NOTAS DE MATEMATICA

Nº 51

" EXISTENCE OF SOLUTIONS CONVERGING TO ZERO FOR THE NON-LINEAR DIFFERENTIAL EQUATION "

$$x''' + p(t,x,x')x' + q(t)x = 0$$

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ABSTRACT

Using a Lyapunov function and a result of Wazewski we show the existence of a non-Trivial solution converging to zero of the Differential Equation

$$x''' + p(t,x,x')x' + q(t)x = 0$$

under the hypotheses: $p(t,x,y)$ continuous in (t,x,y) and $p(t,x,y) \leq -B < 0$, $q \in C'([0,\infty))$ and $q'(t) \geq A > 0$.

§1. We will prove a theorem of existence of a non-Trivial solution converging to zero for the equation

$$x'' + p(t,x,x')x' + q(t)x = 0 \qquad (1)$$

Under certain restrictions on its coefficients. The following lemma will be used in the proof:

LEMMA 1. Let $f \in C'([0,+\infty))$. If $\int_0^\infty f^2(t)dt < +\infty$ and $f'(t)$ is bounded, then $f(t) \to 0$ when $t \to +\infty$.

§2. In all that follows we will use the function:

$$F(t,x_1,x_2,x_3) = -2x_2x_3 - q(t)x_1^2 \qquad (2)$$

Let $x(t)$ be a solution of (1). If we substitute x_1 by $x(t)$, x_2 by $x'(t)$ and x_3 by $x''(t)$ in (2) we get the function of t:

$$F(x(t)) = -2\,x'(t)x''(t) - q(t)x^2(t)$$

LEMMA 2. Let's suppose the coefficients of (1) satisfy the following hypotheses: $p(t,x,y)$ is continuous in (t,x,y) $q \in C'([0,+\infty))$, $p(t,x,y) \leq -B < 0$ and $q'(t) \geq A > 0$. Then, given a solution $x(t)$ of (1) such that $F(x(t)) \geq 0$ for $t \in ([t_o, \omega))$, where ω is the right end of the largest interval of existence of $x(t)$, we have that $\omega = +\infty$, $x^{(i)}(t) \in L^2([t_o, +\infty))$ and $x(t)$ converge to cero when $t \to +\infty$.

PROOF. We have

$$F(x(t)) = \frac{d}{dt}(F(x(t))) = -2(x''(t))^2 + 2p(t,x,x')(x(t))^2$$
$$-q(t)((x(t))^2$$

If we use the hypotheses in p and q we have:

$$F(x(t)) \leq -W(x,t))$$

where

$$W(x(t)) = 2|x''(t)|^2 + 2B|x'(t)|^2 + Ax^2(t)$$

we have then:

$$F(x(t)) \leq - W(x(t)) < 0$$

and integrating,

$$F(x(t)) \leq F(x(t_o) - \int_{t_o}^{t} W(x(\tau))d\tau ,$$

or,

$$\int_{t_o}^{t} W(x(\tau))d\tau \leq F(x(t_o)) - F(x(t)).$$

If now $F(x(t)) \geq 0$, for $t \geq t_o$ we have

$$\int_{t_o}^{t} W(x(\tau))d\tau \leq F(x(t_o)) \qquad (4)$$

From (4) we obtain that $x^{(i)}(t) \in L^2([t_o,\omega))$ $i=0,1,2.$ We will now prove that the functions $x^{(i)}(t)$, $i=0,1,2$ are bounded in $[t_o,\omega)$ and, as a consequence of this , the solution can be extended to $[t_o,+\infty)$: we have

$$\begin{aligned} |x^2(t)| &\leq |x^2(t_o)| + 2\int_{t_o}^{t} |x\ x'|d\tau \\ &\leq |x^2(t_o)| + 2\left(\int_{t_o}^{t} |x^2(\tau)|d\tau\right)^{1/2} \\ &\qquad \left(\int_{t_o}^{t} |x(\tau)|^2 d\tau\right)^{1/2} \end{aligned}$$

For $t \in [t_o,\omega)$. In a similar way it is proved that $x'(t)$ is bounded in $[t_o,\omega)$.

To see that $x''(t)$ is bounded in $[t_o,\omega)$ we use (1)

$$x''' = -p(t,x,x')x' - q(t)x$$

so, by integration,

$$|x''(t)| \leq |x''(t_o)| + \left(\int_{t_o}^{t} |p(\tau,x,x'|^2 d\tau\right)^{1/2} \left(\int_{t_o}^{t} |x'(\tau)|^2 d\tau\right)^{1/2}$$

$$+ \left(\int_{t_o}^{t} g^2(\tau)d\tau\right)^{1/2} \left(\int_{t_o}^{t} x^2(\tau)d\tau\right)^{1/2} .$$

Using now lemma 1 we have that $x(t) \to 0, x'(t) \to 0$ when $t \to \infty$.

THEOREM 1. Let's suppose that the coefficients of (1) satisfies the hypotheses of Lemma 1. Then theres exist a non trivial solution $\phi(t)$ of (1) such that

$\phi^{(i)}(t) \in L^2([t_o,\infty)))$ for $i=0,1,2$; $\phi^{(i)}(t) \to 0$ for $i=0,1$ when $t \to +\infty$.

PROOF. Let's consider the following subset of $\mathbb{R}^4$:

$$H = \{(t,x_1,x_2,x_3) \in \mathbb{R}^4 \mid F(t,x_1,x_2,x_3) > 0,\ t > t_o\}$$

with t_o big enough so that $q(t) > 0$ if $t \geq t_o$.

We want to show that every point of the following set

$$C = \text{Boundary of } H - \{(t,x_1,x_2,x_3) \in \mathbb{R}^4 \mid x_1=x_2=x_3=0\}$$

is a strict egress point (see |2|) of the flow

$$\Psi(t) = (t, x_1(t), x_2(t), x_3(t))$$

in $\mathbb{R}^4$ determined by the solutions of the following differential equation:

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -q(t)x_1 - p(t,x_1,x_2)x_2$$

Because

Boundary of $H = \{(t,x_1,x_2,x_3) \in \mathbb{R}^4 \mid F(t,x_1,x_2,x_3)=0\}$

then it suffices to calculate the scalar product

$$(\nabla F, \dot{\Psi}(t))$$

and see that it is negative.

In fact,

$$(\nabla F, \dot{\Psi}(t)) = -q'(t)x_1^2 - 2x_3^2 + 2p(t,x_1,x_2)x_2^2$$

$$\leq -Ax_1^2 - 2x_3^2 - Bx_2^2 < 0,$$

and then we have that every point of C is strict egress point.

If now we suppose that every solution curve of (1) leaves H, then we would have a mapping from H into c which is a retract, that is, continuous and its restriction to C equal the identity (see|2|). But we now from algebraic topology that this is not possible because H is contractible to a point and C has a first homotopy group different from the trivial one.

R E F E R E N C E S

[1] E. Escanglon, Nouvelles recherches sur les fonctions presqueperiodiques, Ann. Obs. Bordeaux, 16 (1917) 15-226.

[2] Hartman, Philip, Ordinary Differencial Equations, John Wiley, New York, 1964.