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ON MODULI OF SMOOTHNESS OF BANACH SPACES

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1. INTRODUCTION.

In the geometric theory of Banach spaces the notions of moduli of convexity and smoothness play a very important and significant role. They appeared to be useful tools in characterization of Banach spaces with respect to the rotundity and to the smoothness of their unit balls. By this regard these notions are applicable in many theories concerning Banach spaces (cf. [1,9,5,6,11,12]).

This paper is devoted mainly to the notion of a modulus of smoothness. The notion of such a modulus was introduced by Day [3] and next was examined and applied by several mathemticians [4,5,10,12,13]. It is worth while to mention that there exists the nice relationship between a modulus of convexity of a given Banach space E and a modulus of smoothness of its dual space E^*[10]. But on the other hand it is very difficult to find some relations among these moduli in the same space E.

In this paper we introduce two new moduli of smoothness. One of them is defined in the manner very close to the modulus of convexity. By this regard it seems to be more natural than the classical modulus

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due to Day . We prove a few properties of the given moduli and a few relations among them. Moreover, we calculate exact formulas for these moduli in the case of some Banach spaces.

Actually we were not able to salve many interesting problems which appear in connection with moduli of smoothness considered here (see: Final Remarks).

2. NOTATION, DEFINITIONS AND PRELIMINARY RESULTS.

Let $(E, \|\cdot\|)$ be a given Banach space. By B,S we will denote the closed ball and the sphere in the space E with the center Θ and radius 1. Let us recall that *the modulus of convexity* of the space E is the function $\delta : \langle 0,2\rangle \to \langle 0,1\rangle$ defined by

$$\delta(\varepsilon) = \inf \left[1 - \frac{\|x+y\|}{2} : x,y \in S,\ \|x-y\| = \varepsilon \right]$$

[2]. The function δ is continuous on the interva $\langle 0,2)$, nondecreasing on the interval $\langle 0,2\rangle$ and increasing on the interval $\langle \varepsilon_0, 2\rangle$, where $\varepsilon_0 = \sup[\varepsilon : \delta(\varepsilon) = 0]$ is the so-called *characteristic of convexity* of the space E. The space is called *uniformly convex* provided $\varepsilon_0 = 0$. For other properties of the modulus of convexity we refer to [5,6,12,13].

New let us recolled two definitions of a uniformly smooth space [3,8].

DEFINITION 1. We say that E is uniformly smooth if for each $\varepsilon > 0$ there is $\eta > 0$ for which $\|x\| \geq 1$, $\|y\| \geq 1$ and $\|x-y\| \leq \eta$ implies

$\|x\| + \|y\| - \|x+y\| \leq \varepsilon \|x-y\|$.

DEFINITION 2. A space E is referred to as uniformly smooth if for any $\varepsilon > 0$ there is $\tau > 0$ such that $\|x\| = 1$, $\|y\| \leq \tau$ implies $\|x+y\| + \|x-y\| \leq 2 + \varepsilon \|y\|$.

It may be shown that the above definitions are equivalent [8].

By means of Definition 2 the notion of *the modulus of smoothness* of a space E was introduced by Day [3] in the following way

$$\delta(\varepsilon) = \sup\left[\frac{\|x+y\| + \|x-y\| - 2}{2} : \|x\| = 1, \|y\| \leq \varepsilon\right],$$

where $\varepsilon \in \langle 0,1\rangle$. The function ρ possesses several properties, for example it is continuous and convex on the interval $\langle 0,1\rangle$ and $\sqrt{1+\varepsilon^2} - 1 \leq \rho(\varepsilon) \leq \varepsilon$ for $\varepsilon \in \langle 0,1\rangle$ [10]. Moreover, it may be shown that E is uniformly smooth iff $\lim_{\varepsilon\to 0} \rho(\varepsilon)/\varepsilon = 0$.

