NOTAS DE MATEMATICAS

Nº 108

FIXED POINT THEORY: A BRIEF SURVEY

BY

W.A. KIRK

UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMATICA
MERIDA - VENEZUELA
1990

1. **INTRODUCTION**. For sets A and B with $A \subseteq B$ and a mapping $T: A \longrightarrow B$, every solution of the equation

$$T(x) = x \tag{1}$$

is called a *fixed point* of T, and the set of all such points is denoted FixT. *Fixed point theory* entails the study of conditions on A and/or T which assure that T always has at least one fixed point, as well as the study of methods of approximating fixed points when they do exist and the study of the structure of FixT.

Fixed point theory is a major branch of nonlinear functional analysis because of its wide applicability. Numerous questions in physics, chemistry, biology, and economics lead to various nonlinear differential and integral equations, and if one ignores the concrete form of these equations one can often reduce them to abstract operator equations of the form

$$F(x,y) = 0, \tag{2}$$

where x and y are, respectively, elements of Banach spaces X and Y. In turn, (2) can often be reformulated as a fixed point problem. In the simplest case, $F(x) = 0$ may be written

$$x = x - F(x).$$

Thus finding a solution to the equation $F(x) = 0$ reduces to finding a fixed point for the mapping T defined by

$$T(x) = x - F(x).$$

More generally, it suffices to find a fixed point of T where T is defined in any of the following ways:

$$T(x) = x - \lambda F(x) \qquad (\lambda \neq 0);$$

$$T(x) = x - \lambda g(F(x)) \qquad (\lambda \neq 0;\ g(u) = 0 \Rightarrow u = 0);$$

$$T(x) = h^{-1}(F(x) - G(x)), \text{ where } F(x) = h(x) + G(x).$$

There are three major branches of fixed point theory in functional analysis, and each branch has its celebrated theorems.

I. Metric: (Banach's Contraction Mapping Principle; The Browder–Göhde–Kirk Theorem).

II. Topological: (Brouwer's Theorem; Schauder's Theorem; Sadovskii's Theorem; The Leray–Schauder Theorem).

III. Set Theoretic: (The Bourbaki – Kneser Theorem; Tarski's Theorem; Amann's Theorem).

2. **THE METRIC THEORY.** A mapping T defined on a metric space (M,d) and taking values in a metric space (N,d) is said to have *Lipschitz constant* k, or to be k–*lipschitzian*, for a real number $k \geq 0$, if for each $x,y \in M$

$$d(T(x),T(y)) \leq kd(x,y).$$

If $k < 1$, then T is said to be a *contraction mapping*, or k–*contraction*, and if $k = 1$, then T is said to be *nonexpansive.*

BANACH'S CONTRACTION MAPPING PRINCIPLE (1922). *Suppose* (M,d) *is a complete metric space and suppose* $T:M \longrightarrow M$ *a contraction mapping with Lipschitz constant* $k < 1$. *Then:*

a) T *has exactly one fixed point, say* $z \in M$.

b) *Moreover, given any* $x \in M$, *the sequence* $\{T^n(x)\}$ *converges to* z, and

c) *for all* $n = 1,2, \dots$, $d(T^n(x),z) \leq k^n(1-k)^{-1}d(x,T(x))$.

d) *Also, for all* $n = 1,2, \dots$, $d(T^{n+1}(x),z) \leq kd(T(x),x)$, *and*

e) $d(T^{n+1}(x),z) \leq k(1-k)^{-1}d(T^n(x),T^{n+1}(x))$.

The great significance of Banach's Principle, and the reason it is one of the most frequently cited fixed point theorems in all of analysis, lies in the fact that a) – e) contain elements of fundamental importance to the theoretical and practical treatment of mathematical equations. Another practical result is the following corollary to Banach's Principle.

COROLLARY (Continuous Dependence on a Parameter). *Suppose* (M,d) *is a complete metric space, suppose* S *is a metric space, and suppose for each* $s \in S$, $T_s:M \longrightarrow M$ *is a* k–*contraction. Let* $s_0 \in S$ *and suppose that for all* $x \in M$,

$$\lim_{s \to s_0} T_s(x) = T_{s_0}(x).$$

Then, for each $s \in S$, *there is exactly one point* $x_s \in M$ *such that* $x_s = T_s(x_s)$ *and moreover,* $\lim_{s \to s_0} x_s = x_{s_0}$.

Modifications of Banach's Theorem.

