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ON DILATION FUNCTIONS AND SOME APPLICATIONS

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ABSTRACT. The aim of this paper is to study some properties of the so called dilation functions ([7]), and applications thereof to questions on Orlicz Spaces and linear bounded operators on them. Some results are part of a Ph.D, dissertation presented by the author at Chelsea College, London yet unpublished.

1.- INTRODUCTION

Let $\phi(u)$, $u \in [0,\infty)$, be a real, increasing function, right continuous on $(0,\infty)$. The function $\Phi(u)$, $u \geq 0$, defined by

$$\Phi(u) = \int_0^u \phi(t)\,dt$$

is called a Young function.

The function $\Psi(v)$, $u \geq 0$, defined by

$$\Psi(v) = \sup_{u\geq 0} \{uv - \Phi(u)\},$$

where sup can be replaced by max if $\Psi(v)$ is finite for finite v, is called the complementary function to $\Phi(u)$. One also has that

$$\Phi(u) = \max_{u\geq 0} \{uv - \Psi(v)\}$$

A Young function satisfies the $\delta_2(\Delta_2)$ condition if there is some $u_o>0$ and $M>0$ such that

$$\Phi(2u) \leq M\,\Phi(u),$$

for all u in $[0,u_o]$ (in $[u_o,\infty)$). If this inequality holds for all $u\geq 0$, then it is said that Φ satisfies the (δ_2,Δ_2) condition ([9]).

A Young function satisfies the $\delta'(\Delta')$ condition if there are $u_o\geq 0$, $M>0$ such that

$$\Phi(uv) \leq M\,\Phi(u)\,\Phi(v)$$

for all u,v in $[0,u_o]$ (in $[u_o,\infty)$). If Φ satisfies both conditions, then it is said that Φ is submultiplicative.

Whenever these inequalities hold in reverse we say that Φ satisfies the $\rho'(\nabla')$ condition and that Φ is supermultiplicative respectively.

The Young functions $\Phi_1(u)$, $\Phi_2(u)$ are said to be equivalent on the set A if for some positive constants k_1,k_2 we have

$$\Phi_1(k_1u) \leq \Phi_2(u) \leq \Phi_1(k_2u)$$

for all u in A.

A Young function $\Phi(u)$ with representation

$$\Phi(u) = \int_0^u \phi(t)\,dt,$$

is called an N-function ([6]) if $\phi(t)$ is positive for positive t, and satisfies the conditions $\phi(0) = 0$, $\lim_{t\to\infty} \phi(t) = \infty$.

One can easily see that the following hold for $\Phi(u)$:

$$\lim_{u\to 0} \frac{\Phi(u)}{u} = 0 \quad \text{and} \quad \lim_{u\to\infty} \frac{\Phi(u)}{u} = \infty.$$

Let $\Phi(u)$ be a Young function that satisfies the (δ_2,Δ_2) condition. Let μ be a totally σ-finite measure on $\mathbb{R}^n$. The Orlicz space $L_\Phi(\mathbb{R}^n,\mu)$ consists of all μ-measurable functions f, such that

$$\int_{\mathbb{R}^n} \Phi(|f|)\,d\mu < \infty.$$

By ℓ_Φ we mean, as usual, the space of all scalar sequences $\{a_n\}_{n=1}^{\infty}$ such that

$$\sum_{n=1}^{\infty} \Phi(|a_n|) < \infty.$$

Conditions for these spaces to be reflexive are known since long ago. Here we give yet another such condition which seems to be new.

Consider the spaces $L_\Phi(\mathbb{R}^n,\mu)$, where μ is a positive Radón measure. For any $h \in \mathbb{R}^n$, the operation of translation is defined by

$$\tau(h)f(x) = f(x-h),$$

for any μ-measurable function f. In this paper we generalize a result in [2] which gives necessary and sufficient conditions for $\tau(h)$ to be defined as an operator on $L_\Phi(\mathbb{R}^n,\mu)$. We also obtain a necessary condition for there to exist a translation invariant operator T,

$$T: L_{\Phi_1}(\mathbb{R}^n,\mu) \longrightarrow L_{\Phi_2}(\mathbb{R}^n,\gamma).$$

When restricted to L_p spaces this condition gives those in [2] and [5].

Moreover, a necessary and sufficient condition for there to exist a linear bounded translation invariant operator T,

$$T: \ell_{\Phi_1} \longrightarrow \ell_{\Phi_2}$$

is obtained.

Let X,Y be normed spaces. A linear bounded operator $T : X \longrightarrow Y$ is said to be strictly singular if for any subspace A of X, the restriction of T to A is not an isomorphism. For a submultiplicative function Φ_1 a sufficient condition for every linear bounded operator

$T : \ell_{\Phi_1} \longrightarrow \ell_{\Phi_2}$ to be strictly singular, is given in this paper.

The following theorem can be easily deduced from [7] (Th. 1.2. p.52).

<u>THEOREM 1</u>.- Let Φ be a submultiplicative Young function. Then, there exist real numbers a α,β such that $1<\alpha<\beta<\infty$ and

$$\Phi(t) \geq t^{\beta} \text{ for } t \in [1, \infty), \ \Phi(t) \geq t^{\alpha} \text{ for } t \in [0,1].$$

Moreover, given $\varepsilon > 0$ there exist real numbers a_ε and b_ε such that

$$\Phi(t) \leq t^{\beta+\varepsilon} \text{ for } t \in [b_\varepsilon,\infty) \text{ and } \Phi(t) \leq t^{\alpha-\varepsilon} \text{ for } t \in [0,a_\varepsilon].$$

2.-

Let $\Phi(u)$ be a non negative, increasing, left continuous real function defined on the interval $[0,\infty)$. Let u_o be a non negative number fixed throughout. Define the function $n(\Phi,u_o;x)$ by

$$n(\Phi,u_o;x) = \sup \{s \geq 0 \,;\, \Phi(su) \leq x\, \Phi(u),\ u \geq u_o\}.$$

The function $n(\Phi,u_o;x)$ is manifestly increasing and the inequality

$$\Phi(n(\Phi,u_o;x)u) \leq x\, \Phi(u), \quad u \geq u_o,$$

holds whenever $n(\Phi,u_o;x)$ be finite.