Now let us remark that the assumptions $\|x\| \geq 1$, $\|y\| \geq 1$ in Definition 1 seem to have no significance. In fact, we may accept the following

DEFINITION 3. A space E is called uniformly smooth if for each $\varepsilon > 0$ there exists $\eta > 0$ such that for any $x,y \in S$ and $\|x-y\| \leq \eta$ the inequality $1 - \frac{\|x+y\|}{2} \leq \varepsilon \|x-y\|$ holds true.

We show that this definition is equivalent to those previous given. Actually if E is uniformy smooth in the sense of Definition 1 then it is also uniformly smooth in the sense of Definition 3. Conversely, let us suppose that E is uniformly smooth with respect to Definition 3 and it is not in the sense of Definition 1. Then there exist $\varepsilon_0 > 0$ and two sequences (x_n), (y_n) such that $\|x_n\| \geq 1$, $\|y_n\| \geq 1$, $\|x_n - y_n\| \to 0$ as n tends to infinity and

$$(1) \qquad \|x_n\| + \|y_n\| - \|x_n + y_n\| > \varepsilon_0 \|x_n - y_n\|$$

for all n=1,2,... Without loss of generality we may assume that $\|y_n\| \leq \|x_n\|$ for $n \in N$. Further, putting $U_n = x_n / \|x_n\|$, $V_n = y_n / \|y_n\|$ we have $\|U_n\| = \|V_n\| = 1$ and

$$\|U_n - V_n\| \leq \left\| \frac{x_n}{\|x_n\|} - \frac{y_n}{\|x_n\|} + \frac{y_n}{\|x_n\|} - \frac{y_n}{\|y_n\|} \right\| \leq \frac{2}{\|x_n\|} \|x_n - y_n\|$$

what implies that $\|U_n - V_n\| \to 0$ as $n \to \infty$ and additionally

$$(2) \qquad \|x_n - y_n\| \geq \frac{\|x_n\|}{2} \|U_n - V_n\| , \; n=1,2,...$$

Next, by virtue of (1) we get

$$2 - \|U_n + V_n\| = \left\| \frac{x_n}{\|x_n\|} \right\| + \left\| \frac{y_n}{\|y_n\|} \right\| - \left\| \frac{x_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right\| =$$

$$= \frac{1}{\|x_n\|} \|x_n\| + \frac{1}{\|x_n\|} \|y_n\| - \frac{1}{\|x_n\|} \|y_n\| + \frac{1}{\|y_n\|} \|y_n\| -$$

$$- \left\| \left(\frac{x_n}{\|x_n\|} + \frac{y_n}{\|x_n\|} \right) + \left(- \frac{y_n}{\|x_n\|} + \frac{y_n}{\|y_n\|} \right) \right\| \geq \frac{1}{\|x_n\|} \left(\|x_n\| + \|y_n\| \right) +$$

$$+ \|y_n\| \left(\frac{1}{\|y_n\|} - \frac{1}{\|x_n\|} \right) - \frac{1}{\|x_n\|} \|x_n + y_n\| - \frac{1}{\|x_n\|} \Big| \|x_n\| - \|y_n\| \Big| =$$

$$= \frac{1}{\|x_n\|} \left\{ \|x_n\| + \|y_n\| - \|x_n+y_n\| \right\} + \|y_n\| \left(\frac{1}{\|y_n\|} - \frac{1}{\|x_n\|} \right) -$$

$$- \frac{1}{\|x_n\|} \Big| \|x_n\| - \|y_n\| \Big| > \frac{1}{\|x_n\|} \varepsilon_0 \|x_n - y_n\| + 1 - \frac{\|y_n\|}{\|x_n\|} - 1 +$$

$$+ \frac{\|y_n\|}{\|x_n\|} = \frac{\varepsilon_0}{\|x_n\|} \|x_n - y_n\| ,$$

what in view of (2) implies

$$2 - \|u_n + v_n\| > \frac{\varepsilon_0}{2} \|u_n - v_n\| , \quad n=1,2,\ldots\ldots$$

But this inequality contradicts to our assumption and the proof is complete.