THEOREM (Krasnoselskii and Zabreiko, 1975). *Suppose* M *is a complete metric space and* T *a selfmapping of* M *which satisfies the following condition: For arbitrary numbers* $0 < a \leq b$ *there exists a number* $k(a,b)$, $0 \leq k(a,b) < 1$, *such that if* $x,y \in M$ *satisfy* $a \leq d(x,y) \leq b$, *then*

$$d(T(x),T(y)) \le k(a,b)d(x,y).$$

Then T *has a unique fixed point.*

THEOREM (Edelstein, 1962). *Suppose* M *is a compact metric space and suppose* T *is a contractive selfmapping of* M *(i.e.,* $d(T(x),T(y)) < d(x,y)$, $x,y \in M$). *Then* T *has a unique fixed point in* M *and, moreover, for any* $x \in M$, *the Picard iterates* $\{T^n(x)\}$ *converge to this fixed point.*

THEOREM (Cf. Browder, 1968/76). *Suppose* M *is a complete metric space and suppose* T *is a selfmapping of* M *which satisfies*

$$d(T(x),T(y)) \le \psi(d(x,y)) \quad \textit{for all } x,y \in M,$$

where $\psi:[0,1] \longrightarrow \mathbb{R}$ *is a monotone, increasing, right–continuous function satisfying* $0 < \psi(r) < r$ *for all* $r > 0$. *Then* T *has a unique fixed point in* M *and, moreover, for any* $x \in M$, *the Picard iterates* $\{T^n(x)\}$ *converge to this fixed point.*

THEOREM (Caristi, 1976). *Let* M *be a complete metric space, suppose* $\varphi:M \longrightarrow \mathbb{R}$ *is a lower semicontinuous function which is bounded below, and suppose* T *is a selfmapping of* M *which satisfies*

$$d(x,T(x)) \le \varphi(x) - \varphi(T(x)) \quad \textit{for all } x \in M.$$

Then T *has at least one fixed point in* M. *Moreover, if* T *is continuous, then the Picard iterates* $\{T^n(x)\}$ *converge to a fixed point of* T *for each* $x \in M$

NONEXPANSIVE MAPPINGS. The nonexpansive mappings can obviously be viewed as natural extensions of the contraction mappings. However fixed point theory for nonexpansive mappings differs sharply from that of the contraction mappings in the sense that additional structure is needed on the underlying space to assure the existence of fixed points. This can be seen be considering the following very simple examples.

Examples. Consider the unit ball B in the Banach space c_0 of all sequences of real numbers with zero limit and supremum norm. Thus if $x = (x_1,x_2, \ldots) \in c_0$, $\lim_{i=\infty} x_i = 0$, and $\|x\| = \max_n |x_n|$. It is easy to check that the mapping T defined by

$$T(x) = (1 - \|x\|,x_1,x_2, \ldots)$$

is nonexpansive, maps B into itself, and has no fixed points.

For another example, consider the space C[0,1] of continuous real valued functions defined on the interval [0,1] and for $f \in \mathcal{C}[0,1]$, set

$$\|f\| = \sup\{|f(t)| : t \in [0,1]\}.$$

It is well known that with this norm C[0,1] is a Banach space. Now let

$$K = \{f \in \mathcal{C}[0,1] : f(0) = 0;\ f(1) = 1;\ 0 \le f(t) \le 1,\ t \in [0,1]\}.$$

It is easy to check that the set K is a closed and convex subset of $C[0,1]$. Now define the mapping $T:K \longrightarrow K$ sy setting

$$T(f)(t) = tf(t),\ \ t \in [0,1];\ \ f \in \mathcal{C}[0,1].$$

Again T is nonexpansive and fixed point free.

As our first positive result for nonexpansive mappings we prove the following:

LEMMA. *Let* K *be a nonempty, bounded, closed, and convex subset of a Banach space and let* $T:K \longrightarrow K$ *be nonexpansive. Then*

$$\inf\{\|x - T(x)\| : x \in K\} = 0.$$

Proof. Fix $z \in K$ and $t \in (0,1)$, and consider the mapping T_t defined by

$$T_t(x) = (1 - t)z + tT(x), \quad x \in K.$$

Since K is convex, $T_t:K \longrightarrow K$. Moreover, T has Lipschitz constant $t < 1$. So by Banach's theorem T_t has a unique fixed point $x_t \in K$. Thus

$$x_t = (1 - t)z + tT(x_t)$$

and

$$\|x_t - T(x_t)\| = (1 - t)\|z - T(x_t)\|.$$

Since K is bounded the right hand side of the above approaches 0 as $t \longrightarrow 1^-$ and the conclusion follows.

COROLLARY. *If* K *is a nonempty, compact, and convex subset of a Banach space and if* $T:K \longrightarrow K$ *is nonexpansive, then* T *has at least one fixed point in* K.