The basic idea behind the function $n(\Phi,u_o;x)$, with $u_o = 0$, seems to go back to D.W. Boyd [1]. The less restrictive definition we use here is taken from [3]. These appear named dilation functions in [7]; and are also considered in in [4].

The following properties of $n(\Phi,u_o;x)$ are easy consequences of the definition.

Let $\Phi(u)$, be as above, then

a) if $n(\Phi,u_o;x)$ is finite on $[0,a)$, then it is right continuous on $[0,a)$.

b) The inequality $n(\Phi,u_o;x) \geq x$, for any $x \in (0,1)$, holds true if and only if

$$\Phi(xu) \leq x\,\Phi(u)$$

for any $u \geq u_o$ and $x \in (0,1)$.

c) For any $x \geq 0$, and $y \geq 1$, we have that

$$n(\Phi,u_o;x)\, n(\Phi,u_o;y) \leq n\,(\Phi,u_o;xy).$$

<u>LEMMA 1.-</u> Let $\Phi(u)$, $u \geq 0$, be an increasing left continuous real function such that $\Phi(0) = 0$ and $\Phi(u) > 0$ for $u > 0$. If for any $y \in (0,1)$ is

$$\Phi(yu) \leq y\,\Phi(u),$$

for all $u \geq u_o$, then $n(\Phi,u_o;x)$ is continuous for any $x \geq 1$.

Proof: Let $\{x_k\}_{k=1}^{\infty}$ be a strictly increasing sequence of real positive numbers whose limit is one, then

$$\Phi(x_k u) \leq \Phi(n(\Phi,u_o;x_k)u) \leq x_k\,\Phi(u)$$

for $k \in \mathbb{N}$ and any $u \geq u_o$. By passing to the limit as $k\to\infty$, we get.

$$\Phi(u) \leq \Phi(n(\Phi,u_o;1^-)u) \leq \Phi(u),$$

that is, $n(\Phi,u_o;1^-) = 1$; so that $n(\Phi,u_o;x)$ is continuous at 1.

Let $x_o \geq 1$, and $\{x_k\}_{k=1}^{\infty}$ be as above, then

$$n(\Phi,u_o,x_o^+) = \lim_{n\to\infty} n(\Phi,u_o;x_o^+)\, n(\Phi,u_o;x_k)$$

$$\leq \lim_{n\to\infty} n(\Phi,u_o;x_o x_k)$$

$$= n(\Phi,u_o,x_o^-),$$

that is $n(\Phi,u_o;x_o^+) = n(\Phi,u_o;x_o^-)$.

If $n(\Phi,u_o;x)$ is supermultiplicative then we also get that, in the conditions of the previous Lemma, it is continuous for all $x \geq 0$.

LEMMA 2.- Let $\Phi(u)$, $u \geq 0$, be an increasing, left continuous real function such that $\Phi(0) = 0$. A necessary and sufficient condition that $n(\Phi,u_o;x)$ tend to infinity as x tend to infinity and be finite for finite values of the argument x, is that $\Phi(u)$ satisfy the Δ_2 condition for $u \geq u_o$, and that

$$\lim_{n\to\infty} \Phi(u) = \infty.$$

Proof: If $\Phi(2u) \leq M\,\Phi(u)$, $u \geq u_o$ then

$$\Phi(2^k u) \leq M^k \Phi(u),\ u \geq u_o,\ k \in \mathbb{N},$$

and consequently

$$n(\Phi,u_o;M^k) \geq 2^k;$$

so that $n(\Phi,u_o;x) \to \infty$ as $x \to \infty$.

Suppose by absurd that, for some $x < \infty$, we have that $n(\Phi,u_o;x) = +\infty$, then for a fixed $u \geq u_o$ and any $y > 0$, we have

$$\Phi(yu) < x\,\Phi(u).$$

However, this contradict the fact that $\Phi(u) \to \infty$ as $u\to\infty$.

Hence, $n(\Phi,u_o;x)$ must be finite for finite x.

Conversely, if $n(\Phi,u_o;x) \to \infty$ as $x \to \infty$ and is finite for finite x, then, given $\lambda > 1$, there is some x_λ such that $n(\Phi,u_o;x_\lambda) > \lambda$.

Thus,

$$\Phi(\lambda u) \leq \Phi(n(\Phi,u_o;x_\lambda)u) \leq x_\lambda \Phi(u),$$

for all $u \geq u_o$. In particular x_λ must be larger than one.

Finally, if $\Phi(u)$ be bounded, say $\Phi(u) < K$ for all $u \geq u_o$, with $K > 1$, then, taking some $\hat{x} > \frac{K}{\Phi(u_o)}$ we would have that

$$\Phi(yu) \leq \hat{x}\, \Phi(u), \quad u \geq u_o$$

for all $y > 1$. However, this contradicts the fact that $n(\Phi,u_o;\hat{x})$ is finite.

If in the previous Lemma we assume further that $\Phi(u)$ is a Young function then $n(\Phi;u_o;x)$ is positive for positive x. Also, $n(\Phi,u_o;x)$ is a concave function of x.

Definition : The function $N(\Phi,u_o;x)$, inverse to the function $n(\Phi,u_o;x)$, will be called the right dilation function of Φ.

