NOTAS DE MATEMATICA

Nº 124

THE OPERATOR DECOMPOSITION METHOD FOR THE NUMERICAL SOLUTION OF A CLASS OF SINGULARLY PERTURBED PROBLEMS

BY

LILIAM ALVAREZ DIAZ

Y

ANTONIO R. RODRIGUEZ SANTIESTEBAN

UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS
DEPARTAMENTO DE MATEMATICA
MERIDA - VENEZUELA
1993

En ocasión de la visita de la Dra. Lilliam Alvarez Díaz de la Academia de Ciencias de Cuba.

Mérida, Enero de 1993.

THE OPERATOR DECOMPOSITION METHOD FOR THE NUMERICAL SOLUTION OF A CLASS OF SINGULARLY PERTURBED PROBLEMS

by

Lilliam Alvarez Díaz
Institute of Mathematics, Cybernetics and Physics
Cuban Academy of Sciences
Calle E esq. 15, Vedado, Habana, Cuba

Antonio R. Rodríguez Santiesteban
Institute of Mathematics, Cybernetics and Physics
Cuban Academy of Sciences
Calle E esq. 15, Vedado, Habana, Cuba

ABSTRACT

In this work, we propose to use the roots of the characteristic equation of the differential operator, to decompose the original boundary value problem (BVP) in a serie of initial value problems, which are easier to solve.

In particular, we have applied the Operator Decomposition Method (ODM) to the numerical solution of a class of linear singularly perturbed problems of the type

$$\varepsilon\, y''(x) + p(x)\, y'(x) + q(x)\, y(x) = f(x)$$

with $\varepsilon << 1$ and general linear or non-linear boundary conditions in [a,b]. We proved that the stiffer the original BVP the faster the algorithm converges, and it needs only a few iterations for $\varepsilon = 10^{-6}$ or 10^{-8}.

About 30 stiff tests reported in the literature (including boundary-layer, rappidly oscilatory and turning-point problems) have been solved with a very simple FORTRAN 77 code. All of them needed only a few seconds of execution time in a personal computer.

The ODM has been also applied to solve some interesting problems like the Orr-Sommerfeld equation, one-dimensional Stefan problem and a class of BVP in an unbounded interval.

1. Introduction

Singularly perturbed boundary value ordinary differential problems have received a big attention during the last years. It is shown by the variety of special methods proposed, difference methods as in [1] to [4], multiple shooting techniques as in [4] to [6], Riccati method as in [4] and [6], etc. The above words are an indication of the difficulty and the importance of the task involved.

In this work we mainly focus the following problem:

$$Py := \varepsilon y''(x) + p(x,\varepsilon)y'(x) + q(x,\varepsilon)y(x) = f(x,\varepsilon), \quad a \le x \le b \qquad (1.a)$$

under the boundary conditions

$\alpha_1 y(a) + \beta_1 y'(a) = \gamma_1(\varepsilon)$ (1.b)

$\alpha_2 y(b) + \beta_2 y'(b) = \gamma_2(\varepsilon)$ (1.c)

where P represents the differential operator, $0 < \varepsilon << 1$, p, q, f $\in C^1[a,b]$ and α_i, β_i, $\gamma_i(\varepsilon) \in \mathbb{R}$ for $i=1,2$. Moreover we supouse that (1) is well-conditioning.

We propose a new iterative method that is based on the roots of the differential operator characteristic equation, to decompose the problem (1) in a set of initial value problems, which are easier to solve.

The main ideas of the ODM were given in [7] for a fourth and second order equations.

In **2.**, we describe the method and the algorithm. In **3.** the convergence of a typical case is analysed, and it is shown under which conditions the method converges. The rate of convergence appears to be not slower than for a geometric progression whose ratio decreases as ε decreases, under certain conditions on the characteristic roots. It is also obtained,that the convergence does not depend on the initial approximation of the solution.

In **4.** some quadrature formulas are specialy obtained to solve the linear initial value problems in which solution the ODM is based. Some good properties of this formulas are given.

