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LIST OF SYMBOLS

δ	delta
Δ	capital delta
ε	epsilon
λ	lambda
μ	mu
ξ	csi
1	one
l	elle
0	zero
o	smoll oh

SOME MATHEMATICAL SYMBOLS

$\cap$	intersection
$\cup$	union
$\sum$	sumatory
ε	belongs to

0. INTRODUCTION. In this paper we consider the system

$$u_i' = u_i\, F_i(t,u), \quad u = (u_1,\ldots,u_n); \quad 1 \le i \le n \tag{0.1}$$

where, from now on, $F:= (F_1,\ldots,F_n): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ denotes a continuous function which is locally Lipschitz with respect to the u variable. These systems arise naturally in population biology.

If F is "diagonal dominant" ; i.e.

$$-c_i \frac{\partial F_i}{\partial x_i}(t,x) \ge m + \sum_{j \in J_i} c_j \left| \frac{\partial F_j}{\partial x_i}(t,x) \right| ; \quad 1 \le i \le n , \tag{0.2}$$

($t \in \mathbb{R}$, $x > 0$) for some positive constants $m, c_1,\ldots,c_n$; $J_i = \{1,\ldots, i-1, i+1,\ldots,n\}$; and the system (0.1) has a positive solution $v^\circ = (v_1^\circ,\ldots,v_n^\circ)$ defined and bounded in $[0,\infty)$, then

$$u(t) - v^\circ(t) \to 0 \quad \text{as} \quad t \to +\infty \tag{0.3}$$

for any solution u of (0.1) with $u(0) > 0$. Moreover, this system has at most one solution u° defined in $\mathbb{R}$ whose components are bounded above and below by positive constants. Further if F is almost periodic and $v_1^\circ,\ldots,v_n^\circ$ are bounded below in $[0,\infty)$ by positive constants, then such u° exists and u° is almos periodic. A parallel result holds for the periodic case. See also remark 3.3. below.

This result has the advantage that we do not assume any sign condition in $\partial F_i/\partial x_j$ for $i \neq j$. So we can study simultaneously several population models: Competing species (competitive systems [10]), predator-prey and cooperative models, [8] pag. 36 (cooperative systems [10]).

Condition (0.2) is quite restrictive but can be applied successfully when $F_i(t,u)$ has the form:

$$F_i(t,u) = a_i(t) - \sum_{j=1}^{n} b_{ij}(t)u_j \tag{0.4}$$

and $a_i, b_{ij}\colon \mathbb{R} \to \mathbb{R}$ are bounded continuous functions. For example, assume $a_{iL} > 0$, $b_{ijL} > 0$ and

$$a_{iL} > \sum_{j\varepsilon J_i} b_{ijM}\, a_{jM}/b_{jjL} \tag{0.5}$$

where, in the next; $g_L(g_M) = \inf(\sup)\ \{g(t)\colon t \in \mathbb{R}\}$ for each bounded function $g\colon \mathbb{R} \to \mathbb{R}$. We shall prove that in this case F satisfies (0.2) and (0.1) has a solution u° as above and so we improve the main results in [1], [2], [6] and [7]. To end this paper we prove a "stable coexistence" theorem for the predator-prey model.

REMARK. If n=2 and F_i is given by (0.4) then (0.2) is implied by $b_{iiL} > 0$, $|b_{21}|_L > 0$ and

$$\sup(|b_{12}|/_{b_{22}}) < (|b_2|/_{b_{11}})$$

1. THE MAIN RESULT. We begin with some notations. Given $x = (x_1,\dots,x_n)$ in $\mathbb{R}^n$ we put $x > 0$ $(x \geq 0)$ if $x_i > 0$ $(x_i \geq 0)$ for $1 \leq i \leq n$. We also define $\|x\| = |x_1| + \dots + |x_n|$.

The domain of a solution u to (0.1) is denoted by dom(u). Notice that if $u(t_o) > 0$ for some t_o then $u(t) > 0$ for all $t \in \mathrm{dom}(u)$.

Since we are interested in the almost periodic case we must study the system (0.1) in the non differentiable case. Thus we shall assume that there are positive constants $c_1,\dots,c_n$ and a non negative continuous function $m\colon \mathbb{R} \to \mathbb{R}$ such that

$$c_i \left[F_i(t,x_h^i) - F_i(t,x_h^{i-1})\right] + m(t)h_i \leq$$

$$\leq -\sum_{j\in J_i} c_j \left|F_j(t_o, x_h^i) - F_j(t_o,x_h^{i-1})\right| \tag{1.1}$$

for $x = (x_1,\dots,x_n) > 0$, $h = (h_1,\dots,h_n)$ 0 and $x_h^i = (x_1 + h_1,\dots,x_i + h_i, x_{i+1},\dots,x_n)$ $(0 \leq i \leq n)$. Notice that (0.2) implies (1.1) in the "C^1-case".

