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EXTENSION OF THE RANGE CONDITION FOR THE CONTROLLABILITY OF NON-AUTONOMOUS LINEAR SYSTEMS

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Abstract. In this paper we extend the wellknown Kalman range condition for the controllability of the non-autonomous linear control system

$$\dot{x}(t) = \left(\sum_{i=1}^{m} a_i(t)A_i\right)x(t) + Bu(t) ,$$

where the state $x(t) \in \mathbb{R}^n$, A_i and B are matrices of $n \times n$ and $n \times \ell$ dimensions respectively, $a_i(t) \in \mathbb{R}$ are fixed functions, $u(t) \in \mathbb{R}^{\ell}$ is the control; besides the Lie algebra generated by $A_1, A_2, \ldots, A_m$ under the commutator product has dimension m. For this system we give range type conditions, necessary and sufficient for the controllability.

Key words: *Non-autonomous system, controllability, range condition*

1. **Introduction.** In this work we study the problem of the controllability asociated with the control system:

$$(1.1) \qquad \dot{x}(t) = \left(\sum_{i=1}^{m} a_i(t)A_i\right)x(t) + Bu(t) , \quad t \geq 0 .$$

Obviously, all the non-autonomous systems

$$(1.2) \qquad \dot{x}(t) = A(t)x(t) + Bu(t) \ , \ t \geq 0 \ ,$$

may be written in the system (1.1) form.

Here we will suppose that A_i, $i = 1,2,\ldots,m$, are real matrices of $n \times n$ dimension and the Lie algebra, generated by these matrices, has m dimension; B is a real matrix of $n \times \ell$ dimension. The control function $u(t) \in \mathbb{R}^{\ell}$ and the functions $a_i(t) \in \mathbb{R}$, $A(t) \in \mathbb{R}^{n \times n}$ are locally integrables on $\mathbb{R}$.

When the system is linear and autonomous it has the form:

$$\dot{x}(t) = Ax(t) + Bu(t) \ , \ t \geq 0 \ ,$$

a necessary and sufficient condition for the controllability is given by the wellknown Kalman criterion, which can be written as

$$(1.3) \qquad \mathrm{Sp}\{B\mathbb{R}^{\ell}, AB\mathbb{R}^{\ell}, \ldots, A^{n-1}B\mathbb{R}^{\ell}\} = \mathbb{R}^n \ ,$$

where $A^iB\mathbb{R}^{\ell}$ is the range of the operator A^iB, $i = 0,1,2,\ldots,n-1$ and Sp{ } is the linear hull of these ranges. See [1,pg 81].

In this work we will extend the controllability condition (1.3) for the linear non-autonomous system (1.1), this is to say, we generalize the Kalman range condition.

2. **Preliminaries and Definitions.** From now on we will suppose that the matrices $A_1, A_2, \ldots, A_m$ generate a Lie algebra of m dimension

$$L = L(A_1, A_2, \ldots, A_m)$$

with the commutator product $[A_i, A_j] = A_i A_j - A_j A_i$.

We shall use the following results:

LEMMA 1.1 (Baker-Hausdorff). If X, Y ε L, then $e^X Y e^{-X} \ \varepsilon \ L$ and its given by the formula

(2.1) $e^X Y e^{-X} = Y + [X,Y] + [X,[X,Y]]/2! + [X,[X,[X,Y]]]/3! + \ldots + \ldots$;

See [2], [3] .

If we use the notation adX, to represent the linear operator defined on L, through

$$(\mathrm{ad}X)Y = [X,Y] \ , \ (\mathrm{ad}X)^2 Y = [X,[X,Y]] \ , \text{ etc.}$$

Then the formula (2.1) can be written as follows:

(2.2) $$e^X Y e^{-X} = (e^{\mathrm{ad}X})Y \ , \qquad \text{See [2],}$$

LEMMA 2.2. Be $A_1, A_2, \ldots, A_m$ a L base which multiplication table is

$$[A_i, A_j] = \sum_{k=1}^{m} \gamma_{ij}^{k} A_k \ , \ i,j = 1,2,\ldots,m \ .$$

Then

$$(2.3) \qquad \prod_{j=1}^{r} \exp(g_j \operatorname{ad} A_j) A_i = \left(\prod_{j=1}^{r} \exp(g_j A_j)\right) A_i \left(\prod_{j=r}^{1} \exp(-g_j A_j)\right) =$$

$$= \sum_{k=1}^{m} \xi_{k_i} A_k \ , \ r = 1,2,\ldots,m \ ;$$

where each $\xi_{k_i} = \xi_{k_i}(g_1,\ldots,g_r)$ is an analytic function of g_1 to g_r .