As we shall see, the above corollary is a special case of Schauder's theorem, discussed in the next section. However, their are positive results for nonexpansive mappings which are not included in Schauder's theorem. We begin by looking at a property commom to all Hilbert spaces. Recall that Hilbert spaces are characterized among all Banach spaces by the fact that any such space H satisfies the so–called parallelogram law:

$$\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2), \quad x,y \in H.$$

Now suppose H is a given Hilbert space (e.g., take H to be ℓ^2 — the space of all sequences of real numbers $x = (x_1,x_2, \dots)$ for which

$$\|x\|^2 = (\sum_{i=1}^{\infty} |x_i|^2)^{1/2} < \infty.)$$

Now suppose $x,y \in H$ satisfy $\|x\| \le 1$, $\|y\| \le 1$, and $\|x-y\| = \epsilon > 0$. Calculating from the parallelogram law we obtain

(3) $$\tfrac{1}{2}\|x+y\| \le (1-(\epsilon/2)^2)^{1/2}.$$

Next define the function $\delta:[0,2] \longrightarrow [0,1]$ by setting

(4) $$\delta(\epsilon) = \inf\{1 - \tfrac{1}{2}\|x+y\| : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon\},$$

and observe that (†) implies that

$$\delta(\epsilon) \ge 1-(1-(\epsilon/2)^2)^{1/2}.$$

Since equality holds if $\|x\| = \|y\| = 1$, it follows that

(5) $$\delta(\epsilon) = 1-(1-(\epsilon/2)^2)^{1/2}.$$

It should be noted that the function δ as defined in (4) is defined for any Banach space X. If $\delta(\epsilon) >$ whenever $\epsilon > 0$ then X is said to be *uniformly convex.* In view of (5) any Hilbert space is uniformly convex.

The fundamental fixed point theorem for nonexpansive mappings is the following.

THEOREM (Browder, Göhde, Kirk, 1965). *Suppose* K *is a nonempty, bounded, closed, and convex subset of a uniformly convex Banach space* X, *and suppose* $T:K \longrightarrow K$ *is nonexpansive. Then* T *has at least one fixed point.*

The theorem, as stated, is actually a special case of a much more abstract result, proved first by J. Penot in 1979 and later, constructively, by Kirk in 1981. We refer the reader to the survey [13] for this more abstract treatment of the theory.

The Browder–Göhde–Kirk theorem also follows from another important result which asserts that in such a setting the mapping $f = I - T$ is demiclosed on K. (A mapping $f:K \longrightarrow X$ is *demiclosed* on K if the conditions (i) $\{x_n\}$ converges weakly to $x \in X$ and (ii) $\{f(x_n)\}$ converges strongly to $y \in X$ together imply $x \in K$ and $f(x) = y$.) The demiclosedness result, first formulated in 1967 and stated below, has not been significantly extended to any wider class of spaces.

THOEREM (Browder–Göhde Demiclosedness Principle). *Let* X *be a uniformly convex Banach space and* K *a nonempty closed and convex subset of X. Then if* $T:K \longrightarrow X$ *is nonexpansive, the mapping* $f = I - T$ *is demiclosed on* K.

In order to see that the existence theorem follows from the above recall that if K is bounded then, as we have seen above,

$$\inf\{\|x - T(x)\|: x \in K\} = 0.$$

Also, since X is uniformly convex, X is reflexive. Therefore K is compact in the topology of weak sequential convergence. These two facts enable one to select a sequence $\{x_n\}$ in K such that (i) $\{x_n\}$ converges weakly to $x \in K$ and (ii) $\|x_n - T(x_n)\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since $I - T$ is demiclosed on K, it follows that $x - T(x) = 0$.

3. **TOPOLOGICAL FIXED POINT THEORY.** Suppose X is a topological space and $M \subseteq X$. Then a continuous map $r: X \longrightarrow M$ is called a *retraction* if $r(x) = x$ for all $x \in M$. When this is the case M is said to be a *retract* of X.

At this point, in order to streamline the exposition, we take the following as axioms. While both are intuitively clear in $\mathbb{R}^n$, the proof of (B) is highly nontrivial.

RETRACTION PRINCIPLES

(A) *Every closed convex subset* M *of a normed linear space* X *is a retract of* X.

(B) *The boundary,* ∂B, *of a nontrivial closed ball* B *in* $\mathbb{R}^n$ *is not a retract of* B.

REMARK. In an infinite dimensional Banach space (B) fails. The famous collection of mathematical problems known as *The Scottish Book* (named after the Scottish Cafe in what was Lwow, Poland) contains a question (Problem 36) raised around 1935 by S. Ulam. It reads: "Can one transform continuously the solid sphere of a Hilbert space into its boundary such that the transformation should be the identity on the boundary of the ball?" An addendum indicates that Tychonoff provided an example which answered the question affirmatively.

Another nice solution to Ulam's problem, and one that holds in an arbitrary Banach space, was given by Victor Klee in 1953. Klee [15] proved that any Banach space X is homeomorphic with the 'punctured' space $X\backslash\{0\}$. Let $h: X \longrightarrow X\backslash\{0\}$ be such a homeomorphism and assume (as one may) that $h(x) = x$ for $x \in X$, $\|x\| \geq 1$. Now define $T: X \longrightarrow X$ by taking

$$T(x) = h^{-1}(-h(x)).$$

Then $T^2 = I$ on the ball $B(0;1) = \{x \in X : \|x\| \leq 1\}$ and, moreover, T does not have any fixed points since $T(x) = x$ implies $h(x) = -h(x) = 0$. The required retraction is now given by the mapping $R: X \longrightarrow B(0;1)$ defined by

$$R(x) = h(x)/\|h(x)\|.$$

(In 1979, Nowak [19] proved that for any infinite dimensional Banach space there exists a lipschitzian retraction of the unit ball onto the unit sphere.)