In what follows, keeping in mind Definition 3, we may define the another modulus of smoothness of a space E by the formula

$$\rho_1(\varepsilon) = \sup \left[1 - \frac{\|x+y\|}{2} : x,y \in S, \|x-y\| = \varepsilon \right] , \quad \varepsilon \in \langle 0,2 \rangle .$$

Actually a space E is uniformly smooth iff $\lim_{\varepsilon \to 0} \rho_1(\varepsilon) / \varepsilon = 0$.

Further, let us introduce another one function being kind of a modulus of smoothness, defined in the following way

$$\rho_2(\varepsilon) = \sup \left[\min\{ \|x+y\| , \|x-y\| \} - 1 : \|x\| = 1, \|y\| \le \varepsilon \right] , \quad \varepsilon \in \langle 0,1 \rangle .$$

In the next section we show that ρ_2 may be also treated as a modulus of smoothness of a Banach space.

REMARK. In the situation when we have to distinguish different Banach spaces we will write $\rho_1(\varepsilon,E_1)$, $\rho_2(\varepsilon,E_1)$ or $\rho_{E_2}(\varepsilon)$, for example.

3. FURTHER RESULTS.

In this section we list some properties of the previous introduced functions ρ_1 and ρ_2 in connection with the classical modulus ρ and the modulus of convexity δ. At the beginning let us notice the simple relations

$$\delta(\varepsilon) \leq \rho_1(\varepsilon) \ , \quad \varepsilon \in \langle 0,2\rangle \ ,$$

$$\rho_2(\varepsilon) \leq \rho(\varepsilon) \ , \quad \varepsilon \in \langle 0,1\rangle$$

being valid for an arbitrary Banach space E.

It is easily to compute the exact formulas for ρ_1, ρ_2 in the case of a Hilbert space H. Indeed, using the parallelogram law we obtain

$$\rho_1(\varepsilon,H) = \delta_H(\varepsilon) = 1 - \sqrt{1-\frac{\varepsilon^2}{4}} \ , \ \varepsilon \in \langle 0,2\rangle$$

$$\rho_2(\varepsilon,H) = \rho_H(\varepsilon) = \sqrt{1+\varepsilon^2} - 1, \quad \varepsilon \in \langle 0,1\rangle \ .$$

Furthermore, we have the following lemma which is true for on arbitrary Banach space E.

LEMMA 1. The function ρ_1 is increasing on the interval $\langle 0,2\rangle$ and

$\rho_1(\varepsilon) \leq \varepsilon/2$ for ony $\varepsilon \in \langle 0,2\rangle$.

The proof of the first part requires only some technical argumentations and is therefore omitted. Moreover, we have

$$\rho_1(\varepsilon) = \sup\left[\frac{2\|x\| - \|x+y\|}{2} : x,y \in S,\ \|x-y\| = \varepsilon\right] \leq$$

$$\leq \sup\left[\frac{\|x+y\| + \|x-y\| - \|x+y\|}{2} : x,y \in S,\ \|x-y\| = \varepsilon\right] = \frac{\varepsilon}{2}$$

what proves the second part of our lemma.

It is worth to mention that in the case $C\langle a,b\rangle$ it is easily to calculate that $\rho_1(\varepsilon) = \varepsilon/2$ and $\rho_2(\varepsilon) = \rho(\varepsilon) = \varepsilon$. On the one hond this show that the estimate from Lemma 1 and the earlier given estimate $\rho(\varepsilon) \leq \varepsilon$ are exact. On the other hand the example given by Poulsen [5] suggest that the equality $\rho_2 = \rho$ is not valid in the case of an arbitrary Banach space.

LEMMA 2. The function ρ_2 is nondecreasing on the interval $\langle 0,1\rangle$ and $\rho_2(\varepsilon) \leq \varepsilon$ for $\varepsilon \in \langle 0,1\rangle$.

The proof is obvious.

In the sequel we will also use the following lemma being analogous to Lemma 4 from [5].