If $u = (u_1,\dots,u_n)$ and $v = (v_1,\dots,v_n)$ are positive solutions to (0.1), we define

$$r(t) := r(t,u,v) := \sum_{j=1}^{n} c_j \left|\ln(u_j(t)/v_j(t))\right| \tag{1.2}$$

1.1. THEOREM. If (1.1) holds and $I := \mathrm{dom}(u) \cap \mathrm{dom}(v)$ is nonempty, then there exists a subset I_o of I such that $I - I_o$ is a denumerable set; r is differentiable at all points of I_o and

$$r'(t) \leq -m(t) \| u(t) - v(t) \| \quad ; \quad t \in I_o \tag{1.3}$$

PROOF. Let $f:(a,b) \to \mathbb{R}$ be a c^1-function defined in an open interval of $\mathbb{R}$ and define $g(t) = |f(t)|$. Let D be the subset of (a,b) consisting of points to such that g is not differentiable at t_o, then $f(t_o) = 0$ and $f'(t_o) \neq 0$ and hence D is a discrete set. So, there is a subset I_o of I such that I minus I_o is a denumerable set and $\ln(u_j/v_j)$ is differentiable in I_o for $1 \leq j \leq n$.

Let us fix t_o in I_o and define

$$S_+ = \{j: u_j(t_o) > v_j(t_o)\}, \quad S_- = \{j: u_j(t_o) < v_j(t_o)\} \ ;$$

$$S_o = \{j: u_j(t_o) = v_j(t_o)\}, \quad S = S_+ \cup S_-$$

and notice that S is non empty and

$$\frac{d}{dt} |\ln(u_j/v_j)| = 0 \quad \text{in } t_o \quad \text{for all } j \in S_o$$

Consequently

$$r'(t_o) = \sum_{j \in S_+} c_j \left[F_j(t_o, u(t_o)) - F_j(t_o, v(t_o))\right]$$

$$- \sum_{j\varepsilon S_-} c_j \left[F_j(t_o, u(t_o)) - F_j(t_o, v(t_o))\right].$$

Now let us define $x, h, k \varepsilon \mathbb{R}^n$ and $\Delta^p_{ji} \varepsilon \mathbb{R}$; $1 \leq i, j \leq n$, $p = h, k$; by: $x_j = v_j(t_o)$ if $j \varepsilon S_+$; $x_j = u_j(t_o)$ if $j \varepsilon S_o \cup S_-$, $x = (x_1, \ldots, x_n)$; $h = u(t_o) - x$ $k = v(t_o) - x$ and $\Delta^p_{ji} = F_j(t_o, x^i_p) - F_j(t_o, x^{i-1}_p)$. Then

$$F_j(t_o, u(t_o)) - F_j(t_o, v(t_o)) =$$

$$= \left[F_j(t_o, x + h) - F_j(t_o, x)\right] - \left[F_j(t_o, x+k) - F_j(t_o, x)\right]$$

$$= \sum_{i\varepsilon S_+} \Delta^h_{ji} - \sum_{i\varepsilon S_-} \Delta^k_{ji}$$

(Notice that $h_i = 0$ for all $i \varepsilon S_+ \cup S_o$, so $x^i_h = x^{i-1}_h$ for $i \varepsilon S_o \cup S_+$ and hence $\Delta^h_{ji} = 0$ for $i \varepsilon S_o \cup S_+$. Analogously $\Delta^k_{ji} = 0$ for $i \varepsilon S_o \cup S_-$).