Using (2.3) in [3] proves that in an interval of the form $[0,T]$, the fundamental matrix Φ of the system (1.1) can be written as

$$(2.4) \qquad \Phi(t) = \prod_{j=1}^{m} \exp(g_j(t) A_j) \ , \ 0 \leqq t \leqq T \ ,$$

and the functions g_j are determined by the relation

$$(2.5) \qquad \sum_{i=1}^{m} a_i(t) A_i = g_1' A_1 + \sum_{i=2}^{m} g_i' \left(\prod_{j=1}^{i-1} \exp(g_j \operatorname{ad} A_j)\right) A_i \ ,$$

which is equivalent to

$$a_k(t) = \sum_{i=1}^{m} g_i'(t) \xi_{k_i}(g(t)) \ ; \ g = (g_1,\ldots,g_m) \ .$$

We shall also use the following theorem, which is a consequence of the Cayley Hamilton theorem.

THEOREM 2.1. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$, the eigenvalues of a matrix A of $n \times n$ dimension.

Let us define the polynomi succession in A as follows

$$(2.6) \qquad P_0(A) = I \ , \ P_k(A) = \prod_{j=1}^{k} (A - \lambda_j I) \ , \ k = 1,2,\ldots,n-1 \ .$$

Then

$$(2.7) \qquad e^{tA} = \sum_{k=0}^{n-1} \beta_{k+1}(t) P_k(A) \ ,$$

where the coefficients $\beta_1, \beta_2, \ldots, \beta_n$ are determined by recurrence, beginning with the following linear differential equation system

$$(2.8) \qquad \begin{aligned} &\dot{\beta}_1(t) = \lambda_1 \beta_1(t) \ , \ \beta_1(0) = 1 \\ &\dot{\beta}_{k+1}(t) = \lambda_{k+1} \beta_{k+1}(t) + \beta_k(t) \ , \ \beta_{k+1}(0) = 0 \end{aligned}$$

where $k = 1,2,3,\ldots,n-1$.

Therefore for each matrix A_j, $j = 1,2,\ldots,m$ with their eigenvalues $\lambda_1^j, \lambda_2^j, \ldots, \lambda_n^j$, results that

$$(2.9) \qquad e^{tA_j} = \sum_{k_j=0}^{n-1} \alpha_{k_j}^j(t) A_j^{k_j} \ , \ \alpha_{k_j}^j(t) = \sum_{k=k_j}^{n-1} \beta_{k+1}^j(t) \gamma_{k_j}$$

where

$$\gamma_{k_j} = \gamma_{k_j}(\lambda_1^j, \lambda_2^j, \ldots, \lambda_n^j) \ .$$

Then for each $j = 1,2,\ldots,m$ we have that

$$(2.10) \qquad e^{-g_j(t)A_j} = \sum_{k_j=0}^{n-1} \alpha_{k_j}^{j}(-g_j(t))A_j^{k_j} .$$

DEFINITION 2.1. The non-autonomous system (1.1) is controllable on $[0,T]$ in case: for each pair $x_0, x_1 \in \mathbf{R}^n$, there is an admissible control, $u \in L_1(0,T;\mathbf{R}^\ell)$ so that the corresponding solution $x_u(\cdot)$, satisfies the condition, $x_u(0) = x_0$ and $x_u(T) = x_1$.

From (2.4) results that the solution $x(\cdot)$ from (1.1) corresponding to the control $u \in L_1(0,T;\mathbf{R}^\ell)$ with $x(0) = x_0$, can be written as

$$(2.11) \qquad x(t) = e^{g_1(t)A_1} \cdot e^{g_2(t)A_2} \cdot \ldots \cdot e^{g_m(t)A_m}[x_0 + \int_0^t e^{-g_m(s)A_m} \cdot \ldots \cdot e^{-g_1(s)A_1} Bu(s)ds] , \; 0 \leq t \leq T .$$

Consider the operator $G : L_1(0,T;\mathbf{R}^\ell) \to \mathbf{R}^n$ defined by the relation

$$(2.12) \qquad Gu = \int_0^T \Phi^{-1}(s)Bu(s)ds$$

where $\Phi^{-1}(t) = e^{-g_m(t)A_m} \cdot \ldots \cdot e^{-g_1(t)A_1}$; obviously G is a linear and continuous operator.