LEMMA 3. If $\|x\| \geq 1$, $\|y\| \geq 1$ and $\|x-y\| \leq 2$ then

$$\|x+y\| \geq 2\,(1 - \rho_1(\|x-y\|)).$$

The proof is similar to the proof of the above mentioned lemma from [5] and we omit it.

Now we prove two theorems showing some relations among discussed moduli.

THEOREM 1. $\rho_1(\varepsilon) \leq (1-\rho_1(\varepsilon))\ \rho(\frac{\varepsilon}{2(1-\rho_1(\varepsilon))})$.

PROOF. Let $\varepsilon \in \langle 0,2\rangle$ be fixed. Take on arbitrary $\eta > 0$ (sufficiently small) and $x,y \in S$, $\|x-y\| \leq \varepsilon$ such that

$$\rho_1(\varepsilon) - \eta \leq 1 - \frac{\|x+y\|}{2} \leq \rho_1(\varepsilon).$$

Then

$$\frac{\|x-y\|}{\|x+y\|} \leq \frac{\varepsilon}{2(1-\rho_1(\varepsilon))} .$$

Further we get

$$\rho_1(\varepsilon)-\eta \leq \frac{\|x\|+\|y\|-\|x+y\|}{2} = \frac{\|x+y\|}{2}\left(\frac{1}{2}\left\|\frac{x+y}{\|x+y\|}+\frac{x-y}{\|x+y\|}\right\| + \right.$$

$$\left. + \frac{1}{2}\left\|\frac{x+y}{\|x+y\|}-\frac{x-y}{\|x+y\|}\right\|-1\right) \leq (1-\rho_1(\varepsilon)-\eta)\sup\left[\frac{\|u+v\|+\|u-v\|-2}{2}:\right.$$

$$\left. \|u\| = 1,\ \|v\| \leq \frac{\varepsilon}{2(1-\delta_1(\varepsilon))}\right]$$

what implies the desired inequality.

THEOREM 2. $\rho_1(\varepsilon) = (1-\rho_1(\varepsilon)\ \rho_2(\frac{\varepsilon}{2(1-\rho_1(\varepsilon))})$.

The below given proof is patterned on the proof of Lemma 6 from [5].

Without lass of generality we may assume that dim $E < \infty$. Further, take $\varepsilon \in \langle 0,2\rangle$ and $x,y \in S$, $\|x-y\| = \varepsilon$ such that $\|x+y\| = 2(1-\rho_1(\varepsilon))$. Denote by $s = \varepsilon/2(1-\rho_1(\varepsilon))$ and put $u=(x+y)/\|x+y\|$, $v=(x-y)/\|x+y\|$. Then $\|u\| = 1$, $\|v\| = s$ and $\|u+v\| = \|u-v\| = 1/(1-\rho_1(\varepsilon))$. Hence $\min\{\|u+v\|, \|u-v\|\} = 1/(1-\rho_1(\varepsilon))$ and finally $\rho_2(s) \geq (1/(1-\rho_1(s)))-1=$ $= \rho_1(\varepsilon)/(1-\rho_1(\varepsilon))$. Thus we get

$$(3) \qquad \rho_1(\varepsilon) \leq (1-\rho_1(\varepsilon))\, \rho_2\left(\frac{\varepsilon}{2(1-\rho_1(\varepsilon))}\right).$$

In order to prove the reverse inequality let us choose $u \in S$ and v such that $\|v\| = s$ and $\min\{\|u+v\|, \|u-v\|\} = 1 + \rho_2(s) = \frac{1}{a}$. Putting $x = a(u+v)$, $y = a(u-v)$ we have $\|x\| \geq 1$, $\|y\| \geq 1$ and $\|x-y\| = 2as$.

Then using Lemma 3 we obtain

$$\rho_1(2as) \geq 1 - \frac{\|x+y\|}{2} = 1 - a = \frac{\rho_2(s)}{1+\rho_2(s)}.$$

The above inequality together with (3) implies

$$\rho_1(2as) \geq \rho_1(\varepsilon)$$

what in view of Lemma 1 allows us to infer

$$2as \geq \varepsilon .$$

Hence

$$\rho_2(s) = \frac{1}{a} - 1 \leq \frac{2s}{\varepsilon} - 1 = \frac{\rho 1(\varepsilon)}{1-\rho_1(\varepsilon)}$$

what completes the proof.