From here

$$r'(t_o) = \sum_{j\varepsilon S_+} c_j \Big(\sum_{j\varepsilon S_+} \Delta^h_{ji} - \sum_{j\varepsilon S_-} \Delta^k_{ji}\Big) - \sum_{j\varepsilon S_-} c_j \Big(\sum_{j\varepsilon S_+} \Delta^h_{ji} - \sum_{i\varepsilon S_-} \Delta^k_{ji}\Big)$$

$$= \sum_{i\varepsilon S_+} \Big(\sum_{j\varepsilon S_+} c_j \Delta^h_{ji} - \sum_{j\varepsilon S_-} c_j \Delta^h_{ji}\Big) - \sum_{i\varepsilon S_-} \Big(\sum_{j\varepsilon S_+} c_j \Delta^k_{ji} - \sum_{j\varepsilon S_-} c_j \Delta^k_{ji}\Big) =$$

$$= \sum_{j\varepsilon S_+} \lambda_{ih} - \sum_{i\varepsilon S_-} \lambda_{ik}$$

where

$$\lambda_{ip} = \sum_{j\varepsilon S_+} c_j \Lambda^p_{ji} - \sum_{j\varepsilon S_-} c_i \Lambda^p_{ji} \; ; \quad p = h,\, k$$

For $i \varepsilon S_+$ we have (see (1.1)).

$$\lambda_{ih} \leq c_i \Delta_{ii} + \sum_{j\varepsilon J_i} c_j |\Delta^h_{ji}| \leq - m(t_o) h_i$$

Analogously $\lambda_{ik} \geq m(t_o) k_i$ for all $i \varepsilon S_-$ and the proof is complete.

In the following C_+ denotes the set of all bounded continuous functions $g: \mathbb{R} \to \mathbb{R}$ such that $g_L > 0$.

1.2. THEOREM. Assume (1.1) and suppose that (0.1) has a solution $v = (v_1,\dots,v_n)$ defined and bounded in $[t_o,\infty)$ for some t_o, such that

$$M: = \sup \{v_i(t):\ 1 \leq i \leq n,\quad t \geq t_o\} < +\infty \tag{1.4}$$

If $u = (u_1,\dots,u_n)$ is a solution to (0.1) such that $I: = \mathrm{dom}(u) \cap \mathrm{dom}(v)$ is nonempty and $u(t_*) > 0$ for some t_*, then u is defined and bounded in $[t_*,\infty)$ We have also the following facts:

a) If m is a positive constant then (0.3) holds with $v°=v$

b) If

$$\varepsilon: = \inf \{v_i(t):\ 1 \leq i \leq n,\ t \geq t_o\} > 0 \tag{1.5}$$

then there are positive constants λ,μ such that

$$\| u(t)-v(t) \| \leq \lambda \|u(t_1)-v(t_1)\| \exp(-\mu \int_{t_1}^{t} m(s)ds) \qquad (1.6)$$

for $t_* \leq t_1 \leq t$. In particular (0.3) holds if

$$\int_0^\infty m(s)ds = +\infty \qquad (1.7)$$

c) The problem

$$u_i' = u_i F_i(t,u), \quad u_i \in C_+, \quad 1 \leq i \leq n. \qquad (1.8)$$

has at most one solution if

$$\int_{-\infty}^{0} m(s)ds = +\infty \qquad (1.9)$$

PROOF. From (1.3) we know that r is a decreasing function in I; in particular $r(t) \leq r(t_*)$ for all $t \geq t_*$, $t \in I$. Hence there are positive constants p,q such that

$$q\, v_j(t) \leq u_j(t) \leq p v_j(t); \quad 1 \leq j \leq n, \quad t \geq t_*, \quad t \in I \qquad (1.10)$$

Thus, u is defined and bounded in $[t_*,\infty)$.

From (1.3) we also have, for $t \geq t_*$,

$$\int_{t_*}^{t} \|u(s)-v(s)\|\, ds \leq \frac{1}{m} [r(t_*)-r(t)]$$

and hence

$$\int_{t_*}^{\infty} \| u(s) - v(s)\| \, ds < + \infty \tag{1.11}$$

But u,v are bounded in $[t_*,\infty)$ and then, then same holds for u',v'. In particular $\| u(s)-v(s) \|$ is a Lipschitz function in $[t_*,\infty)$ and so (1.11) implies (0.3). Thus the proof of a) is complete.

REMARK. Let $f: [0,\infty) \to \mathbb{R}$ be an increasing and bounded c^1-function such that f' is Lipschitz continuous. If is not hard to prove that $f'(t) \to 0$ as $t \to +\infty$.