Paragraph 5. displays the numerical solution of five test problems and some especific applications such as, the solution of the one-dimensional Stefan problem, the Orr-Sommerfeld equation and BVP in an unbounded interval of the type $[0, \infty)$.

2. The Operator Decomposition Method

If D and D^2 denote the first and second derivatives, then (1) can be written as

$$Py := (D^2 + p(x,\varepsilon)/\varepsilon\, D + q(x,\varepsilon)/\varepsilon)y = f(x,\varepsilon)/\varepsilon \qquad (2)$$

and its characteristic equation is

$$\mu^2 + p(x,\varepsilon)/\varepsilon\, \mu + q(x,\varepsilon)/\varepsilon = 0 \qquad (3)$$

whose roots are

$$\mu_{1,2}(x,\varepsilon) = \frac{-p(x,\varepsilon)/\sqrt{\varepsilon} \pm \sqrt{p^2(x,\varepsilon)/\varepsilon - 4q(x,\varepsilon)}}{2\sqrt{\varepsilon}} \qquad (4.a)$$

and we set

$$\mu_{\imath 0}(x,\varepsilon) := \sqrt{\varepsilon}\ \ \mu_{\imath}(x,\varepsilon) \qquad (4.b)$$

for $\imath=1,2$.

Then by means of the characteristic roots, if μ_1' exists in $[a,b]$, we have

$$Py := (D - \mu_2(x,\varepsilon))(D - \mu_1(x,\varepsilon))y = f(x,\varepsilon)/\varepsilon - \mu_1'(x,\varepsilon)y \qquad (5)$$

multipling (5) by ε, considering (4.b) and denoting

$$z := \sqrt{\varepsilon}\ (D - \mu_1(x,\varepsilon))y \qquad (6)$$

we obtain the equivalent system

$$\sqrt{\varepsilon}\ z' - \mu_{20}(x,\varepsilon)z = f(x,\varepsilon) - \sqrt{\varepsilon}\ \mu_{10}'(x,\varepsilon)y \qquad (7.a)$$

$$\sqrt{\varepsilon}\ y' - \mu_{10}(x,\varepsilon)y = z(x) \qquad (7.b)$$

It is easy to see that if z_0, z_1, y_0, y_1 and y_2 are complex functions such that satisfy the following problems

$$\sqrt{\varepsilon}\ z_0' - \mu_{20}(x,\varepsilon)z_0 = f(x,\varepsilon) - \sqrt{\varepsilon}\ \mu_{10}'(x,\varepsilon)y,\ z_0(a) \text{ or } z_0(b) = 0 \qquad (8.a)$$

$$\sqrt{\varepsilon}\ z_1' - \mu_{20}(x,\varepsilon)z_1 = 0,\ z_1(a) \text{ or } z_1(b) = 1 \qquad (8.b)$$

$$\sqrt{\varepsilon}\ y_0' - \mu_{10}(x,\varepsilon)y_0 = z_0(x),\ y_0(a) \text{ or } y_0(b) = 0 \qquad (8.c)$$

$$\sqrt{\varepsilon}\ y_1' - \mu_{10}(x,\varepsilon)y_1 = z_1(x),\ y_1(a) \text{ or } y_1(b) = 0 \qquad (8.d)$$

$$\sqrt{\varepsilon}\ y_2' - \mu_{10}(x,\varepsilon)y_2 = 0,\ y_2(a) \text{ or } y_2(b) = 1 \qquad (8.e)$$

the solution of (1) can be written as

$$y(x) = \mathrm{Re}(y_0(x) + C_1 y_1(x) + C_2 y_2(x)) \qquad (9)$$

where C_1, C_2 can be found from (1.b) and (1.c), solving a linear system. Here Re represents the real part of a complex number.

We choose the condition in a or b according to the sign of the real part of (4.a) in order to get stable initial value problems.

Observing (8.a), we note that the right hand side of the equation depends on the solution of the original problem (1). We solve this difficulty by means of an iterative process as follows:

Step1. We solve the problems (8.b), (8.d) and (8.e).

Step2. Given an approximation for y, we solve the problem

(8.a).

Step3. Given z_0, we solve (8.c).