Let us observe that the equality from Theorem 2 permits us to deduce the following inequalities

$$\frac{1}{2}\rho_2(\varepsilon) \leq \rho_1(\varepsilon) \leq \rho_2\left(\frac{\varepsilon}{2-\varepsilon}\right).$$

Hence we obtain the following

COROLLARY. A Banach space is uniformly smooth if and only if

$$\lim_{\varepsilon\to 0} \rho_2(\varepsilon)/\varepsilon = 0.$$

4. THE CASE OF THE SPACE ℓ^p.

We are going now to calculate the exact formula for the modulus ρ_1 in the cas of ℓ^p space, $p > 1$. We will use the following inequalities due to Hanner [7,12]:

LEMMA 4. Let $x,y \in \ell^p$. Then for $p > 2$

$$(\|x\|+\|y\|)^p + \left|\|x\|-\|y\|\right|^p \geq \|x+y\|^p+\|x-y\|^p \geq 2\|x\|^p+2\|y\|^p.$$

If $1 < p \leq 2$ then the inequalities have the converse signs.

Now, putting in the above inequalities $x+y$ instead of x and $x-y$ instead of y we derive.

$$(\|x+y\|^p+\|x-y\|^p)+\left|\|x+y\|-\|x-y\|\right|^p \geq 2^p(\|x\|^p+\|y\|^p) \geq$$

$$\geq 2(\|x+y\|^p+\|x-y\|^p).$$

Hence, dividing all inequalities by 2^p and considering the case $1 < p \leq 2$ we have

$$(4) \qquad \left(\frac{\|x+y\|}{2} + \frac{\|x-y\|}{2} \right)^p + \left| \frac{\|x+y\|}{2} - \frac{\|x-y\|}{2} \right|^p \leq \|x\|^p + \|y\|^p \leq$$

$$\leq \quad \left\{ \left| \frac{\|x+y\|}{2} \right|^p + \left(\frac{\|x-y\|}{2} \right)^p \right\} .$$

Let us consider firstly the case $1 < p \leq 2$. Take an arbitrary $\varepsilon \in <0,2>$. Without loss of generality (cf. [5,12]) we may assume that there exist $x,y \in S_{\ell^p}$, $\|x-y\| = \varepsilon$ with the property

$$\rho_1(\varepsilon) = 1 - \frac{\|x+y\|}{2} \quad ,$$

what gives

$$\frac{\|x+y\|}{2} = 1 - \rho_1(\varepsilon) \ .$$

Substituting the above relationship to (4) and using only the right inequality we get

$$2 \leq 2 \ \{(1-\rho_1(\varepsilon))^p + (\frac{\varepsilon}{2})^p\} \ .$$

Hence

$$(5) \qquad \rho_1(\varepsilon) \leq 1 - (1-(\frac{\varepsilon}{2})^p)^{1/p}.$$

Now we show that the equality in (4) is atlained. Indeed, let us take

$$x = ((1-(\frac{\varepsilon}{2})^p)^{1/p}, \frac{\varepsilon}{2}, 0,0,\dots.)$$

$$y = ((1-(\frac{\varepsilon}{2})^p)^{1/p}, -\frac{\varepsilon}{2}, 0,0,\dots).$$

Then $\|x\| = \|y\| = 1$, $\|x-y\| = \varepsilon$ and finally

$$\|x+y\| = 2\ (1-(\tfrac{\varepsilon}{2})^p)^{1/p}\ .$$

Hence we infer

$$2(1-(\tfrac{\varepsilon}{2})^p)^{1/p} = x+y \geq 2\ (1-\rho_1(\varepsilon))$$

what gives

(6) $$\rho_1(\varepsilon) \geq 1-(1-(\tfrac{\varepsilon}{2})^p)^{1/p}\ .$$

Now combining (5) and (6) we have

(7) $$\rho_1(\varepsilon,\ell^p) = 1-(1-(\tfrac{\varepsilon}{2})^p)^{1/p}$$

for $1 < p \leq 2$.