From (1.4), (1.5) and (1.10) there are positive constants α_o,α_1 such that

$$\frac{1}{\alpha_o} |u_j(t)-v_j(t)| \le |\ln(u_j(t))-\ln(v_j(t))| \le \frac{1}{\alpha_1} |u_j(t)-v_j(t)|$$

for all $t \ge t_*$; $1 \le j \le n$. From this, there are positive constants α_2, α_3 such that

$$\alpha_2 \|u(t) - v(t)\| \le r(t) \le \alpha_3^{-1} \| u(t)-v(t) \| , \text{ for } t \ge t_* \tag{1.12}$$

Consequently $r'(t) \le - \alpha_3 \, m(t) \, r(t)$ $(t \ge t_*)$ and hence

$$r(t) \le r(t_1) \exp(- \alpha_3 \int_{t_1}^{t} m(s)ds) \; ; \quad t \ge t_1 \ge t_*$$

The proof of b) follows now from (1.12).

Assume now that u,v are solutions to (1.8); from the arguments above we conclude that (1.6) holds for all $t_1 \leq t$. In particular

$$\| u(t) - v(t) \| \geq \lambda \| u(0)-v(0) \| \exp(-\mu \int_t^0 m(s)ds) \text{ for } t \leq 0$$

and hence $u(0) = v(0)$ since (1.9) holds and u,v are bounded. Thus the proof is complete.

1.3. COROLLARY. Suppose that F is periodic in time with period $T > 0$; and assume that there is a solution v of (0.1) defined in $[t_o,\infty)$ which satisfies (1.4) - (1.5). If (1.1) and (1.7) hold; then the system (0.1) has exactly a T-periodic solution $u = (u_1,...,u_n)$ such that $u_i \in C_+$ for $1 \leq i \leq n$.

PROOF. Let us define $v^k(t) = v(t + kT)$ for all integers $k \geq 1$ $(t \geq t_o - kT)$; and choose a subsequence $\{v^m(0)\}$ of $\{v^k(0)\}$ such that $v^m(0) \to \xi$ for some ξ in $\mathbb{R}^n$. Since $\varepsilon \leq v_i^k(t) \leq M$ for all components of v^k, if is easy to prove (see Lemma 1 of [1]) that the solution u of (0.1), having the initial condition $u(0) = \xi$, is defined in $\mathbb{R}$ and $\varepsilon \leq u_i \leq M$ for all components u_i of u.

On the other hand, by theorem 1.2, we have that $v^1(t)-v(t) \to 0$ as $t \to +\infty$ and hence $v^{m+1}(0) \to \xi$. Therefore $v^m(T) \to \xi$ and then $u(T) = u(0)$. So the proof is complete.

Let $P = \{x \in \mathbb{R}^n : x > 0\}$; we say that F is almost pe - **riodic in P** if for all sequences (s_k) of $\mathbb{R}$ and all com -

pact subsets K of P; there exists a subsequence (t_k) of (s_k) and a continuous function $G: \mathbb{R} \times K \to \mathbb{R}^n$ such that

$$F(t+t_k, x) \to G(t,x) \quad \text{as} \quad k \to +\infty, \text{ uniformly on } \mathbb{R} \times K \tag{1.13}$$

1.4. THEOREM. Suppose that F is almost periodic in P and assume that every $x \in P$ has a neigborhood N such that F is Lipschitz continuous in $x \in N$. Let v be as in Corollary 1.3 and suppose that (1.1) holds for a positive constant m. Then the problem (1.8) has exactly one solution u and u is almost periodic.

PROOF. Let U be an open and convex nonempty subset of P whose closure $cl(U)$ is contained in P, and let (t_k) be a sequence of $\mathbb{R}$. We can assume that (1.13) holds with $K = cl(U)$. Notice that $G(t,x)$ is locally Lipschitz with respect to the x variable and G satisfies (1.1) for $x, x+h \in K$. In particular, the problem

$$u_i' = u_i \, G_i(t,u), \quad u_i \in C_+, \quad 1 \leq i \leq n \tag{1.14}$$

has at most one solution.

Now choose U as above such that $[\varepsilon, M]^n$ is contained in U and let (t_k) be a sequence of $\mathbb{R}$ such that $t_k \to +\infty$. Without loss of generality we can assume that (1.13) holds for $K = cl(U)$. Define, $v^k(t) = v(t+t_k)$ for all integer

$k \geq 1$; $t \geq t_k - t_o$; then $\varepsilon \leq v_i^k(t) \leq M$ for all component v_i^k of v^k ($k = 1,2,\ldots,$ $1 \leq i \leq n$, $t + t_o \geq t_k$) and v^k is a solution to the system

$$u_i' = u_i\, F_i(t + t_k, u), \quad 1 \leq i \leq n.$$

Since $\{v^k(0)\}$ is a bounded sequence of $\mathbb{R}^n$ we can assume that $v^k(0) \to \xi \in U$ as $k \to +\infty$. Hence, the solution w of the initial value problem

$$u_i' = u_i\, G(t,u), \quad u(0) = \xi, \quad 1 \leq i \leq n$$

is defined in $\mathbb{R}$ and w is a solution to (1.14).