Step4. We compute C_1 and C_2 solving the linear system which results from (1.b), (1.c) and the expression (9).

Step5. We form a new approximation of y by (9) and if the imposed convergence condition is not fulfilled, we go back to step number 2.

Note that the functions z_1, y_1 and y_2 are independent of the approximation of y, thus they may be compute only once.

3. Convergence Analysis

In this section we give a general idea of under which conditions we can guarantee the convergence of the method.

The analysis of the convergence must be considered for several cases depending on the values and the sign of the real part of the characteristics roots, here we limitate the analysis to a typical case. More detailed information about this topic is given in [8].

Let's supose that $\mu_{10}(x,\varepsilon)$ and $\mu_{20}(x,\varepsilon)$ have positive real parts, and that constants $K_1(\varepsilon)$, $K_1'(\varepsilon)$ and $M_1(\varepsilon)$ exist such that

$$0< K_1(\varepsilon)\le \mathrm{Re}(\mu_{i0}(\varepsilon))\le K_1'(\varepsilon),\ i=1,2 \tag{10.a}$$

$$|\ \mu_{10}'(x,\varepsilon)\,|\le M_1(\varepsilon) \tag{10.b}$$

Moreover, set σ_n holding

$$\|\ y^n - y^{n-1}\|\le \sigma_n,\ \sigma_n\in \mathbb{R} \tag{10.c}$$

where y^i represents the approximation of y in the iteration $\imath$ of the method. Analogously, we denote by z_0^i, y_0^i, C_1^i the solution of (8.a), (8.c) and the constant C_1 in the iteration $\imath$ (C_2 does not change during the iterative process because we are taking $\beta_1=\beta_2=0$ in order to simplify the analysis).

Taking into consideration that the solution of

$$w' + g(x)w= h(x),\ w(0)= 0$$

can be written as

$$w(x)=\int_a^x h(\tau)\, e^{\int_x^{\tau t} g(\tau')d\tau'}\, d\tau$$

we can obtain the following bounds:

$$\left\| z_0^{n+1} - z_0^n \right\| \le \frac{\sqrt{\varepsilon}\ \ M_1(\varepsilon)\ \sigma_n}{K_1(\varepsilon)} \tag{11.a}$$

$$\left\| y_0^{n+1} - y_0^n \right\| \le \frac{\sqrt{\varepsilon}\ \ M_1(\varepsilon)\ \sigma_n}{\left(K_1(\varepsilon)\right)^2} \tag{11.b}$$

and because in this case

$$c_1^1 = \frac{\gamma_2\ y_2(a) - \gamma_1 + y_0^n(a)}{-y_1(a)}$$

we have

$$\left| c_1^{n+1} - c_1^n \right| \le \frac{\sqrt{\varepsilon}\ \ M_1(\varepsilon)\ \sigma_n}{\left(K_1(\varepsilon)\right)^2 \,|y_1(a)|} \tag{11.c}$$

now in the same way that for (11.a) and (11.b), we get

$$\left\| y_1 \right\| \le \frac{1}{K_1(\varepsilon)} \tag{11.d}$$

so, using (9) and the bounds (11) we finally obtain that

$$\left\| y^{n+1} - y^n \right\| \le \left(1 + \frac{1}{|y_1(a)|\ K_1(\varepsilon)}\right) \frac{\sqrt{\varepsilon}\ \ M_1(\varepsilon)\ \sigma_n}{\left(K_1(\varepsilon)\right)^2} \tag{12}$$

Then we conclude that, if the multiplicative factor of σ_n in (12) is less than one, the method converges and evenmore if this factor tends to zero as ε tends to zero, the stiffer of (1) the faster algorithm converges.

Observing (12) we see that, if the above convergence condition holds, it does not matter which is the initial approximation for y, so the method has global convergence.

4. Quadrature Formulas for the intial value problems

We are left with the numerical solution of the problems (8). In this section we will obtain explicit quadrature formulas [7], that allow us to solve efficiently our auxiliar Cauchy problems.