PROPOSITION 2.1. The following statements are equivalent:

a) The system (1.1) is controllable on $[0,T]$

b) $GL_1(0,T;\mathbb{R}^{\ell}) = \text{Range } G = \mathbb{R}^n$

c) If $B^*(\Phi^{-1}(t))^*\xi = 0$, $0 \leq t \leq T \Rightarrow \xi = 0$

3. **Range Conditions.** In this section we will present the results of this work.

THEOREM 3.1. If the system (1.1) is controllable on $[0,T]$, then it hold the following range condition

$$\text{(3.1)} \quad Sp\{A_m^{k_m} \cdots A_1^{k_1} B\mathbb{R}^{\ell} / k_i = 0,1,\ldots,n-1 \,;\, i = 1,2,\ldots,m\} = \mathbb{R}^n \,.$$

<u>Proof</u>. If the system (1.1) is controllable on $[0,T]$, then by the Proposition 2.1 the operator G defined by (2.12) has range $\mathbb{R}^n$. From (2.10) this operator can be written as follows:

$$Gu = \int_0^T \Big(\sum_{k_m=0}^{n-1} \alpha_{k_m}^m(-g_m) A_m^{k_m}\Big)\Big(\sum_{k_{m-1}=0}^{n-1} \alpha_{k_{m-1}}^{m-1}(-g_{m-1}) A_{m-1}^{k_{m-1}}\Big)$$

$$\cdots\Big(\sum_{k_1=0}^{n-1} \alpha_{k_1}^1(-g_1) A_1^{k_1}\Big) Bu(s)\,ds =$$

$$= \sum_{k_m,\ldots,k_1=0}^{n-1} A_m^{k_m} \cdots A_1^{k_1} BU(k_m, k_{m-1},\ldots,k_1) \;,$$

where

$$(3.2)\quad U(k_m, K_{m-1}, \dots, k_1) = \int_0^T \prod_{i=0}^{m} \alpha_{k_i}^{i}(-g_i)u(s)ds \in \mathbf{R}^{\ell} .$$

This implies that the operator

$$\overline{G} : Y \to \mathbb{R}^n \; ; \; Y = \bigtimes_{i=1}^{n^m} \mathbb{R}^{\ell} ,$$

defined by:

$$\overline{G}y = \sum_{k_m,\dots,k_1=0}^{n-1} A_m^{k_m} \dots A_1^{k_1} By(k_m,\dots,k_1) ,$$

where, $y(k_m,\dots,k_1)$ are the components of the vector $y \in Y$; has full range, which is equivalen to:

$$Sp\{A_m^{k_m} \dots A_1^{k_1} B\mathbf{R}^{\ell} / k_i = 0,1,\dots,n-1 \; ; \; i = 1,2,\dots,m\} = \mathbf{R}^n .$$

Q.E.D.

OBSERVATION 3.1. The condition (3.1) is not sufficient for the controllability of the system (1.1). Indeed, let us consider the case in which $a_i(t) \equiv 0$, $i = 1,2,\dots,m$; this is to say, the system (1.1) is simply $\dot{x}(t) = Bu(t)$, and the operator G becomes

$$Gu = \int_0^T Bu(s)ds = B\int_0^T u(s)ds \; , \; u \in L\,(0,T;\mathbf{R}^{\ell}) ,$$

which maps L_1 onto the whole $\mathbb{R}^n$, if and only if B has range equal to $\mathbf{R}^n$.

The condition (3.1) implies the controllability of the system (1.1) if we add some type of condition to the $g_i(\cdot)$, $i = 1,2,\ldots,m$, functions.

According to (3.2) we introduce the following definition:

DEFINITION 3.1. Given a partition π of $[0,T]$ into n^m intervals:

$$0 = t_0 < t_1 < t_2 < \ldots < t_{n^m} = T .$$

The following matrix will be called the matrix of controllability associated with (1.1)

$$C_\pi = \left(\left(\int_{t_{i-1}}^{t_i} \prod_{j=1}^{m} \alpha_{k_j}^{j} \, ds \right) I \right)_{n^m \times n^m}$$

where I - identity matrix of $\mathbf{R}^{\ell \times \ell}$, $i = 1,2,\ldots,n^m$, $k_j = 0,1,2,\ldots,n-1$; $j = 1,2,\ldots,m$; $\alpha_{k_j}^{j} = \alpha_{k_j}^{j}(-g_j(\cdot))$.