For any young function Φ, which satisfies the Δ_2 condition for $u \geq u_o$, we have that $N(\Phi,u_o;x)$ is a convex function such that

$$N(\Phi,u_o;xy) \leq N(\Phi,u_o;x)\; N(\Phi,u_o;y)$$

for any $x \geq 0$, and $y \geq 1$. Also, $N(\Phi,u_o;x)$ satisfies the (δ_2,Δ_2) condition.

For $u_o = 0$ $N(\Phi,u_o;x) = N(\Phi,x)$ is submultiplicative, that is

$$N(\Phi,xy) \leq N(\Phi;x)\, N(\Phi;y),$$

for any $x,y \geq 0$. Also,

$$\Phi(xu) \leq N(\Phi,u_o;x)\, \Phi(u),$$

for all $u \geq u_o$.

The following proposition gives an answer to an elementary question posed by Krasnoselskii and Rutitskii [6] p.30.

PROPOSITION 1.-

In each class of functions which satisfy the Δ' condition there is a submultiplicative function.

Proof: Let $\Phi(u)$ be a Young function which satisfies the Δ' condition for $u \geq u_o$. We assume, as we may, that $\Phi(u)$ satisfies the (δ_2,Δ_2) condition.

The function $\Phi(u)$, defined by

$$\hat{\Phi}(u) = \Phi(u_o\, u),\ u \geq 0,$$

is equivalent to Φ and satisfies the Δ' condition for $u \geq 1$. Then, the function $N(\hat{\Phi};u)$ is equal to $\hat{\Phi}(u)$ in $[1,\infty)$; and

$$N(\hat{\Phi};xy) \leq N(\hat{\Phi},x)\, N(\hat{\Phi},y)$$

for all $x,y \geq 0$.

One can also see that

$$N(N(\Phi;x);u) = N(\Phi;u)$$

Definition: Let Φ be a Young function that satisfies the (δ_2,Δ_2) condition. The function $K(\Phi;x)$ defined by

$$K(\Phi;x) = \inf_{0<u<\infty} \frac{\Phi(xu)}{\Phi(u)}$$

will be called the left dilation function of Φ.

It is easy to see that $K(\Phi;x) = \dfrac{1}{N(\Phi;\frac{1}{x})}$ for all $x>0$; so that

$$\frac{K(\Phi;x)}{x} = \frac{\frac{1}{x}}{N(\Phi;\frac{1}{x})}$$

is strictly increasing and

$$\int_0^x \frac{K(\Phi;t)}{t}\,dt$$

is convex. That is, $K(\Phi;x)$ is equivalent to a Young function that satisfies the (δ_2,Δ_2) condition. Also, $K(\Phi;x)$ is supermultiplicative, that is $K(\Phi;xy) \geq K(\Phi,x)\, K(\Phi;y)$ for all $x,y \geq 0$. Moreover, $K(K(\Phi,u),x) = K(\Phi,x)$.

One can also see that

$$\Phi(xu) \geq K(\Phi;x)\ \Phi(u)$$

for all $x,u \geq 0$.

For a Young function Φ not satisfiyng the (δ_2,Δ_2) condition, the study of the function $K(\Phi;x)$ is more complicate as the example of $K(e^u-1;x)$ shows. This is a concave function discontinuous at zero.

There is no mention of this function in [1]. However, it is safe to think that this author already studied this function.

LEMMA 3.- Let $\Phi(u)$, $u \geq 0$, be a Young function. If Φ satisfies the δ' and ρ' conditions, then it is equivalent to x^P for some $P \geq 1$.

Proof: We have that $N(\Phi;x)$, $K(\Phi;x)$ and $\Phi(x)$ are all equivalent. It now follows from Theorem 1. That, for some $k \geq 1$, $\alpha \geq 1$ and all $x \geq 0$.

$$x^{\alpha} \leq N(\Phi,x) \leq k\, x^{\alpha} \; .$$

PROPOSITION 2.- Let $\Phi(u)$, $u \geq 0$, be an N-function which satisfies the (δ_2,Δ_2) condition. A necessary and sufficient condition that the complementary function Ψ of Φ satisfy the (δ_2,Δ_2) condition is that for some $x > 1$, $K(\Phi;x) > x$.

Proof: If, for some $x > 1$, $K(\Phi,x) > x$, then $K(\Phi,x) > \alpha x$, for some $\alpha > 1$; so that

$$\Phi(xu) > \alpha x\ \Phi(u),\ u > 0.$$

Thus,

$$\Psi(2v) = \sup_{0<u} \{\alpha x\, u\, v - \Phi(xu)\}$$

$$= \alpha x\ \Psi(v)\ ,\ v \geq 0.$$

If, on the other hand, Ψ satisfies the (δ_2,Δ_2) condition then, there exist $\alpha > 1$, $x > 1$ such that

$$N(\Psi;\alpha) < \alpha x;$$

consecuently

$$\Phi(xu) = \sup_{0<v} \{\alpha v\, x\, u - \Psi(\alpha v)\}$$

$$= \alpha x \sup_{0<v} \{uv - \frac{\Psi(\alpha v)}{\alpha x}\}$$

$$> \alpha x \sup_{0<v} \{uv - \frac{N(\Psi;\alpha)}{\alpha x}\ \Psi(v)\}$$

$$> \alpha x \sup_{0<v} \{uv - \Psi(v)\}$$

$$= \alpha x\ \Phi(u).$$

Therefore $K(\Phi,x) > \alpha x$.

Corollary: The complementary Ψ to the function Φ satisfies the (δ_2,Δ_2) condition if and only if, for some $x < 1$, $N(\Phi;x) < x$.

In terms of Orlicz spaces this result can be restated as follows:

THEOREM 2.- Let ℓ_Φ be separable space. We have that ℓ_Φ is reflexive if and only if $N(\Phi;x) < x$ for some x.