Similarly, using the left hand side inequality (4) (with the reverse sign, of course) we can compute that for $p > 2$

(8) $$(1-\rho_1(\varepsilon)-\tfrac{\varepsilon}{2})^p + \left|\ 1-\rho_1(\varepsilon)\ -\tfrac{\varepsilon}{2}\right|^p = 2.$$

Let us pay attention to the fact that the equality (7) implies

(9) $$\rho_1(\varepsilon,\ell^p) > \delta_{\ell^p}(\varepsilon)$$

for $1 < p < 2$, so that in this case we can also write (cf.[13])

(10) $$\rho_1(\varepsilon,\ell^p) > \rho_1(\varepsilon,\ell^2) = \delta(\varepsilon,\ell^2) = 1-(1-(\tfrac{\varepsilon}{2})^2)^{1/2}.$$

It may be also shown that for $2 < p$ the following inequality holds true

$$(11) \qquad \rho_1(\varepsilon,\ell^p) > \delta_{\ell^p}(\varepsilon) = 1-(1-(\tfrac{\varepsilon}{2})^p)^{1/p}.$$

In the case p=2 we have the equality $\rho_1(\varepsilon,\ell^2) = \delta_{\ell^2}(\varepsilon)$.

5. FINAL REMARKS

At first, let us pay attention to some facts connected with the inequalities (9), (10) and (11). In the section 2 we have noticed that $\rho_1(\varepsilon) \geq \delta(\varepsilon)$ for any Banach space E. The inequalities (9) and (11) show that in same cases the inequality sign may be strong. That is caused by the fact that for $p > 1$, $p \neq 2$ the space ℓ^p does not have such "nice" geometrical structure as the Hilbert space (ℓ^2, for example). The above discussion motivates the introduction of the notion of *the deformation* of a Banach space.

DEFINITION 4. For an arbitrary Banach space E the function $d:\langle 0,2\rangle \to \langle 0,1\rangle$, defined by the formula

$$d(\varepsilon) = \rho_1(\varepsilon) - \delta(\varepsilon)$$

will be called the deformation of the space E.

Now we raise a few of open problems connected with the considerations of this paper.

PROBLEM 1. Is the function ρ_1 continuous? The same problem for the function ρ_2.

PROBLEM 2. Are the functions ρ_1 and ρ_2 convex?. Let us mention that Liokoumovich [9] showed that it is not always true for the case of the function δ.

PROBLEM 3. It is well known [10,13] that for any Banach space E the following evaluations are valid

$$\rho_E(\varepsilon) \geq \rho_H(\varepsilon) = \sqrt{1+\varepsilon^2} - 1$$

$$\rho_E(\varepsilon) \leq \rho_H(\varepsilon) = 1- \sqrt{1-(\frac{\varepsilon}{2})^2},$$

where H denotes a Hilbert space. That suggests, for example, the validity of the following inequalities

$$\rho_1(\varepsilon,E) \geq \rho_1(\varepsilon,H) = \rho_H(\varepsilon)$$

$$\rho_2(\varepsilon,H) \geq \rho_H(\varepsilon)$$

(compare the inequality (10)). Is that true?.

PROBLEM 4. Compute the exact formulas for the moduli ρ_1 and ρ_2 in the case of other Banach space, for example L^p, the Day's space and the James space (cf. [12]).

PROBLEM 5. Examine the properties of the deformation $d(\varepsilon)$. Let us remark that this function allows us to classify Banach spaces with respect to "beauty" their geometrical structure. For example

$d(\varepsilon, C\langle a,b\rangle) = \varepsilon/2$ for $\varepsilon \in \langle 0,2\rangle$ (very "bad" space) and $d_H(\varepsilon) \equiv 0$, so that H is very "nice" space.

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