Define now $w^k(t) = w(t-t_k)$ for $t \in \mathbb{R}$ and $k = 1,2,\ldots$; we can assume that $w^k(0) \to \eta$ and hence the solution to (0.1) given by $u(0) = \eta$ is a solution to (1.8). The proof follows now from the arguments in [1], theorem 2. See also [5], theorem 1.17.

REMARK. Let $H = (H_1,\ldots,H_n): \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that $H(t,0) \equiv 0$. Assume further that the partial derivatives $\partial H_i/\partial x_j$ are defined and continuous in $\mathbb{R} \times \mathbb{R}^n$ and

$$\frac{\partial H_i}{\partial x_i}(t,x) + \sum_{j \in J_i} \left| \frac{\partial H_j}{\partial x_i}(t,x) \right| \leq -\, m(t); \quad 1 \leq i \leq n.$$

If (1.7) holds, then the trivial solution of the system

$$x' = H(t,x) \tag{1.15}$$

is globally exponentially stable. To show this, let $Z(t)$ be a non trivial solution to (1.15); from the arguments in theorems 1.1, 1.2 we get

$$\|z(t)\| \le \|z(t_o)\| \exp(-\int_{t_o}^{t} m(s)ds); \quad t_o \le t, \quad t,t_o \in dom(z)$$

and hence $dom(z) = (\alpha,\infty)$ for some $-\infty \le \alpha < +\infty$ and $z(t) \to 0$ as $t \to +\infty$. Moreover, if (1.9) holds then the trivial solution to (1.15) is the only solution to this system, bounded in $\mathbb{R}$.

When (1.15) is a linear system and m is constant, this result becomes complement to theorem 2 of [9].

An "unstable" result is obtained from the change of variables $z(t) \to z(-t)$.

2. COMPETITION SYSTEMS. In this section we consider the system

$$u_i' = u_i \left[a_i(t) - \sum_{j=1}^{n} b_{ij}(t)u_j\right] \quad ; \quad 1 \le i \le n \tag{2.1}$$

where $a_i, b_{ij} \in C_+$; $1 \le i,j \le n$. This system models the competition between n biological spaces in a closed environment.

2.1. PROPOSITION. Suppose that there are positive constants $\varepsilon, M_1,\ldots,M_n$; $\varepsilon \leq M_1,\ldots, M_n$; such that

$$F_i(t, 0,\ldots 0, M_i, 0,\ldots,0) \leq 0$$
$$(1 \leq i \leq n)$$
$$F_i(t, M_1,\ldots,M_{i-1}, \varepsilon, M_{i+1},\ldots,M_n) \geq 0$$

Assume further that $F_i(t,x) \leq F_i(t,y)$ if $0 \leq y \leq x$. If u is a solution to (0.1) such that $u(t_o) > 0$ for some to, then u is defined in $[t_o,\infty)$ and

$$\min\,\{u_1(t_o),\ldots,u_n(t_o),\ \varepsilon\} \leq u_i(t) \leq \max\,\{u_1(t_o),\ldots,u_n(t_o),\ M_i\}; \tag{2.2}$$

for $1 \leq i \leq n$ and $t \geq t_o$. Moreover, the system (0.1) has a solution $u = (u_1,\ldots,u_n)$ such that

$$u_i \in C_+\ ,\quad 1 \leq i \leq n \tag{2.3}$$

PROOF. Fix $1 \leq i \leq n$ and let N_i be the max in (2.2). Obviously $u_i(t_o) \leq N_i$. Assume now that $u_i(t_2) > N_i$ for some $t_2 > t_o$. Then, there is t_1; $t_o < t_1 < t_2$ such that $u_i(t_1) > N_i$ and $u_i'(t_1) > 0$. From this we have

$$0 < F_i(t_1, u(t_1)) \leq F_i(t_1, 0,\ldots, M_i,\ldots,0) \leq 0$$

and this contradiction proves the second inequality in (2.2)

$$\varepsilon < M_i^{-1} \left[1 - \sum_{j \in J_i} \sup(b_{ij}/a_i)\right] ; \quad 1 \leq i \leq n$$

If F_i is defined by (0.4) then the assumptions in proposition 2.1 are satisfied and the proof will follow from theorem 1.2 if we show that (1.1) holds.