In some instances the following theorem may also be of interest. We asume, as we may, that $\Phi_1(1) = \Phi_2(1) = 1$.

THEOREM 3.- Let ℓ_{Φ_1}, be a separable space. Assume that $K(\Phi;x)$ is convex. Then a necessary and sufficient condition that ℓ_{Φ_1} be reflexive is that there exist a Young function Φ_2 which satisfies the (δ_2,Δ_2) condition and such that

$$\Phi_1(u) \leq \Phi_2(u),\ u \in [0,1]$$

Proof. If the property holds. Then, for some $x \in (0,1)$ $\Phi_1(x) < \Phi_2(x)$; so that $K(\Phi_1;x) < N(\Phi_2;x) \leq x$, for this x. Since $K(\Phi_1;x)$ is convex and $K(\Phi_1;1) = 1$, then we must have that $K(\Phi_1;x) < x$, for all x in $(0,1)$. This in turn implies that $K(\Phi_1;x) > x$ for $x > 1$.

Thus, Ψ_1, the complementary to Φ_1, satisfies the (δ_2,Δ_2) and ℓ_{Φ_1} is reflexive.

If, on the other hand, ℓ_{Φ_1} is reflexive, then Ψ_1 satisfies the (δ_2,Δ_2) condition and this implies that $N(\Phi_1,x) < x$ for all x in $(0,1)$. We see thus that $\Phi_1(x) < x$, $x \in [0,1]$.

The case when the N-function Φ is submultiplicative is particulary simple.

PROPOSITION 3.-

If $\Phi(x)$, $x \geq 0$, is a submultiplicative N-function; then ℓ_Φ is reflexive.

Proof: Since Φ is submultiplicative, then the complementary function Φ is supermultiplicative, so that $\overline{\Psi} = 1/\Psi(\frac{1}{x})$ satisfies the (δ_2, Δ_2) condition, that is $\overline{\Psi}(2x) \leq M\, \overline{\Psi}(x)$, all $x \geq 0$.

Therefore $\Psi(2x) = \dfrac{1}{\Psi(\frac{1}{2x})} \leq \dfrac{1}{\frac{1}{M}\,\overline{\Psi}(\frac{1}{x})} = M\,\Psi(x)$, for all $x \geq 0$

From now on let us write K_i and N_i for the dilation functions of the Young function Φ_i.

THEOREM 4.- Let Φ_1, Φ_2 be non equivalent Young functions that satisfy the (δ_2, Δ_2) condition and such that ℓ_{K_2} is continuously embedded in ℓ_{Φ_1}. If Φ_1 is submultiplicative, then every linear bounded operator T,

$$T : \ell_{\Phi_2} \longrightarrow \ell_{\Phi_1}$$

is strictly singular.

Proof: According to theorem 2 the space ℓ_{Φ_1} happens to be reflexive. Let T be a linear bounded operator

$$T : \ell_{\Phi_1} \longrightarrow \ell_{\Phi_2}$$

and suppose that there exist subspaces $X \subset \ell_{\Phi_1}$, $Y \subset \ell_{\Phi_2}$ such that

$$T : X \longrightarrow Y$$

is a isomorphism, then there exist normalized block basic sequences $\{B_k\}_{k=1}^{\infty}$,

$$B_k = \sum_{i=P_k+1}^{i=P_{k+1}} t_i e_i,\ \{A_k\}_{k=1} ,\ A_k = \sum_{j=q_k+1}^{j=q_{k+1}} r_j e_j$$

in X e Y respectively, where $\{e_i\}_{i=1}^{\infty}$ is the unit basis, such that

$$T(B_k) = A_k,\ k \in \mathbb{N}.$$

Since Φ_1, Φ_2 are non equivalent, then there is a sequence $a = \{a_n\}_{r=1}^{\infty}$ such that $\sum_{k=1}^{\infty} K_2(|a_k|)$ converges and $\sum_{n=1}^{\infty}\Phi_1(|a_n|)$ diverges.

Let $x = \sum_{k=1}^{\infty} a_n \sum_{i=P_k+1}^{i=P_{k+1}} t_i e_i$, then we have

$$\sum_{k=1}^{\infty} \sum_{i=P_k+1}^{i=P_{k+1}} \Phi_2(|a_k|\ |t_i|) \geq \sum_{k=1}^{\infty} K_2(|a_k|) \sum_{i=P_k+1}^{i=P_{k+1}} \Phi_2(|t_i|)$$

$$= \sum_{k=1}^{\infty} K_2(|a_k|);$$

that is $\sum_{k=1}^{\infty} \sum_{i=p_k+1}^{i=P_{k+1}} \Phi_2(|a_k|\ |t_i|)$ diverges. On the other hand T(x) is in Y. Indeed,

$$\sum_{k=1}^{\infty} \sum_{j=q_k+1}^{j=q_{k+1}} \Phi_1(|a_k r_j|) < \sum_{k=1}^{\infty} \Phi_1(|a_k|) \sum_{j=q_k+1}^{j=q_{k+1}} \Phi_1(|r_j|)$$

$$= \sum_{k=1}^{\infty} \Phi_1(|a_k|) < \infty.$$

Contradiction.

Related results can be found in [8] and [10].