BROUWER'S FIXED POINT THEOREM (1912). *Suppose* M *is a nonempty, compact, convex subset of* $\mathbb{R}^n$ *and suppose* $f:M \longrightarrow M$ *is continuous. Then* f *has a fixed point.*

Proof. First take $M = B(0;\rho)$. If $f(x) \neq x$ for each $x \in B(0;\rho)$ then one could construce a retraction $r:B(0;\rho) \longrightarrow S(0;\rho)$ as follows. For each x follow the directed line segment from f(x) through x to its intersection with S(0;1), and let this intersection point be r(x). The existence of such a retraction contradicts (B).

Now, for a general M select ρ sufficiently large that $M \subseteq B(0;\rho)$. By (A) there exists a retraction $r: B(0;\rho) \longrightarrow M$. Then the composition map $f \circ r$ is a continuous mapping of $B(0;\rho)$ into itself and therefore must have a fixed point x. Moreover, x must lie in M. Since $r(x) = x$ for points in M, we have

$$x = f(r(x)) = f(x).$$

REMARK. For the special case $M = [0,1]$ in $\mathbb{R}^1$, the *intermediate value theorem* yields a proof of Brouwer's Theorem.

THE SCHAUDER FIXED POINT THEOREM (1938). *Let* M *be a nonempty, compact, convex subset of a Banach space* X, *and suppose* $T:M \longrightarrow M$ *is continuous. Then* T *has a fixed point.*

Proof. Since T(M) is relatively compact, for each n there exist elements $y_i \in T(M)$, $i = 1, \dots, N_n$ such that for any $x \in M$,

$$\min_i \|T(x) - y_i\| < 1/n$$

Now define the so–called *Schauder operator* P_n by taking

$$P_n(x) = [\sum_{i=1}^{N_n} a_i(x) y_i]/[\sum_{i=1}^{N_n} a_i(x)],$$

where $a_i(x) = \max(n^{-1} - \|T(x) - y\|, 0)$. (Note that for each x, $a_i(x) \neq 0$ for at least one i. Also, the functions a_i are continuous.) Let $M_n = \text{conv}(\{y_1, \dots, y_{N_n}\})$. Then

$$M_n \subseteq \text{conv}(T(M)) \subseteq M$$

and $P_n:M_n \longrightarrow M_n$ is continuous. By Brouwer's Theorem each mapping P_n has a fixed point $x_n \in M_n \subseteq M$ and, since M is compact $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ — say $x_{n_k} \longrightarrow x \in M$ as $k \longrightarrow \infty$.

Now observe that

$$\|P_n(x) - T(x)\| = \|[\Sigma_i a_i(x)(y_i - T(x))]/\Sigma_i a_i(x)\| \leq [\Sigma_i a_i(x)n^{-1}]/\Sigma_i a_i(x) \leq n^{-1}.$$

We now conclude that x is a fixed point of T since

$$\|x_n - T(x)\| \leq \|P_n(x_n) - T(x_n)\| + \|T(x_n) - T(x)\|,$$

and the right side vanishes as $n \longrightarrow \infty$.

LERAY–SCHAUDER DEGREE. One of the principal tools in nonlinear functional analysis is the concept of a *degree of a mapping*. Classical degree theory, in its most general sense, is the study of mapping degree for various classes of continuous mappings with domains contained in one Banach space X which take values in a (possibly different) Banach space Y. Such a theory consists of the *algebraic count* of the number of solutions of the equation $f(x) = y_0$, where f is defined on the closure $\overline{G}$ of an open set G in X and y_0 is a given point in Y which is assumed to lie in the complement of the image of the boundary ∂G of G in Y (i.e., $y_0 \notin f(\partial G)$). The value of the degree function for any such count is an ordinary integer, either positive or negative.

Degree theory in $\mathbb{R}^n$ was introduced by L. E. J. Brouwer in 1912 ([4]). We begin with the case $n = 2$. Consider a continuous mapping f defined on the closed disk $D = B(0;\rho)$ of radius ρ centered at the origin in $\mathbb{R}^2$ and taking values in $\mathbb{R}^2$. As x travels once around the boundary of the disk in a counterclockwise sense $f(x)$ travels along an oriented curve C. Assume $0 \notin C$, and let ω_+ and ω_- denote the number of respective windings C makes about 0 in the counterclockwise and clockwise sense, and defing the degree of f (at) by

$$\deg(f,D) = \omega_+ - \omega_-.$$

It is intuitively clear that the function deg has two important properties:

(i) *Kronecker's existence principle.* If $\deg(f,D) \neq 0$ then there exists an $x_0 \in D$ such that $f(x_0) = 0$.

(ii) *Homotopy invariance.* If f is changed continuously in such a way that none of the corresponding curves C pass through the origin, then $\deg(f,D)$ remains unchanged.