To make this, let us fix δ; $0 < \delta < 1$, such that the matriz $M_\delta := M + \delta(\text{identity})$ has no reigenvalues in $[1,\infty)$. From the Perron-Frobenius theory of positive matrices we know that M has a positive eigenvalue $\lambda < 1$ and $M_\delta(c) = \lambda c$ for some column vector $c = \mathrm{col}(c_1,\ldots,c_n) > 0$. From here, $M(c) < (1-\delta)c$ and hence

$$c_i\, b_{ii}(t) \geq m + \sum_{j \in J_i} b_{ji}(t) c_j \; , \quad 1 \leq i \leq m$$

where $m := \delta \min\{b_{ii}(t) : t \in \mathbb{R} ; \; 1 \leq i \leq n\} > 0$. This implies (1.1) and so the proof is complete.

2.3. COROLLARY. If (0.5) holds then the assertions in theorem 2.2 are true.

PROOF. Obviously condition (0.5) implies (2.4). Define now the real $n\times n$ matriz $P = (p_{ij})$ by $p_{ii} = 0$ and $p_{ij} = b_{ijM}/b_{jjL}$ for $i \neq j$; then (0.5) implies $P(d) < d$, where d is the column vector $\mathrm{col}(a_{1L},\ldots, a_{nL})$.

Let us fix $\delta > 0$ ($\delta < 1$) such that $P_\delta(d) < d$; where $P = P + \delta(\text{identity})$; from Perron's theorem we get a real positive eigenvalue λ of P_δ such that $\lambda < 1$ and $\lambda \geq |\mu|$ for all eigenvalues μ of P_δ. See [4] pg. 227. Hence, the adjoint matrix P^* of P has no eigenvalues in $[1, +\infty)$ and the proof follows easily.

REMARK. The second assumption in theorem 2.2 is satisfied for $n = 2,3$ if $\det(I - M) > 0$; where I = identity matrix. In particular we have:

2.4. COROLLARY. Assume $n = 2$ and

$$\inf(a_1/b_{12}) > \sup(a_2/b_{22})$$

$$\inf(a_2/b_{21}) > \sup(a_1/b_{11})$$

$$\inf(b_{11}/b_{21}) > \sup(b_{12}/b_{22})$$

then the assertions in theorem 2.2 holds.

REMARKS.

(a) Corollary 2.4 was proved in [3] for the periodic case. In the almost periodic case this corollary improves a theorem of [1].

(b) Corollary 2.3 generalizes the main results in [1], [2], [6] and [7] and [11].

3. THE PREDATOR-PREY MODEL. In this section we consider the system

$$u' = u\left[-a(t)-b(t)u+c(t)v\right], \quad v' = v\left[d(t)-e(t)u-f(t)v\right] \tag{3.1}$$

where $a,\dots,f \in C_+$. The following result justifies the assumption (3.3) in the main result of this section.

3.1. PROPOSITION. If $\inf(f/d) \geq \sup(c/a)$ then the system (3.1) has no solution (u,v) such that

$$u,v \in C_+ \tag{3.2}$$

PROOF. To simplify our statements let us define, in the next; $B = b/a$, $C = c/a$, $E = e/d$ and $F = f/d$. Assume now that (u,v) is a solution to (3.1) - (3.2); it is not hard to prove that for all bounded differentiable functions $g: \mathbb{R} \to \mathbb{R}$ there exists a sequence (t_n) of $\mathbb{R}$ such that $g(t_n) \to g_L$ (resp. g_M) and $g'(t_n) \to 0$. Choose a sequence (t_n) such that $u(t_n) \to u_L$ and $u'(t_n) \to 0$ then $-B(t_n)u(t_n) + C(t_n)v(t_n) \to 1$ and hence $1 \leq -B_L u_L + C_M v_M$. Analogously; $1 \geq E_L u_L + F_L v_M$ and thus $0 \geq C_M - F_L \geq (C_M E_L + F_L B_L)u_L > 0$. This contradiction ends the proof.