<u>THEOREM 5.-</u> Let μ be a positive a Radon measure defined on $\mathbb{R}^n$. Let $\Phi(u)$ be a Young function that satisfies the (δ_2, Δ_2) condition. Then the following condition are equivalent.

a) if $f \in L_\Phi(\mathbb{R}^n, \mu)$ then $\tau(h)f(x) \in L_\Phi(\mathbb{R}^n, \mu)$ for all h in $\mathbb{R}^n$,

b) $\tau(h)$ is a continuous map of $L_\Phi(\mathbb{R}^n, \mu)$ to itself for any h,

c) there is a positive Lebesgue measurable function $\lambda(x)$ bounded with $\lambda(x)^{-1}$ over any compact set of values of x such that $\lambda(x)dx = d\mu$ and

$$K^{-1}(||\tau(h)||) \leq \sup \frac{\lambda(x+h)}{\lambda(x)} \leq N^{-1}(||\tau(h)||).$$

Proof: If (a) holds, then $\mu(E) = 0$ implies that $\mu(E+h) = 0$ for all h. For, let $\mu(E) = 0$ and let $f(x) = \infty$ for $x \in E$, $f(x) = 0$ otherwise, so that $\int \Phi(|f(x)|)d\mu = 0$; and since $\tau(h)f(x) = f(x-h)$ is infinity on $E + h$, then we must have that $\mu(E+h) = 0$.

Let us now write $\tau(h)\mu = \mu_h$, that is

$$\int_{\mathbb{R}^n} f(x)d\mu_h = \int f(x+h)d\mu.$$

We see that μ_h is absolutely continuous with respect to μ and that μ is absolutely continuous with respect to μ_h; whence $d\mu_h = \phi(x,h)d\mu$ with $\phi(x,h)$ and $\phi(x,h)^{-1}$ locally summable.

Therefore

$$\int f(x+h)d\mu = \int f(x)\phi(x,h)d\mu.$$

Let us define

$$\phi_n(x,h) = \min\{\phi(x,h), 2^n\},$$

and

$$F_{h,n}(f) = f(x(N^{-1}(\phi_n(x,h)), \quad f \in L_\Phi(\mathbb{R}^n, \mu)$$

Then

$$\int \Phi(|F_{h,n}(f)|)d\mu \leq \int \Phi(2^n|f(x)|)d\mu$$

$$\leq M^n \int \Phi(|f(x)|)d\mu,$$

so that $F_{h,n}$ is a linear bounded transformation of $L_{\phi}(\mathbb{R}^n,\mu)$ to itself for any fixed n and h. Moreover $||F_{h,n}|| \leq 2^n$.

It now follows that $\sup||\phi_n(x,h)||_\infty < \infty$ and $\phi(x,h)$ is bounded for each h; whence $\tau(h)$ is bounded for each h.

We have thus proved (a) => (b). The converse is inmediate.

Let us now assume that (b) holds. Then for any $f \in L_\phi$ with $||f|| = ||f||_{L_\phi} > 0$, we have that

$$1 = \int_{\mathbb{R}^n} \Phi\left(\frac{|\tau(h)f|}{||\tau(h)f||} \right) d\mu = \int_{\mathbb{R}^n} \Phi\left(\frac{|f|}{||\tau(h)f||} \right) \phi(x,h)d\mu$$

$$\leq \int_{\mathbb{R}^n} \Phi\left(\frac{K^{-1}(||\phi(x,h)||_\infty)|f|}{||\tau(h)f||} \right) d\mu,$$

so that $||\tau(h)f|| \leq K^{-1}(||\phi(x,h)||_\infty)||f||$.

Given $\varepsilon > 0$ there is a set E such that

$$\int_{\mathbb{R}^n} \Phi(|\tau(h)\chi_E|)d\mu = \int_{\mathbb{R}^n} \Phi(|\chi_E|)\phi(x,h)d\mu$$

$$\geq \int_{\mathbb{R}^n} \Phi(N^{-1}(||\phi(x,h)||_\infty - \varepsilon)|\chi_E|)d\mu,$$

thus

$$||\tau(h)\chi_E|| \geq |N^{-1}(||\phi(x,h)||_\infty - \varepsilon| \, ||\chi_E||$$

We have now proved that

$$N^{-1}(||\phi(x,h)||_\infty) \leq ||\tau(h)|| \leq K^{-1}||\phi(x,h)||_\infty),$$

hence $||\tau(h)||$, is bounded or unbounded over any compact set of values of h together with $||\phi(x,h)||_\infty$.

It now follows from the previous Lemma and the fact that log $||\tau(h)||$ is subadditive that $||\phi(x,h)||_\infty$ is bounded over any compact set of values of h.

Since μ is a Radon measure, it follows from the Radon-Nikodym theorem that

$$d\mu = \lambda(x)dx$$

with λ bounded over any compact. Thus

$$d\mu_h = \lambda(x+h)dx,$$

so that $\phi(x,h) = \frac{\lambda(x+h)}{\lambda(x)}$, and

$$K(||\tau(h)||) \leq \sup_x \frac{\lambda(x+h)}{\lambda(x)} \leq N(||\tau(h)||).$$

This proves that (b) implies (c). It is easy to see that (c) implies (b).

Necessary conditions for the existence of non-trivial, linear, translation invariant operators acting on L_p

spaces with general Radon measures subject to some conditions of regularity have been studied by J.L.B. Cooper [2].

We now pass on to examine the existence of operators acting on Orlicz spaces $L_{\Phi_1}(\mathbb{R}^n,\mu)$ and $L_{\Phi_2}(\mathbb{R}^2,\nu)$ where Φ_1 and Φ_2 satisfy the (δ_2,Δ_2) condition, $\mu = e^{a||x||}$ and $\nu = e^{b||x||}$.

In some important particular instances the condition that Φ_1 and Φ_2 satisfy the (δ_2,Δ_2) condition is necessary. For example, D. BOYD [1] has proved that, a necessary condition that the Hilbert transform be a map of the space of Lebesgue measurable functions $L_\Phi(\mathbb{R}^n)$ to itself, is that Φ satisfy the (δ_2,Δ_2) condition. This condition turns out to be sufficient.