REMARK. The above principles yield an interesting geometric proof of the fundamental theorem of algebra. Let

$$h(z,t) = z^n + (a_{n-1}z^{n-1} + \cdots + a_0)t,$$

where $z \in \mathbb{C}$ and $0 \leq t \leq 1$. If R is sufficiently large, then it is the case that for all z with $|z| = R$ and all $t \in [0,1]$,

$$|h(z,t)| \geq R^n - (|a_{n-1}|R^{n-1} + \cdots + |a_0|) > 0.$$

As z traverses the boundary of the disk $D = \{z \in \mathbb{C} : |z| < R\}$ in a counterclockwise sense, $h(z,0) = z^n$ winds about the origin n times in the same sense. As t varies from 0 to 1, $h(z,0)$ is transformed continuously into $h(z,1)$ without touching the origin (since $|h(z,t)| > 0$). Thus is follows from (ii) that

$$\deg(h_1,D) = \deg(h_0,D) = n, \text{ where } h_t(z) = h(z,t).$$

(i) now implies that there exists $z_0 \in D$ for which $h(z_0,1) = 0$. This is the fundamental theorem of algebra.

MAPPING DEGREE IN $\mathbb{R}$. Suppose $-\infty < a < b < \infty$ and let $f:[a,b] \subset \mathbb{R} \longrightarrow \mathbb{R}$ be continuous with $f(a) \neq 0 \neq f(b)$. By perturbing f an arbitrarily small amount one may obtain (by Weierstrauss's approximation theorem) a function $\overline{f}$ such that

(a) $\overline{f}$ is continuously differentiable on $[a,b]$;

(b) $\overline{f}$ has at most a finite number of zeros, $x_1, \ldots, x_m$, in (a,b) and $\overline{f}'(x_i) \neq 0$ for all i.

Now set

$$\deg(f,G) = \sum_{1=1}^{m} \operatorname{sgn}(\overline{f}'(x_i)$$

where $G = (a,b)$, and define

$$\deg(f,G) = \deg(\overline{f},G).$$

If $\overline{f}$ has no zeros on (a,b), let $\deg(f,G) = 0$. It is possible to show that $\deg(f,G)$ is independent of the choice of the approximating function $\overline{f}$. In fact, in general:

$$\deg(f,G) = 0 \text{ if } f(a)f(b) > 0;$$
$$\deg(f,G) = 1 \text{ if } f(a) < 0,\ f(b) > 0;$$
$$\deg(f,G) = -1 \text{ if } f(a) > 0,\ f(b) < 0.$$

Example: Let $[a,b] = [-1,1]$ and consider, respectively, $f(x) = x^2$; $f(x) = x$; $f(x) = x^3$.

MAPPING DEGREE IN $\mathbb{R}^n$. For a fixed $y \in \mathbb{R}^n$ $\deg(f,G,y)$ is the number of solutions of the equation

$$f(x) = y, \quad (x \in \overline{G}),$$

With positive and negative values assigned according to whether orientation is preserved or reversed. When $y = 0$, we simply write $\deg(f,G)$.

Let G be a bounded open subset of $\mathbb{R}^n$ with $f:\overline{G} \longrightarrow \mathbb{R}^n$ continuous. As in the case $n = 1$, it is possible to approximate f with a function $\overline{f}$ which is continuously differentiable on G and which has at most finitely many zeros, $x_1, \ldots, x_m$, in G, each of which is regular $(\det f'(x_i) \neq 0)$. Then define

$$\deg(f,G,y) = \deg(\overline{f},G,y)$$

where

$$\deg(\overline{f},G,y) = \sum_{i=1}^{m} \operatorname{sgn}(\det f'(x_i)).$$

Specifically, set $\deg(f,G,y) = 0$ if $G = \phi$ and assume

(a) $$f(x) \neq y \text{ for } x \in \partial G,$$

and approximate f with a mapping $\overline{f}:G \longrightarrow \mathbb{R}^n$ which is C^1 on G and which satisfies

(b) $$\sup_{x \in \partial G} \|f(x) - \overline{f}(x)\| < \inf_{x \in \partial G} \|f(x) - y\|,$$

and so that the equation

$$\overline{f}(x) = y$$

either has no solutions or it has finitely many solutions each of which is regular. Note that (b) insures $\overline{f}(x) \neq y$ for all $x \in \partial G$. If there are no solutions, set $\deg(f,G,y) = 0$.

It is possible to show that the number $\deg(f,G,y)$ is independent of the approximating function $\overline{f}$ and that the function deg is the *unique* function defined for all continuous $f:\overline{G} \longrightarrow \mathbb{R}^n$ which satisfies:

1) If $y \notin \partial G$, $\deg(f,G,y)$ is an integer, and if $\deg(f,G,y) \neq 0$ then the equation

$f(x) = y$ has a solution.

2) If $f = I$, $\deg(f,G,y) = 1$ if $y \in G$ and $\deg(f,G,y) = 0$ if $y \notin \overline{G}$.