3.2. THEOREM. Assume

$$\inf(c/a) > \sup(f/d) \tag{3.3}$$

$$\inf(b/e) > \sup(c/f) \tag{3.4}$$

If $a,\dots,f$ are periodic with period $T > 0$ then the problem (3.1) - (3.2) has exactly one solution (u_o,v_o) and

$$(u(t) - u^\circ(t),\ v(t) - v_o(t)) \to 0 \quad \text{as} \quad t \to +\infty$$

for any solution (u,v) to (3.1) with $(u(t_o), v(t_o)) > 0$ for some t_o. Moreover; (u_o, v_o) is T-periodic.

PROOF. Notice first that (3.3) is equivalent to $C_L > F_M$ and let us define $\alpha = (C_M - F_L) / (B_L F_L)$. If (u,v) is a solution to (3.1) with $v(0) \leq 1/F_L$ and $u(0) \leq \alpha$ then $v(t) \leq 1/F_L$ and $u(t) \leq \alpha$ for all $t \in \text{dom}(u,v)$. In particular (u,v) is defined and bounded in $[0,\infty)$.

Let us write $I = \inf(b/e)$ and $S = \sup(c/f)$; then the system (3.1) satisfies the hypothesis in theorem 1.1 with $c_1 = 1$, $c_2 = (I + S)/2$ and $m = (I - S)\min\{e_L, f_L\}/2$. Thus theorem 1.2 implies:

$$(u(t) - u(t+T),\ v(t) - v(t+T)) \to 0 \quad \text{as} \quad t \to +\infty$$

where (u,v) is a solution to (3.1) as above.

Define now $P = \{p \in \mathbb{R}^2 : p \geq 0\}$ and $P_+ = \{p \in P : p > 0\}$, and for all $p \in P$ let $(u(t,p), v(t,p))$ be the solution to (3.1) given by $(u(0,p), v(0,p)) = p$. By theorem 1.2 we know that this solution is defined for all $t \geq 0$, and $p > 0$. Notice that the same holds if p has a trivial coordinate. So the Poincaré map $T: P \to P$; $T(p) = (u(T,p), v(T,p))$ is well defined and

$$T^{n+1}(p) - T^n(p) \to 0 \quad \text{as} \quad n \to \infty; \quad p \in P_+ \qquad (3.5)$$

Assume now that T has no fixed points in P_+; then $p_o := (0,0)$ and $p_1 := (0, V(0))$ are the unique fixed points of T where $V(t)$ is the unique positive and T-periodic solution to the logistic equation $x' = x [d - fx]$.

Define $W_i = \{p \varepsilon P : T^n(p) \to p_i \text{ as } n \to \infty\}$; $i = 0,1$; it is easy to prove that

$$W_o = (0,\infty) \times \{0\} \tag{3.6}$$

On the other hand, the eigenvalues λ,μ of the Frechet derivative $T'(p_1)$ are given by

$$\lambda = \exp(- \int_0^T f(s)\ V(s) ds); \quad \mu = \exp(\int_0^T [c(t)\ V(t) - a(t)] dt)$$

and $1/F_M \leq V(t) \leq 1/F_L$. Thus $c(t)\ V(t) - a(t) =$ $= a(t)\ [C(t)\ V(t) - 1] \geq a_L\ [(C_L/F_M) - 1] > 0$ and hence $0 < \lambda < 1 < \mu$. So, p_1 is a hyperbolic fixed point of T and there exists an open ball B of $\mathbb{R}^2$, centered at p_1, such that $B \cap W_1$ is an open interval of the positive vertical axis. Assume now that $p \varepsilon W_1$, then $T^N(p) \varepsilon B \cap W_1$ for some integer $N \geq 1$ and hence $u(NT,p) = 0$. Consequently $u(0,p) = 0$ and therefore

$$W_1 = \{0\} \times (0,\infty) \tag{3.7}$$

Let us fix $p \varepsilon P_+$ and open balls B_o, B_1 in $\mathbb{R}^2$ such that p_i belongs B_i $(i = 0,1)$ and $cl(B_o) \cap cl(B_1) = \emptyset$. (cl = clousure). Since $T^n(p)$ is a bounded sequence then there is a subsequence $\{T^{n_k}(p)\}$ of $\{T^n(p)\}$ such that $T^{n_k}(p) \to q$ for some $q \in P$, and by (3.5); $T(q) = q$. Conse - quently either $q = p_o$ or $q = p_1$.

For $i = 0,1$, let N_i be the set consisting of all integers $n \geq 1$ such that $T^n(p) \in B_i$ and let N_2 be the set of all integers $n \geq 1$ such that $n \notin N_o \cup N_1$. From the argument mentioned above we have that N_2 is a finite set; and by (3.6) (3.7) we get that N_o, N_1 are infinite sets.