From (2.5) results that the matrix C_π only depends of the coefficients $a_i(\cdot)$ and the Lie algebra L.

THEOREM 3.2. If the range condition (3.1) is true and there is a partition π of $[0,T]$ into n^m intervals, such that the matrix C_π defined by (3.3) is nonsingular, then the system (1.1) is controllable on [0,T].

Proof. The condition (3.1) implies that the operator $\overline{G}$ defined by the relations

$$\overline{G} : Y \to \mathbf{R}^n , \quad Y = \mathop{\times}_{i=1}^{n^m} \mathbf{R}^\ell$$

$$\overline{G}y = \sum_{k_m,\ldots,k_1=0}^{n-1} A_m^{k_m} \ldots A_1^{k_1} B y(k_m,\ldots,k_1) ,$$

where $y(k_m,\ldots,k_1)$, $k_i = 0,1,2,\ldots,n-1$; $i = 1,2,\ldots,m$ are the components of the vector $y \in Y$, has full range: $\overline{G}Y = \mathbf{R}^n$.

We must prove that the operator G maps $L_1(0,T;\mathbf{R}^\ell)$ onto the whole of $\mathbf{R}^n$, for which is suficient to prove that if G is restricted to the class of piecewise constant controls has full range. According to the hypotheses, there is a partition π of [0,T]

$$\pi : 0 = t_0 < t_1 < t_2 < \ldots < t_{n^m} = T$$

such that the matrix C_π is nonsingular.

Consider piecewise constant controls on this partition:

$$u(t) = U_i \in \mathbf{R}^\ell , \quad t_{i-1} \leq t \leq t_i , \quad i = 1,2,\ldots,n^m .$$

Then

$$Gu = \int_0^T \Big(\sum_{k_m=0}^{n-1} \alpha^m_{k_m} A_m^{k_m}\Big)\ldots\Big(\sum_{k_1=1}^{n-1} \alpha^1_{k_1} A_1^{k_1}\Big)Bu(s)ds =$$

$$= \sum_{k_m,\ldots,k_1=0}^{n-1} A_m^{k_m} A_{m-1}^{k_{m-1}} \ldots A_1^{k_1} B \int_0^T \prod_{j=1}^{m} \alpha^j_{k_j} \; u(s)ds =$$

$$= \sum_{k_m,\ldots,k_1=0}^{n-1} A_m^{k_m} A_{m-1}^{k_{m-1}} \ldots A_1^{k_1} B \sum_{i=1}^{n^m} \int_{t_{i-1}}^{t_i} \prod_{j=1}^{m} \alpha^j_{k_j} U_i ds$$

This implies that

$$Gu = \overline{G}\cdot C\cdot \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_{n^m} \end{pmatrix}_{n^m \times 1}$$

As $C_\pi Y = Y$ and $\overline{G}Y = \mathbf{R}^n$, we obtain that Rang $G = \mathbf{R}^n$.

Q.E.D.

COROLLARY 3.1. If the system (1.2) is controllable on $[0,T]$, then

$$Sp\{ \prod_{i=n}^{1} (\prod_{j=n}^{1} A_{ij}^{k_{ij}})B\mathbf{R}^{\ell}/k_{ij} = 0,1,2,\ldots,n-1 \; ; \; i,j = 1,2,\ldots,n\} = \mathbf{R}^n \; ,$$

where $\{A_{ij}/i,j = 1,2,\ldots,n\}$ is the canonic base of the matrices space

Proof. Results from the fact

$$A(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}(t) A_{ij}$$

and Lie algebra generated by A_{ij}, has n^n dimension.

Q.E.D.

OBSERVATION 3.2. It is possible that in the practice we find the system (1.1) in the following forms:

$$(3.5)\qquad \dot{x}(t) = (\sum_{i=1}^{m} a_i(t,x(t),u(t))A_i)x(t) + Bu(t)\ ,\ t \geq 0\ ,$$

moreover we will suppose that functions $a_i : \mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^\ell \to \mathbb{R}$, $i = 1,2,\ldots,m$, are locally integrable and for each $x(\cdot) \in C(0,T,\mathbb{R}^n)$, $u(\cdot) \in L_1(0,T;\mathbb{R}^\ell)$, $a_i(\cdot,x(\cdot),u(\cdot))$ is integrable on $[0,T]$.