3) (Additivity): $\deg(f,\cup G_i,y) = \Sigma_i \deg(f,G_i y)$ if G_i, $i = 1, ..., m$, are disjoint open sets with $y \notin \partial G_i$.

4) (Homotopy): If $h:[0,1] \times G \longrightarrow \mathbb{R}^n$ is continuous, and if $h_t(x) \equiv h(t,x) \neq y$ for all $x \in \partial G$, then

$$\deg(h_0,G,y) = \deg(h_1,G,y).$$

5) (Excision): If K is a closed subset of $\overline{G}$ and if $y \notin f(K)$, then

$$\deg(f,G,y) = \deg(\tilde{f},G\backslash K,y)$$

where $\tilde{f}$ denotes the restriction of f to $g\backslash K$.

We now use degree theory to prove the following simple extension of the Brouwer fixed point theorem.

THEOREM. *Suppose* B *is the unit ball in* $\mathbb{R}^n$ *and suppose* $T:B \longrightarrow \mathbb{R}^n$ *is a continuous mapping which satisfies*

(LS) $$T(x) \neq \lambda x \quad \textit{for all } x \in \partial B \text{ and } \lambda > 0.$$

Then T *has a fixed point in* B.

Proof. For $t \in [0,1]$ let h_t be the mapping defined by taking $h_t(x) = x - tT(x)$, $x \in B$, i.e., $h_t = I - tT$. First note that $h_t(x) \neq 0$ for $x \in \partial B$ and $t \in [0,1]$ since $h_t(x) = 0$ implies $tT(x) = x$, i.e., $T(x) = t^{-1}x$, and this contradicts (LS). Therefore $0 \notin h_t \partial B$ and by 1) $\deg(h_t,B,0)$ is defined. By 4) and 2)

$$1 = \deg(h_0,B,0) = \deg(h_1,B,0),$$

and again by 1) $0 \in h_1(B)$. Clearly this implies that T has a fixed point in B.

MAPPING DEGREE IN INFINITE DIMENSIONS. We now turn to the theory developed by Leray and Schauder. Let X be an arbitrary Banach space. A mapping T defined on a subset D of X and taking values in X is said to be *compact* if T is continuous and if T maps bounded sets into sets whose closures are compact. Using approximations similar to the one used in the proof of the Schauder fixed point theorem it is possible to extend degree theory to mappings of the form $I - T$ where T is a compact mappings. This degree theory inherets all the properties of the degree function in $\mathbb{R}^n$ and yields the following theorem.

THE LERAY–SCHAUDER THEOREM (1934). *Let* G *be a nonempty bounded open subset of a Banach space* X *and let* $T:\overline{G} \longrightarrow X$ *be a compact mapping which satisfies for some* $x_0 \in G$ *the condition*

(*LS*) $$T(x) - x_0 \neq \lambda(x - x_0) \text{ for all } x \in \partial G \text{ and } \lambda > 1.$$

Then T *has a fixed point in* G.

Proof. By a simple translation there is no loss in generality in assuming $x_0 = 0$. The proof now follows precisely the one given above for Brouwer's thoerem.

A GENERALIZATION OF SCHAUDER'S THEOREM. We now turn to a result which bridges the gap between the geometric and the topological theory.

Definition. Let M be a bounded subset of a complete metric space (X,d). The *Kuratowski measure of noncompactness* $\chi(M)$ of M is defined as follows:

$$\chi(M) = \inf\{\epsilon > 0 : \exists \ n \text{ such that } M \subseteq \bigcup_{i=1}^{n} A_i \text{ where } \operatorname{diam}(A_i) \leq \epsilon\}.$$

Note in particular that $\chi(\overline{M}) = \chi(M)$ and that $\chi(M) = 0$ if and only if $\overline{M}$ is compact. Also, if M is a subset of a Banach space it is possible to show that

$$\chi(\text{conv}M) = \chi(M) = \chi(\text{convcl}M).$$

We use these facts to prove a theorem which includes both Schauder's theorem as well as the existence part of Banach's theorem.

Let D be a subset of a Banach space X. A mapping $T:D \longrightarrow X$ is said to be a k–*set contraction*, $k \geq 0$, if T is bounded and continuous, and for all bounded subsets M M of D:

$$\chi(T(M)) \leq k\chi(M).$$

T is said to be *condensing* if T is bounded and continuous, and for all bounded subsets M of D for which $\chi(M) > 0$:

$$\chi(T(M)) < \chi(M).$$

Example: Suppose $T_1:D \longrightarrow X$ is a contraction mapping with Lipschitz constant $k < 1$ and suppose $T_2:D \longrightarrow X$ is a compact mapping. Then the mapping $T = T_1 + T_2$ is a k–set contraction and hence also condensing. The following generalization of Schauder's theorem was proved for k–set contractions, $k < 1$, by Darbo in 1955.