Choose now an open ball U_1 such that $U_1 \cap B_o = U_1 \cap \{T^n(p) : n \in N_2\} = \phi$ and $cl(B_1) \subset U_1$ and let us write $N_1 = \{n_1 < n_2 < \ldots\}$. Since

$$T^{n_r+1}(p) - T^{n_r}(p) \to 0 \quad \text{as} \quad r \to +\infty$$

then there exists an integer $r_o \geq 1$ such that $T^{n_r+1}(p) \in U_1$ for $r \geq r_o$. Then $n_r + 1 \in N_1$ for all $r \geq r_o$ and hence N_1 contains all integer $n \geq r_o$. So N_o is a finite set and this contradiction ends the proof.

REMARK. Theorem 3.2 remains valid for a system of the form

$$u' = u\,F(t,u,v) \ , \quad v' = v\,G(t,u,v)$$

if we assume that $F(t,0,0) < 0$ and

(i) $\partial G/\partial u \leq 0$, $\partial G/\partial v < 0$, $\partial F/\partial u \leq 0$, $\partial F/\partial v \geq 0$;

$\partial F/\partial u - c\, \partial G/\partial u \leq -m$, and $c\, \partial G/\partial v + \partial F/\partial v \leq m$

for some constants $c,m > 0$.

(ii) There are positive constants $0 < \varepsilon \leq M$, $\alpha > 0$ such that $G(t,0,N) \leq 0 \leq G(t,0,\varepsilon)$ and $F(t,\alpha, N) \leq 0$.

3.3. REMARK. Suppose that F is T-periodic: $F(t+T,u) = F(t,u)$; if F satisfies (0.2) and (0.1) has a positive solution v° defined and bounded in $[0,\infty)$ then there exists a non negative T-periodic solution u° of (0.1) such that

$$u(t) - u^\circ(t) \to 0 \quad \text{as} \quad t \to +\infty.$$

for any solution u of (0.1) with $u(0) > 0$. Proof. By theorem 1.2 (a) we know that system (0.1) has at most a T-periodic positive solution. From this and induction on n, the set of all T-periodic non negative solutions of (0.1) is finite. Now let (n_k) be a sequence of positive integers such that $v^\circ(n_k T) \to q$ for some $q \in \mathbb{R}^n_+$. By the arguments in theorem 3.2 we have that

$$u^\circ(t) - u^\circ(t) \to 0 \quad \text{as} \quad t \to +\infty$$

where u° is the T-periodic solution of (0.1) given by $u^\circ(0)=q$ The proof follows now from theorem 1.2 (a).

R E F E R E N C E S

[1] S. AHMAD. On almost period solutions of the competeing species problems. Proc. Amer. Math. Soc. Vol. 102, Nº 4 (1988).

[2] C. ALVAREZ AND A.C. LAZER. An application of topological degree to the periodic competing species problem. Jour. Austral. Math. Soc. Ser. B, 28 (1986).

[3] C. ALVAREZ AND A. TINEO. Asymptotically stable solutions of Lotka-Volterra equations. Radovi Matematicki Vol. 4 (1988) 309 - 319.

[4] R. BELLMAN. Introduction to matrix analysis New York. Mc Graw-Hill 1960.

[5] FINK A.M. Almost periodic differential equations. Lecture Notes in Math. Springer-Verlag. Vol. 377.

[6] K. GOPALSAMY. Global asymptotic stability in a periodic Lotka-Volterra system. J. Austral. Math. Soc. Ser B, 27 (1985) 66-72.

[7] K. GOPALSAMY. Global asymptotic stability in an almost periodic Lotka-Volterra system. J. Austral. Math. Soc. Ser. B. 27 (1986) 346-360.

[8] G.S. LADDE, V. LAKSMIKANTHAN AND A.S. VATSALE. Monotone iterative techniques for nonlinear differential

equations. Pitman advance publishing program. Boston London-Melbourne.

[9] A. LAZER. Caracteristic exponents and diagonally dominant linear differential systems. Jour. Math. An. and App. 35, 215-229 (1971).

[10] H. SMITH. Periodic solutions of periodic competitive and cooperative systems. SIAM. Math Anal. Vol. 17, Nº 6, 1986.

[11] A. TINEO AND C. ALVAREZ. A different consideration about the globally asymptotically stable solution of the periodic n-competing species problems. To appear in this Journal.