COROLLARY 3.2. If the system (3.5) is controllable on $[0,T]$, then

$$\mathrm{Sp}\{A_m^{k_m}A_{m-1}^{k_{m-1}}\ldots A_1^{k_1}B\mathbb{R}^\ell / k_i = 0,1,2,\ldots,n-1\ ;\ i = 1,2,\ldots,m\} = \mathbb{R}^n\ .$$

The following Theorems, 3.3 and 3.4, are proved in analogous form to the Theorems 3.1 and 3.2, respectively.

THEOREM 3.3. If the system (1.1) is controllable on $[0,T]$, then the range condition is true:

$$(3.6)\qquad \mathrm{Sp}\{P_{k_m}(A_m)P_{k_{m-1}}(A_{m-1})\ldots P_{k_1}(A_1)B\mathbb{R}^\ell / k_i = 0,1,\ldots,n-1\ ;\ i = 1,2,\ldots,m\} = \mathbb{R}^n\ ,$$

where $P_{k_i}(A_i)$ are given by (2.6)

OBSERVATION 3.3. According to the Theorem 2.1 the operator G defined by (2.12) may also be written as follows:

$$Gu = \sum_{k_m,\ldots,k_1=0}^{n-1} P_{k_m}(A_m)\ldots P_{k_1}(A_1)B\int_0^T \prod_{i=1}^{m} \beta_{k_i}^{i}(-g_i(s))u(s)ds ,$$

and the analogous matrix to C_π defined in (3.3) adopts the following form:

$$\text{(3.7)} \qquad \overline{C}_\pi = \left(\left(\int_{t_{i-1}}^{t_1} \prod_{i=1}^{m} \beta_{k_i}^{i}(-g_i(s))ds\right)I\right)_{n^m \times n^m}$$

I - Identity Matrix of $\mathbf{R}^{\ell \times \ell}$.

THEOREM 3.4. If the range condition (3.6) is true and there is a partition π of [0,T] into n^m intervals, such that the matrix $\overline{C}_\pi$ defined by (3.7) is nonsingular then the system (1.1) is controllable on [0,T].

Next we will give condition sufficient for the system (1.1) controllability, different from those given in the Theorems 3.2 and 3.4, respectively.

THEOREM 3.5. If the range condition (3.1) is true and the set of functions

$$(3.8) \qquad \{\prod_{i=1}^{m} \alpha^{i}_{k_i}(-g_i(\cdot))/k_i = 0,1,\ldots,n-1\} ,$$

is linearly independent on [0,T], then the system (1.1) is controllable on [0,T].

<u>Proof</u>. If the system (1.1) is not controllable on [0,T], then, by the Proposition 2.1 there is $\xi \in \mathbf{R}^n$ such that:

$$B^*(\Phi^{-1}(t))^*\xi = 0 , \; 0 \leqq t \leqq T \quad \text{and} \quad \xi \neq 0 ,$$

this implies that

$$\langle \xi , \Phi^{-1}(t)Bv\rangle = 0 , \; (t \in [0,T] , \; v \in \mathbf{R}^{\ell}) .$$

From (2.4) and (2.10) results

$$\langle \xi , \sum_{k_m,\ldots,k_1=0}^{n-1} \alpha^m_{k_m}(-g_m(t)\ldots\alpha^1_{k_1}(-g_1(t))A_m^{k_m}\ldots A_1^{k_1}Bv\rangle =$$

$$= \sum_{k_m,\ldots,k_1=0}^{n-1} \alpha^m_{k_m}(-g_m(t))\ldots\alpha^1_{k_1}(-g_1(t))\langle \xi , A_m^{k_m}\ldots A_1^{k_1}Bv\rangle = 0$$

($t \in [0,T]$, $v \in \mathbf{R}^{\ell}$). Since the given set (3.8) is linearly independent on [0,T], it follows

$$\langle \xi , A_m^{k_m}A_{m-1}^{k_{m-1}}\ldots A_1^{k_1}Bv\rangle = 0 , \; (k_i = 0,1,\ldots,n-1 \; ; \; v \in \mathbf{R}^{\ell}) ,$$

this is to say:

$$\mathrm{Sp}\{A_m^{k_m}\ldots A_1^{k_1}B\mathbf{R}^\ell / k_i = 0,1,\ldots,n-1 \; ; \; i = 1,2,\ldots,m\} \subsetneqq \mathbf{R}^n \; ,$$

which is a contradiction. Therefore the system (1.1) is controllable on [0,T].