Let $I(0,\frac{m}{2})$ be the closed cube in $\mathbb{R}^n$ centred at 0 and having side m. Let h(k,r) be the element in $\mathbb{R}^n$ whose components are all equal to $\frac{m|kr-k-1|}{r-1}$, where k is a natural number greater than or equal to one and $r>1$. We also write $H(k,r,m) = ||h(h,r,m)||$.

LEMMA 4.-

a) For any $x \in I(0,\frac{m}{2})$ we have that

$$||x+h(k,r,m)|| \geq ||x|| + \frac{H(k,r,m)}{r} .$$

b) For any $y \in I(0,\frac{m}{2}) + h(k,r,m)$, we have that

$$||x + h(k+1,r,m)|| \geq ||y||,$$

holds for any $x \in I(0,\frac{m}{2})$.

Proof:

a) The minimum of $||x + h(k,r,m)||$ with $x \in I(o,\frac{m}{2})$ is attained at $x_m = (-\frac{m}{2},\ldots,\frac{m}{2})$ and its value is

$$||x_m + h(k,r,m)|| = \sqrt{n}\left\{ -\frac{m}{2} + m\,\frac{|kr - (k-1)|}{r - 1} \right\}.$$

On the other hand, the maximum of $||x|| + \frac{H(k,r,m)}{r}$ is attained at $x = x_m$ and at $x = -x_m$ and its value is

$$||x_m|| + \frac{H(k,r,m)}{r} = \sqrt{n}\left\{\frac{m}{2} + \frac{m|kr - (k-1)|}{r(r-1)}\right\}.$$

Thus

$$||x_m + h(k,r,m)|| - ||x_m|| - \frac{H(k,r,m)}{r} = \sqrt{n}\;\frac{m(r-1)(k-1)}{r}$$

$$\geq 0.$$

b) The maximum of $||y||$ is attained at $y = -x_m + h(k,r,m)$ and

$$||-x_m + h(k,r,m))|| = \sqrt{n}\left\{\frac{m}{2} + m\,\frac{|kr - (k-1)|}{r - 1}\right\}$$

The minimum of $||x + h(k+1,r,m)||$ is attained at $x = x_m$ and,

$$||x_m + h(k+1,r,m)|| = \sqrt{n}\ \left\{-\frac{m}{2} + m\ \frac{|(k+1)r-k|}{r-1}\right\}.$$

Therefore, $||x_m + h(k+1,r,m)|| - ||-x_m + h(k,r,m)||$

$$= \sqrt{n}\left\{-m + \frac{m(r-1)}{r-1}\right\} = 0.$$

The same results follows if we replace h by -h throughout.

THEOREM 6.- Let $L_{\Phi_1}(\mathbb{R}^n,\mu)$ and $L_{\Phi_2}(\mathbb{R}^n,\nu)$ be Orlicz spaces defined by the Young functions $\Phi_1(u)$, $\Phi_2(u)$ that satisfy the (δ_2,Δ_2) condition, where $\mu = e^{a||x||}$ and $\nu = e^{b||x||}$. Then, in order that there should exist a nonzero, translation invariant, bounded operator

$$T : L_{\Phi_1}(\mathbb{R}^n,\mu) \to L_{\Phi_2}(\mathbb{R}^n,\nu),$$

it is necessary that, for any natural number $s \geq 1$ and any real number $r, r > 1$,

$$\liminf_{m\to\infty} \frac{K_1(1 + \sum_{k=1}^{s} e^{aH(k,r,m)})}{N_2(1 + \sum_{k=1}^{s} e^{b/r\,H(k,r,m)})} \geq 1,$$

where the expression $H(k,r,m)$ is defined as in Lemma 4.

Proof: Let $I(0,\frac{m}{2})$ and $h(k,r,m)$ be as in the previous Lemma. For any function $f(x)$ we write $f_m(x)$ for $\chi_{m(x)/2}f(x)$, where $\chi_{m/2}(x)$ stands for the characteristic function of $I(0,\frac{m}{2})$.

On account of part (b) of the previous Lemma we have that, for any $f \in L_{\Phi_1}(\mathbb{R}^n,\mu)$

$$\int_{\mathbb{R}^n} \Phi_1(|f_m(x) + \sum_{k=1}^{s} \tau(h(k,r,m))f_m(x)|)d\mu =$$

$$= \int_{\mathbb{R}^n} \Phi_1(|f_m(x)|)d\mu + \sum_{k=1}^{s} \Phi_1(|f_m(x)|)e^{a||x+h(k,r,m)||}dx$$

$$\leq (1 + \sum_{k=1}^{s} e^{aH(k,r,m)}) \int_{\mathbb{R}^n} \Phi_1(|f_m|)d\mu. \qquad (1)$$

Similarly,

$$\int_{\mathbb{R}^n} \Phi_2(|(Tf_m)_m + \sum_{k=1}^{s} \tau(h(k,r,m))(Tf_m)_m|)d\nu =$$

$$= \int_{\mathbb{R}^n} \Phi_2(|(Tf_m)_m|)d\nu + \sum_{k=1}^{s} \int_{\mathbb{R}^n} \Phi_2(|\tau(h(k,r,m))(Tf_m)_m|)d\nu,$$

and, by virtue of part (a) of the previous Lemma, this expression is greater than or equal to

$$(1 + \sum_{k=1}^{s} e^{\frac{b}{r}H(k,r,m)}) \int_{\mathbb{R}^n} \Phi_2(|(Tf_m)_m|)d\nu. \qquad (2)$$