For the problem

$$\sqrt{\varepsilon}\ u' + \lambda(x)u = v(x),\ u(0) = 0 \qquad (13.a)$$

where $\mathrm{Re}\lambda(x) \geq \alpha_0 > 0$ and $\varepsilon \ll 1$.

We define a grid $0 = x_0 < \ldots < x_n = a$ in $[0,a]$. In what follows, for any function $f(x)$, we put $f_k = f(x_k)$. We need to compute $u_1, \ldots, u_n$, while $v_0, \ldots, v_n$, $\lambda_0, \ldots, \lambda_n$ and $\Lambda_0, \ldots, \Lambda_n$ are known; here

$$\Lambda(x) = \int_0^x \lambda(z)\,dz$$

The computations are carried out from left to right. We will give a recursive formula that expresses u_{k+1} in terms of u_k and the given quantities.

From (13.a) for $k=0,1,\ldots,N-1$ we have

$$u_{k+1} = u_k \exp\left[-\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)\right] + \phi_k \qquad (13.b)$$

where

$$\phi_k = \int_{x_k}^{x_{k+1}} \exp\left\{-\sqrt{\varepsilon}\ \left[\Lambda_{k+1} - \Lambda(t)\right]\right\} \frac{v(t)}{\lambda(t)}\ d\left\{-\sqrt{\varepsilon}\ \left[\Lambda_{k+1} - \Lambda(t)\right]\right\}$$

Put $\varphi(t) = v(t)/\lambda(t)$. Let $v(t)$ be a "good" function. Represent $\varphi(t)$ in the form

$$\varphi(t) = \varphi_{k+1} + \frac{\Lambda_{k+1} - \Lambda(t)}{\Lambda_{k+1} - \Lambda_k}\left[(\varphi_k - \varphi_{k+1}) + O\left((t_{k+1} - t_k)^2\right)\right] \qquad (13.c)$$

Then for the principal part of $\varphi(t)$ integration can be carried out explicitly, and we obtain

$$\phi_k \approx \frac{v_{k+1}}{\lambda_{k+1}}\left\{1 - \exp\left[-\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)\right]\right\} +$$

$$\frac{\dfrac{v_k}{\lambda_k} - \dfrac{v_{k+1}}{\lambda_{k+1}}}{\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)}\left\{1 - \left[1 + \sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)\right]\exp\left[-\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)\right]\right\}$$

From (13.b) we now obtain the required recursive formula

$$u_{k+1} = u_k \exp\left[-\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)\right] + \frac{v_{k+1}}{\lambda_{k+1}}\left\{1 - \exp\left[-\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)\right]\right\} +$$

$$\frac{\dfrac{v_k}{\lambda_k} - \dfrac{v_{k+1}}{\lambda_{k+1}}}{\sqrt{\varepsilon}\ (\Lambda_{k+1} - \Lambda_k)}\left\{1 - \left[1 + \sqrt{\varepsilon}(\Lambda_{k+1} - \Lambda_k)\right]\exp\left[-\sqrt{\varepsilon}(\Lambda_{k+1} - \Lambda_k)\right]\right\} \qquad (13.d)$$

which has the following properties

a) it is stable for any x-step and ε (provided, of course, $\mathrm{Re}\lambda(x) \geq \alpha_0 > 0$);

b) it gives a global error $O(h \min(h, \sqrt{\varepsilon}))$ on $[0,a]$, where

$$h = \max_k (x_{k+1} - x_k)$$

If $\mathrm{Re}\lambda(x) \leq \alpha_0 < 0$ an anologous backward formula can be obtained.

This result was obained by integrating the error in (13.c) over the interval $[x_k, x_{k+1}]$ and then summing over all intervals.

For $\varphi(t)$ we used an interpolation formula linear in $\Lambda(t)$. If (13.c) is replaced with an expression which is a polynomial of higher degree in $\Lambda(t)$, then we obtain an explicit formula with property a) that has an accuracy of higher order (an analogue of Adams formulas).

5. Numerical examples and applications

All the following examples were solved in a PC, IBM AT using a very simple FORTRAN 77 code. In all cases the exact or the numerical error were less than 10^{-4}. The execution time was always of a few seconds.