Q.E.D.

THEOREM 3.6. If the range condition (3.6) is true and the set of functions

$$\{\prod_{i=1}^{m} \beta^i_{k_i}(-g_i(\cdot))/k_i = 0,1,2,\ldots,n-1\} \; , \tag{3.9}$$

is linearly independent on [0,T], then the system (1.1) is controllable on [0,T].

Proof. Is analogous to the previous theorem, because from (2.4) and (2.7) we can write $\Phi^{-1}(t)$ as follows:

$$\Phi^{-1}(t) = \sum_{k_m,\ldots,k_1=0}^{n-1} \beta^m_{k_m}(-g_m(t))\ldots\beta^1_{k_1}(-g_1(t))P_{k_m}(A_m)\ldots P_{k_1}(A_1) \; .$$

Q.E.D.

4. **Aplications.** Let us consider now some examples to apply the just obtained results.

EXAMPLE 4.1. Consider the control system in $\mathbf{R}^2$

$$(4.1) \qquad \dot{x}(t) = a(t)Ax(t) + Bu(t) \ , \ t \geq 0 \ ,$$

where

$$a(t) = \frac{1}{t+1} \ , \ A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} , \ B = \begin{pmatrix} 1 \\ 2 \end{pmatrix} , \ u(t) \in \mathbf{R} \ .$$

Then $Sp\{B\mathbf{R}, AB\mathbf{R}\} = \mathbf{R}^2$.

The eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 0$, according to the Theorem 2.1, it results that:

$$\exp(tA) = I + \frac{1}{2} e^{2t} A$$

and the fundamental matrix of (4.1) is

$$\Phi(t) = \exp(\int_0^T a(s)dsA) = \exp(\ln(t+1)A) \ ,$$

this implies that:

$$\Phi^{-1}(t) = I + \frac{1}{2} (t+1)^{-2} A \ .$$

Therefore the operator G defined by (2.12) can be written as

$$Gu = B\int_0^T u(s)ds + AB\int_0^T \frac{1}{2} (s+1)^{-2} u(s)ds \ .$$

Lets consider the following partition of [0,T]

$$\pi : 0 = t_0 \ , \ t_1 = T/2 \ , \ t_3 = T$$

and the matrix

$$C_\pi = \begin{pmatrix} \int_0^{T/2} ds I & \int_{T/2}^{T} ds I \\ \int_0^{T/2} 1/2(s+1)^{-2} ds I & \int_{T/2}^{T} 1/2(s+1)^{-2} ds I \end{pmatrix}$$

Obviously C_π is nonsingular, and from the Theorem 3.2 we have that the system (4.1) is controllable on each interval of the form [0,T], (T > 0). This result is obtained inmediatelly applying the Theorem 3.5, because

$$\Phi^{-1}(t) = \alpha_0(-g(t))I + \alpha_1(-g(t))A \ ,$$

where

$$g(t) = \int_0^t a(s)ds = \ln(t + 1) \ ,$$

$$\alpha_0(-g(t)) = 1$$

and

$$\alpha_1(-g(t)) = 1/2(t + 1)^{-2} \ ;$$

it is easy to prove that $\{\alpha_0(g(\cdot)), \alpha_1(g(\cdot))\}$ is a linearly independent set on [0,T].