From (2) and (3) above, it follows that

$$N_2(1 + \sum_{k=1}^{s} e^{\frac{b}{r}H(k,r,m)})||Tf_m)_m||_{L_{\Phi_2}} \leq$$

$$\leq ||(Tf_m)_m + \sum_{k=1}^{s} \tau(h(k,r,m))(Tf_m)_m||_{L_{\Phi_2}}$$

$$\leq ||Tf_m + \sum_{k=1}^{s} \tau(h(k,r,m))Tf_m||_{L_{\Phi_2}}$$

$$\leq ||T|| \; ||f_m + \sum_{k=1}^{s} \tau(h(k,r,m))f_m||_{L_{\Phi_1}},$$

so that

$$||(Tf_m)_m||_{L_{\Phi_2}} \leq ||T|| \frac{K_1(1 + \sum_{k=1}^{s} e^{aH(k,r,m)})}{N_2(1 + \sum_{k=1}^{s} e^{b/r\,H(k,r,m)})} ||f_m||_{\Phi}, \qquad (3).$$

We prove next that $||(Tf_m)_m||_{L_{\Phi_2}} \to ||Tf||_{L_{\Phi_2}}$ as $m \to \infty$.

In fact.

$$||(Tf_m)_m - Tf||_{L_{\Phi_2}} \leq ||(Tf_m)_m - (Tf)_m||_{L_{\Phi_2}} + ||(Tf)_m - Tf||_{L_{\Phi_2}}$$

$$\leq ||Tf_m - Tf||_{L_{\Phi_2}} + ||(Tf)_m - Tf||_{L_{\Phi_2}}$$

$$\leq ||T|| \; ||f_m - f||_{L_{\Phi_1}} + ||(Tf)_m - Tf||_{L_{\Phi_2}}$$

and this expression tends to 0 as $m \to \infty$.

Therefore, by passing to the limit as $m \to \infty$ on both sides of the expression (3) above, we see that, if for some natural number s and a real number $r > 1$,

$$\liminf_{m\to\infty} \frac{K_1(1 + \sum_{k=1}^{s} e^{aH(k,r,m)})}{N_2(1 + \sum_{k=1}^{s} e^{b/r\, H(k,r,m)})} < 1,$$

then $T = 0$.

Thus, in order that T be different from zero it is necessary that

$$\liminf_{m\to\infty} \frac{K_1(1 + \sum_{k=1}^{s} e^{aH(k,r,m)})}{N_2(1 + \sum_{k=1}^{s} e^{b/r\, H(k,r,m)})} \geq 1$$

for any natural number s and any real $r > 1$.

In particular, if $a = b = 0$, then the condition above becomes

$$\frac{K_1(1+s)}{N_2(1+s)} \geq 1.$$

If $\Phi_1(u) = u^p, p > 1$ and $\Phi_2(u) = u^q$, $q > 1$ the condition is $(1+s)^{1/p} > (1+s)^{1/q}$, that is $q \geq p$. (Hörmander [5] p.96).

If $a \neq 0$, $b \neq 0$, $\Phi_1(u) = u^p$, $p>1$ and $\Phi_2(u) = u^q, q > 1$, then the condition that T be different from zero is $\frac{a}{p} - \frac{b}{rq} \geq 0$ for any $r > 1$, as becomes apparent from writting out explicitly the condition found in the theorem above. In this case we see that $\frac{a}{p} \geq \frac{b}{q}$ (Cooper [2],p.44).

A more clear picture emerges when we consider the same problem by replacing the spaces L_Φ with Orlicz spaces ℓ_Φ.

Let us recall that, given a Young function Φ, the indices α_Φ an β_Φ are defined as follows (see [8])

$$\alpha_\Phi = \sup \left\{ p > 0; \sup_{0<x,t\leq 1} \frac{\Phi(tx)}{\Phi(t)x^p} < \infty \right\}$$

$$\beta_\Phi = \inf \left\{ p > 0; \inf_{0<x,t\leq 1} \frac{\Phi(tx)}{\Phi(t)x^p} > 0 \right\}.$$

We now prove:

<u>LEMMA 5.-</u> Let us write F(t) for any of the functions N,K,Φ. Let α,β be as in theorem 1. Then, the interval $[\alpha_F,\beta_F]$ is contained in $[\alpha,\beta]$.

Proof: Let $\varepsilon > 0$, then for some $a_\varepsilon > 0$,

$$\frac{F(\lambda t)}{F(t)\lambda^{\alpha-\varepsilon}} \leq \frac{F(t)\ N(\lambda)}{F(t)\lambda^{\alpha-\varepsilon}} \leq \frac{\lambda^{\alpha-\varepsilon}}{\lambda^{\alpha-\varepsilon}}, \quad \lambda \in [0,\ \alpha_\varepsilon],$$

that is $\alpha-\varepsilon \leq \alpha_F$ and so $\alpha \leq \alpha_F$.

Also,

$$\frac{F(\lambda t)}{F(t)\lambda^{\beta+\varepsilon}} \geq \frac{F(t)\ K(\lambda)}{F(t)\lambda^{\beta+\varepsilon}} \geq \frac{\lambda^{\beta+\varepsilon}}{\lambda^{\beta+\varepsilon}},\ \lambda \in (0, \frac{1}{b_\varepsilon}).$$

It follows that $\beta \geq \beta_F$.

Let $\Phi(u)$, $u \geq 0$, be an N-function such that $\sup\limits_{0<u<\infty} \frac{\Phi(u\lambda)}{\Phi(u)}$ is attained on the interval $[0,1]$. Then

$$\sup_{0<\lambda_1 t} \frac{\Phi(\lambda t)}{\Phi(\lambda)t^{\alpha+\varepsilon}} \geq \sup_{0<x\leq 1} \left\{ \sup_{0<x\leq 1} \frac{\Phi(t\lambda)}{\Phi(\lambda)t^{\alpha+\varepsilon}} \right\}$$

$$= \sup_{0<t\leq 1} \frac{N(t)}{t^{\alpha+\varepsilon}} \geq \sup_{0<t\leq 1} \frac{t^{\alpha}}{t^{\alpha+\varepsilon}} = \infty,$$

that is, $\alpha \geq \alpha_\Phi$. It now follows from the above Lemma that $\alpha = \alpha_\Phi$.

