove the following theorem:

THEOREM A. $\omega_X(f^*,t) \leq 7\, \omega_X(f,t)$.

In [8] we used estimates of the type (1.1) to obtain embedding theorems for r.i. spaces. These results generalize some classical inequalities by Sobolev and more recent results by Ul'janov [13], Storoženko [12] and Garsia (cf. [2] and the references quoted in this paper). In a very interesting paper Garsia [3] used an inequality of the type (1.1) (for $X= L^P$) to obtain conditions for the uniform convergence of Fourier series.

In some recent articles inequalities of the type described in Theorem A have been proved in the context of L^P spaces (c f. [4],[9], [10], [14], [15]).

Our result combined with (1.1) allows us to extend the interesting work by Garsia to a wider class of spaces which includes the L(p,q) spaces as well as the Orlicz spaces. Moreover, our methods, which are based on work by the Russian school, seem to be simpler (cf. [3], p.87).

The paper is organized as follows: §2 contains the basic set of Lemmas on which we base our proof of Theorem A in §3. The reader is then referred to Garsia [3] for applications to the theory of Fourier series and Probability theory. We hope to return to these questions at a later occasion.

2. AUXILIARY RESULTS. Our method of proof of Theorem A follows closely ideas of [13] and [14]. We start with a result available in the literature (cf. [11] , p. 455).

(2.1) Lemma. Let f be differentiable almost everywhere in [0,1], then

$$\| f^{*`} \|_X \leq \| f` \|_X \; (= \| f^{`*} \|_X).$$

Remark. In [11], (2.1) is stated and proved for $X = L^P$, however the same proof applies in the more general case.

Using (2.1) we get the following

(2.2) Lemma. Let $f \in X$, and $g \in W^1_X$ (i.e. g' belongs to X) then

$$\omega_X(f^*, t) \leq 2 \| f^* - g^* \|_X + t \| g` \|_X .$$

Proof. By the triangle inequality we get

$$\omega_X(f^*, t) \leq \omega_X(f^* - g^*, t) + \omega_X(g^*, t)$$

$$\leq 2 \| f^* - g^* \|_X + t \| (g^*)` \|_X$$

and the result follows by (2.1).

Our next result allows us to replace $\| f^* - g^* \|_X$ by $\| f-g \|_X$ in (2.2)

(2.3) Lemma. Let $X(0,1)$ be a r.i. space, then

$$\| f^* - g^* \|_X \leq \| f-g \|_X \qquad \forall\ f,g \in L^1.$$

Proof. Follows from Day [1](p.941, example (ii)).

Let $f \in X$, and define f_s, $0<s<1$, as follows

$$f_s(x) = \begin{cases} \frac{1}{s} \int_x^{x+s} f(u)\,du & \text{if } 0 \leq x \leq 1-s \\ \frac{1}{s} \int_{1-s}^{1} f(u)\,du & \text{if } 1-s \leq x \leq 1. \end{cases}$$

Estimates for $\| f-f_s \|_p$ have been given for example in [13] . Similar estimates hold if we replace $\| \cdot \|_p$ by an arbitrary r.i. norm.

(2.4) Lemma. Let $f \in X$, then

$$\| (f-f_s)\, \chi_{(0,1-s)} \|_X \leq \omega_X(f,s).$$

Proof.

$$|f(x)-f_s(x)|\chi_{(0,1-s)}(x) = \left|\frac{1}{s}\int_0^s f(x)\,du - \frac{1}{s}\int_0^s f(x+u)\,du\right|\chi_{(0,1-s)}(x)$$

$$\leq \frac{1}{s}\int_0^s |f(x)-f(x+u)|\,\chi_{(0,1-s)}(x)\,du.$$

Therefore by Minkowski's inequality we get

$$\| (f-f_s)\chi_{(0,1-s)} \|_X \le \frac{1}{s} \int \omega_X(f,u)\, du$$

$$\le \omega_X(f,s).$$

We can also estimate $\| (f-f_s)\chi_{(1-s-1)} \|_X$ using the following Lemma (cf. [13]).

(2.5) Lemma. Let $f \in X$, then

$$\sup_{\|g\|_{X'} \le 1} \int_a^b \int_a^b |f(u)-f(t)|du|g(t)|dt \le 2 \int_0^{b-a} \omega_X(f,s)ds$$

where, $0 \le a < b \le 1$, and X' denotes the associate space of X ([6]).

Proof. Let $g \in X'$, $\| g \|_{X'} \le 1$, then

$$\int_a^b \int_a^b |f(u)-f(t)|du|g(t)|dt = \int_a^b \int_{a-t}^{b-t} |f(x+t) - f(t)|dx|g(t)|dt$$

$$= \int_0^{b-a} \{\int_a^{b-x} |f(x+t)-f(t)|| g(t)|dt\}dx + \int_{a-b}^0 \{\int_{a-x}^b |f(x+t)-f(t)||g(t)|dt\}dx$$

$$= J_1 + J_2 .$$

Now,

$$J_1 \le \int_0^{b-a} \sup_{\| g\|_{X'} \le 1} \{ \int_0^{b-x} |f(x+t) - f(t) ||g(t)|dt\}dx$$

$$\le \int_0^{b-a} \omega_X(f,x)dx,$$

and

$$J_2 \leq \int_{a-b}^{0} \{ \int_{a-x}^{b} |f(x+t) - f(t)|| g(t) | dt\} dx$$

$$\leq \int_{0}^{b-a} \{ \int_{a+v}^{b} |f(t-v) - f(t)|| g(t) | dt\} dv$$

$$\leq \int_{0}^{b-a} \{ \sup_{\| g_v \|_{X'} \leq 1} \int_{0}^{b-v} |f(t) - f(t+v)||g_v(t) | dt\} dv$$

$$\leq \int_{0}^{b-a} \omega_X (f,v) dv$$

where $g_v(t) = g(t+v)$. Collecting inequalities we obtain the desired result.

Using (2.5) we obtain the following

(2.6) Lemma. $\|(f-f_s)\chi_{(1-s,1)}\|_X \leq 2\omega_X(f,s)$.

Proof.

$$|f-f_s|(x)\chi_{(1-s,1)}(x) = |\frac{1}{s} \int_{1-s}^{1} f(x)du - \frac{1}{s}\int_{1-s}^{1} f(u)du|\chi_{(1-s,1)}(x)$$

therefore,

$$\|(f-f_s)\chi_{(1-s,1)}\|_X \leq s^{-1} \sup_{\| g\|_{X'} \leq 1} \int_{1-s}^{1} |g(x)| \int_{1-s}^{1} |f(x)-f(u)|dudx$$

$$\leq 2s^{-1} \int_{0}^{s} \omega_X (f,u)du \qquad \text{(by (2.5))}$$

$$\leq 2\omega_X(f,s)$$

as required.

3. Proof of Theorem A. We are now ready to prove our main result. Firstly observe that $f_s \in W_X^1$ whenever $f \in X$ and moreover

$f_s'(x) = \frac{1}{s}(\Delta_s f)(x)$, thus $\| f_s' \|_X \le s^{-1}\omega_X(f,s)$. Now, by (2.2), (2.3) we get

$$\begin{aligned}\omega_X(f^*,s) &\le 2\| f-f_s \|_X + s\| f_s' \|_X \\ &\le 6\omega_X(f,s) + \omega_X(f,s) \quad \text{(by (2.4) and (2.6))} \\ &\le 7\omega_X(f,s).\end{aligned}$$

The theorem is established.

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