EXAMPLE 4.2. Lets consider the control system in $\mathbb{R}^2$, given by

$$(4.2) \qquad \dot{x}(t) = A(t)x(t) + Bu(t) \;,\; t \geqq 0$$

where

$$A(t) = \begin{pmatrix} a_1(t) & a_2(t) \\ a_3(t) & a_4(t) \end{pmatrix} \;,\; B \in \mathbf{R}^{\ell \times \ell} \;.$$

In this case we can write A(t) as follows

$$A(t) = S(t) + \frac{a_1(t) + a_4(t)}{2} I \;,\; I \text{ - identity matrix}$$

of $\mathbb{R}^{2\times 2}$. Hence the solution Φ of the system

$$\dot{\Phi}(t) = A(t)\Phi(t) \;,\; \Phi(0) = I \;,$$

is given by

$$\Phi(t) = V(t)e^{g_4(t)} \;,\; g_4(t) = 1/2\int_0^t (a_1(s) + a_4(s))ds$$

and

$$\dot{V}(t) = S(t)V(t) \;,\; V(0) = I \;,$$

moreover

$$S(t) = \frac{a_1(t) - a_4(t)}{2} A_1 + a_2(t)A_2 + a_3(t)A_3$$

where

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Thus, from (2.5), we obtain:

$$\frac{1}{2}(a_1 - a_4)A_1 + a_2A_2 + a_3A_3 = g_1'A_1 + g_2'\exp(g_1\,\mathrm{ad}A_1)A_2 +$$

$$+ g_3'\exp(g_1\,\mathrm{ad}A_1)\exp(g_2\,\mathrm{ad}A_2)A_3 =$$

$$= (g_1' + g_3'g_2)A_1 + (g_2'e^{2g_1} - g_3'g_2^2e^{2g_1})A_2 + g_3'e^{-2g_1}A_3 ,$$

which implies that:

$$g_1' + g_2g_3' = \frac{1}{2}(a_1(t) - a_4(t))$$

$$(4.3) \qquad g_2'e^{2g_1} - g_3'g_2e^{2g_1} = a_2$$

$$g_3'e^{-2g_1} = a_3$$

$\Phi^{-1}(\cdot)$ adopts the following form

$$\Phi^{-1}(t) = e^{-g_4}(I - g_3A_3)(I - g_2A_2)(\cosh g_1 I - \operatorname{senh} g_1 A_1) =$$

$$= e^{-g_4}[\cosh g_1 I - \operatorname{senh} g_1 A_1 - g_2e^{g_1}A_2 - g_3e^{-g_1}A_3 +$$

$$+ g_3g_2\cosh g_1 A_3A_2 - g_3g_2\operatorname{senh} g_1 A_3A_2A_1] ,$$

because $A_2A_1 = -A_2$ and $A_3A_1 = A_3$.

If $B \in \mathbf{R}^{2\times\ell}$ is such that

$$\mathrm{Rank}[B \mid A_1B \mid A_2B \mid A_3B \mid A_3A_2B \mid A_3A_2A_1B] = 2$$

and $\{\cosh g_1, \mathrm{senh}\, g_1, g_2e^{g_1}, g_3e^{-g_1}, g_3g_2\cosh g_1, g_3g_2\mathrm{senh}\, g_1\}$ is a set linearly independent on $[0,T]$, then from the Theorem 3.4, the system (4.2) is controllable on $[0,T]$.

In case $a_3 \equiv 0$, from (4.3) it is obtained:

$$g_1(t) = 1/2\int_0^t (a_1(s) - a_4(s))ds ,$$

$$g_2(t) = \int_0^t \exp[\int_0^s (a_4(t) - a_1(t))dt]a_2(s)ds$$

$$g_4(t) = 1/2\int_0^t [a_1(s) + a_4(s)]ds ,$$

$$g_3 \equiv 0 .$$

For example, if $a_1(t) = t$, $a_2(t) = -1/2$, $a_3 \equiv 0$ and $a_4(t) = t - 1$, it is obtained that

$$\Phi^{-1}(t) = e^{-g_4}[\cosh \frac{1}{2} t\, I - \mathrm{senh} \frac{1}{2} t\, A_1 - (e^{-t/2} - 1)e^{t/2}A_2] .$$

It is not difficult to prove that the set

$$\{\cosh \frac{1}{2} t , \mathrm{senh} \frac{1}{2} t , (e^{-t/2} - 1)e^{t/2}\}$$

is linearly independent.

R E F E R E N C E S

[1] E.B. LEE and L. MARKUS, *Foundation of Optimal Control Theory*, John Wiley, New York 1967.

[2] W. MAGNUS, *On the Exponential Solution of Differential Equations for a Linear Operator*, Comm. Pure Appl. Math. 7 (1954), 649-673.

[3] J. WEI and E. NORMAN, *On Global Representation of the Solutions of Linear Differential Equations as a Product of Exponentials*, Americ.Math.Society, Vol 15, No. 1 (1964), 327-334.