A similar calculation shows us that, if $\inf\limits_{0<u<\infty} \frac{\Phi(u\lambda)}{\Phi(u)}$ is attained on $[0,1]$, then $\beta = \beta_\Phi$.

These conditions hold for $N(\Phi;x)$ and $K(\Phi,x)$ respectively. We thus have that $\alpha = \alpha_N$ and $\beta = \beta_K$ also

$$\lim_{t\to 0} \frac{L\,N(t)}{L\,t} = \alpha_N,$$

and $$\lim_{t\to\infty} \frac{L\,N(t)}{L\,t} = \beta = \lim_{t\to\infty} \frac{L\,\frac{1}{K(\frac{1}{t})}}{L\,t}$$

$$= \lim_{t\to 0} \frac{L\,K(t)}{L\,t} = \beta_k$$

LEMMA 6.- Let N(t) be submultiplicative and K(t) super multiplicative functions defined on on $[0,1]$; and such that

$$N(0) = K(0) = 0,$$

$$N(1) = K(1) = 1.$$

If N(t) and K(t) are not equivalent in any set $[0,\delta]$ with $\delta \leq 1$, then we must have that either

$$N(t) < K(t),\ x \in (0,1)$$

or

$$K(t) \leq N(t).\ x \in [0,1].$$

Proof: Assume that neither case hold, then we have that, for some decreasing sequence $\{t_n\}_{n=1}^{\infty}$, with $t_1 = 1$ and $\lim_{n\to\infty} t_n = 0$,

$$N(t_n) = K(t_n),\ n \in N.$$

The function $F(t) = \frac{t\,N(t)}{K(t)}$, $t > 0$ is submultiplicative and

$$\alpha_F \geq \lim_{t\to 0} \inf \frac{tF'(t)}{F(t)} \geq 1 + \lim_{t\to 0} \inf \frac{tN'(t)}{N(t)}$$

$$- \lim_{t\to 0} \sup \frac{tK'(t)}{K(t)}$$

$$= 1 + \alpha_N - \beta_K.$$

From $N(t_n) = K(t_n)$ we deduce that

$$\alpha_N = \lim_{t\to 0} \frac{L\,N(t)}{L\,t} = \lim_{t\to 0} \frac{L\,K(t)}{L\,t} = \beta_K \;;$$

that is $\alpha_F \geq 1$. Assume $\alpha_F > 1$, then we deduce from theorem 1 that given $\varepsilon > 0$ there is $\delta > 0$ such that

$$t^{\alpha_F} \leq \frac{t\,N(t)}{K(t)} \leq t^{\alpha_F - \varepsilon} < t, \quad t \in (0,\delta).$$

By placing $t = t_n$, we get

$$t_n \leq t_n^{\alpha_F - \varepsilon} < t_n \,.$$

Contradiction; we must have that $\alpha_F = 1$. This in turn implies that

$$\frac{t\,N(t)}{K(t)} \geq t, \quad t < 1,$$

and so $N(t) \geq K(t)$, $t < 1$.

Proceeding eliminste as in theorem 6, we see that a necessary condition that there exist a linear bounded, translation invariant operator T: $\ell_{\Phi_1} \longrightarrow \ell_{\Phi_2}$ is that

$$\lim_{x \to 0} \inf \frac{K_1(x)}{N_2(x)} \geq 1.$$

In the following Theorem, by ℓ_{Φ} we mean the Banach space of all sequences $\{a_n\}_{n=-\infty}^{n=+\infty}$ such that $\sum_{n=-\infty}^{n=+\infty} \Phi(|a_n|) < \infty$.

<u>THEOREM 7.-</u> A necessary and sufficient condition that there exist a linear bounded, translation invariant operator.

$$T : \ell_{\Phi_1} \longrightarrow \ell_{\Phi_2}$$

is that $N_2(x) < K_1(x)$, for all x in (0,1).

Proof: If N_2 and K_1 are equivalent in $|0,1|$ then there is nothing to prove. Otherwise, according to the previous Lemma que have that either

$$K_1(x) \leq N_2(x) \text{ or } K_1(x) > N_2(x) \text{ on } |0,1|.$$

Assume the first case. If

$$\lim_{x\to 0} \inf \frac{K_1(x)}{N_2(x)} = 1.$$

Then, given $\varepsilon > 0$ there is $\delta > 0$ such that

$$\frac{K_1(x)}{N_2(x)} > 1 - \varepsilon, \quad x \in (0,\delta);$$

that is $(1-\varepsilon)\, N_2(x) < K_1(x) \leq N_2(x)$, $x \in (0,\delta)$, and since $\frac{K_1(x)}{N_2(x)}$ is bounded, and bounded away from zero on $[\delta,1]$, we see that $K_1(x)$ and $N_2(x)$ are equivalent on $[0,1]$. Contradiction; we must have then that in this case

$$\lim_{x\to 0} \inf \frac{K_1(x)}{N_2(x)} < 1.$$

In the second case it is apparent that

$$\lim_{x\to 0} \inf \frac{K_1(x)}{N_2(x)} \geq 1.$$

Also, since $N_1(x) < K_2(x)$, $x \in (0,1)$ implies that $\Phi_1(x) < \Phi_2(x)$, $x \in (0,1)$; we can see that the identity $I : \ell_{\Phi_1} \longrightarrow \ell_{\Phi_2}$ is continuous.

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