SADOVSKII'S THEOREM (1976). *Suppose* K *is a nonempty bounded closed and convex subset of a Banach space* X *and suppose* $T:K \longrightarrow K$ *is condensing. Then* T *has a fixed point.*

Proof. Let $x \in K$ and let Σ denote the family of all subsets D of K for which $x \in D$ and $T:D \longrightarrow D$. Now set

$$B = \bigcap_{D \in \Sigma} D$$

and let

$$C = \text{convcl}\{T(B) \cup \{x\}\}.$$

First we show that $B = C$. Since $x \in B$ and $T:B \longrightarrow B$ we have $C \subseteq B$. This implies that $T(C) \subseteq T(B) \subseteq C$. Hence, since $x \in C$, $C \in \Sigma$. Therefore $B \subseteq C$.

In view of the above, $T(C) = T(B) \subseteq C = B$. Also, $\chi(C) = \chi(T(C))$. Since T is condensing, it follows that $\chi(C) = 0$. Therefore C is compact and convex with $T:C \longrightarrow C$ continuous. By Schauder's theorem, T has a fixed point in C.

4. **SET THEORETIC FIXED POINT THEORY.** *A* set M is said to be *ordered* if M is nonempty and for certain pairs (x,y) in $M \times M$ there is a relation $\leq$ which satisfies:

(i) $x \leq x$ for all $x \in M$;

(ii) if $x \leq y$ and $y \leq x$ then $x = y$;

(iii) if $x \leq Y$ and $y \leq z$ then $x \leq z$.

(The notation $x < y$ means $x \leq y$ and $x \neq y$.)

Let N be a subset of an ordered set M. N is said to be a *chain* if N is nonempty and for all $x,y \in N$, either $x \leq y$ or $y \leq x$. An element $x \in N$ is said to be a *largest* (respectively, *smallest*) element of N if $y \leq x$ (respectively, $x \leq y$) for every $y \in N$, and $x \in N$ is said to be *maximal* if there is no $y \in N$ such that $x < y$. An ordered set M is said to be *well ordered* if every nonempty subset of M has a smallest element.

Note in particular that maximal elements need not be unique. Every greatest element of a set is also a maximal element. If M is an ordered set and $N \subseteq M$, then an element $y \in M$ is said to be the *supremum* (least upper bound) of N if (i) $x \leq y$ for all $x \in N$ and (ii) $x \leq u$ for all $x \in N$ implies $y \leq u$. When this occurs we write $y = \sup(N)$. The *infimum*, $\inf(N)$, is defined analogously as the greatest lower bound of N.

A *lattice* is an ordered set M with the property that $\inf\{x,y\}$ and $\sup\{x,y\}$ exist for all $x,y \in M$. A lattice M is said to be *complete* if $\inf(N)$ and $\sup(N)$ exist for all nonempty subsets N of M.

Examples. The set $\mathbb{R}$ of real numbers with the usual $\leq$ relation is ordered but not well ordered. Moreover, $\mathbb{R}$ is a chain. Also, $\mathbb{R}$ is a lattice but not a complete lattice. The set $\mathbb{N}$ of natural numbers is a well ordered chain. If X is any set then the set $M = 2^X$ of all subsets of X with the order relation: $A \leq B$ iff $A \subseteq B$, $A,B \in 2^X$, is a complete lattice. If $N \subseteq M$, then $\inf(N) = \bigcap_{A \in M} A$ and $\sup(N) = \bigcup_{A \in M} A$.

THE BOURBAKI–KNESER THEOREM (1940/1950). *Suppose* M *is an ordered set and suppose* $f:M \longrightarrow M$ *satisfies:*

(a) $x \leq f(x)$ *for all* $x \in M$;
(b) *every chain of* M *has a supremum.*
Then f *has a fixed point.*

The above theorem is trivial if one uses Zorn's Lemma. However a constructive proof (one that does not require the Axiom of Choice) exists. This proof is rather detailed, but the general idea is the following. A chain A in M is constructed with the properties

$$f(A) \subseteq A \text{ and } \sup(A) \in A.$$

Then if $u = \sup(A)$, $f(u) \leq u$ because $f(u) \in A$. On the other hand, $u \leq f(u)$ by (a). By (ii) of the definition of $\leq$, $u = f(u)$.

Remark. The Bourbaki–Kneser Theorem remains true if one replaces $x \lesssim f(x)$ with $f(x) \leq x$ in (a) and "supremum" with "infimum" in (B).

THEOREM (Amann, 1977). *Suppose* X *is an ordered set and* $f:X \longrightarrow X$ *satisfies*:
(a) f *is monotone increasing* ($x \leq y$ *implies* $f(x) \leq f(y)$);
(b) *every chain in* X *has a supremum*;
(c) $x_0 \leq f(x_0)$ *for some* $x_0 \in M$.
Then f has a smallest fixed point in the set $\{x \in X : x_0 \leq x\}$.

Proof. We set

$$M = \{x \in X : x_0 \leq x \leq f(x)\},$$

and apply the Bourbaki–Kneser Theorem. $M \neq \emptyset$ because $x_0 \in M$. Also, if $x \in M$ then by monoticity $f(x) \leq f(f(x))$, i.e., $f(x) \in M$.

Now let C be a chain in M. Then C has a supremum in X by (b). Since $x \leq u$ for all $x \in C$ implies $x \leq f(x) \leq f(u)$ for all $x \in M$. Therefore $f(u)$ is an upper bound for C and since $u = \sup(C)$, we have $u \leq f(u)$. And since $x_0 \in M$, $x_0 \leq u$, we conclude $u \in M$.

The existence of a fixed point for f now follows from the Bourbaki–Kneser Theorem.

Next let F denote the fixed point set of f in M. Set

$$N = \{y \in M : y \leq z \text{ for each } z \in F\}.$$

$N \neq \emptyset$ again because $x_0 \in N$. It is now possible to repeat the above argument to conclude that f has a fixed point in N. Obviously this is the smallest fixed point in M and hence in $\{x \in X : x_0 \leq x\}$.

Remark. Amann's Theorem may be modified to show that f has a greatest fixed point in the set $\{x \in X : x \leq x_0\}$ if one replaces "supremum" with "infimum" in (b) and $x_0 \leq f(x_0)$ with $f(x_0) \leq x_0$ in (c).

TARSKI'S THEOREM (1955). *Let* X *be a complete lattice and* $f:X \longrightarrow X$ *a monotone increasing mapping. Then* f *has a smallest and a greatest fixed point in* X.

Proof. Since X is a complete lattice every nonempty subset of X has both an infimum and a supremum. Let $u_0 = \inf(X)$ and $v_0 = \sup(X)$. Then $u_o \leq f(u_0)$ and $f(v_0) \leq v_0$. The conclusion now follows from Amann's Theorem.

BIBLIOGRAPHY

[1] Amann, H., *Order Structures and Fixed Points*, Ruhr Universität Lecture Notes, Bochum, 1977.

[2] Banach, S., Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fund. Math. 3(1922), 133–181.

[3] Bourbaki, N., *Topologie générale*, Hermann, Paris, 1940.

[4] Brouwer, L., Über Abbildungen von Mannnigfaltigkeiten, Math. Ann. 70(1912), 97–115.

[5] Browder, F., Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci. USA 54(1965), 1041–1044.

[6] Browder, F., *Nonlinear Operators and Nonlinear Equations of Evolution*, Proc. AMS Symp Pure Math., vol. 18, pt2, Providence, RI, 1976.

[7] Caristi, J., Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc. 215(1976), 241–251.

[8] Clarkson, J. A., Uniformly convex spaces, Trans. Amer. Math. Soc. 40(1936), 396–414.

[9] Darbo, G., Punti uniti in transformazioni a codominio non compatto, Rend. Sem. Univ. Padua 24(1955), 84–92.

[10] Edelstein, M., On fixed and periodic points under contractive mappings, J. London Math. Soc. 37(1962), 74–79.

[11] Göhde, D., Zum Prinzip der kontraktiven Abbildung, Math. Nachr. 30(1965), 251–258.

[12] Kirk, W. A., A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly 72(1965), 1004–1006.

[13] Kirk, W. A., Metric fixed point theory: Nonexpasive mappings, Proceedings of the Fourth International Workshop in Analysis, Dubrovnik, 1990.

[14] Kneser, H., Eine direkte Ableitung des Zornschen Lemmas aus dem Auswahlaxiom, Math. Z. 53(1950), 110–113.

[15] Klee, V., Some topological properties of convex sets, Trans. Amer. Math. Soc. 78(1955), 30–45.

[16] Kuratowski, C., *Topologie*, PWN, Warsaw, 1940.

[17] Leray, J. and Schauder, J., Topologie et équations fonctionelles Ann. Sci. Ecole Norm. Sup. 51(1934), 45–78.

[18] Mauldin, D. (ed.), *The Scottish Book: Mathematical Problems from the Scottish*

Cafe, Birkhauser, Boston, 1981.

[19] Nowak, B., On the Lipschitz retraction of the unit ball in infinite dimensional Banach spaces onto boundary, Bull. Acad. Polon. Sci. 27(1979), 861–864.

[20] Sadovskii, B., On a fixed–point principle, Analiz i Prilozen. 1(1967), 74–76 (Russian).

[21] Schauder, J., Der fixpunktsatz in Funktionalraumen, Studia Math. 2(1930), 171–180.

[22] Tarski, A., A lattice–theoretical fixed–point theorem and its applications, Pacific J. Math. 5(1955), 285–309.

[23] Zabrieko, P. and Krasnoselskii, M., Iteration of operators and fixed points, Doklady Akad. Nauk SSSR 196(1971), 1006–1009 (Russian).

[24] Zeidler, E., *Nonlinear Functional Analysis and its Applications* I: *Fixed Point Theorems*, Springer–Verlag, New York, Berlin, Heidelberg, Tokyo, 1986.