5.1 Test problems

a) Boundary Layer Problems

Example (5.1.1), Fig.1, [1].

$\varepsilon y'' - xy' - 1/2y = 0$

$y(-1)=1, \; y(1)=2$

This example was solved for $\varepsilon = 10^{-3}$, N=121, relative error=10^{-6} and absolute error=10^{-10}.

Example (5.1.2), Fig.2, [4].

$\varepsilon y'' + (2+\cos(\pi x))y' - y = F(x)$

$y(0)=0, \; y(1)=-1$

where

$$F(x) = -(1+\varepsilon\pi^2)\cos(\pi x) - \pi(2+\cos(\pi x))\mathrm{sen}(\pi x) + (1+ \pi^2 x^2/\varepsilon)e^{-3x/\varepsilon e}$$

Its exact solution is

$$y(x) = \cos(\pi x) - e^{-3x/\varepsilon e} + o(\varepsilon^2)$$

This example was solved for $\varepsilon = 10^{-6}$, N=121, the maximal order of disagreement with the exact solution was 10^{-4}.

b) Turning Point Problems

Example (5.1.3), Fig.3, [9].

$\varepsilon y'' + xy' - 1/2y = 0$

$y(-1)=1,\ y(1)=2$

Its turning point is for x = 0. We took $\varepsilon = 10^{-3}$, N =221, relative error=10^{-6} and absolute error=10^{-10}.

Example (5.1.4), Fig.4, [9].

$\varepsilon y'' + xy' - y = -(1+\varepsilon\pi^2)\cos(\pi x) + \pi x\ \mathrm{sen}(\pi x)$

$y(-1)=-1,\ y(1)=1$

Its turning point is in x = 0 too and was solved for $\varepsilon = 10^{-3}$ N =241, relative error=10^{-6} and absolute error=10^{-10}.

c) Rapidly Oscillarory Problems

Example (5.1.5), Fig.5, [10].

$\varepsilon y'' + y = \cos(80x)$

$y(0)=1,\ y(1)=0$

Its exact solution is

$$y(x) = a\cos(100x) + b\ \mathrm{sen}(100x) + 10^4\cos(80x)(\gamma-6400)^{-1}.$$

where

$$a = 1-\gamma/(\gamma-6400)$$

$$b = ((\cos(100)-\cos(80))/\varepsilon(1/\varepsilon-6400)-\cos(100))/\mathrm{sen}(100)$$

This example was solved for $\varepsilon = 10^{-4}$, N=501, the maximal order of disagreement with the exact solution was 10^{-2}.

For this example we used the quadrature formulas in spite off that $\mathrm{Re}\lambda(x)=0$. This was possible because of the constant coefficient of the equation. If not the code access to a Runge-Kutta routine, RK4, [11].

The code was proved for about 30 stiff tests reported in the literature and all the numerical experiments are shown in [12].

5.2 Stefan Problem

This problem appears in many fields of the science when we study phenomena like evaporation, fusion, solidification, etc. An important characteristic of this problem is the unknowledge of the boundary position for each value of time.

The one-dimensional Stefan problem consists of

$$\frac{\partial}{\partial x}\left[k(x)\frac{\partial u}{\partial x}\right] - \rho c\frac{\partial u}{\partial t} = f(x,t)- \qquad (14.a)$$

$$u(0,t) = \Gamma \frac{\partial u}{\partial x}(0,t) + \alpha(t) \qquad (14.b)$$

$$u(s(t),t) = 0 \qquad (14.c)$$

$$k(s(t))\frac{\partial u}{\partial x}(s(t),t) = -\lambda\rho\frac{\partial s}{\partial t} + \mu(s(t),t) \qquad (14.d)$$

μ is a given boundary source which may depends on the position of the free boundary $s(t)$. $k(x)$ and all other data functions are assumed to be piece-wise continuos in this model. At points of discontinuity of the data the natural one-sided limits are to be taken.

As it is known the method of lines is one of the main sources from where singularly perturbed BVPs arise. We solved the Stefan problem combining the ODM and the method of lines like was done for the sweep method in [13].

The first step is to approximate the parabolic equation by a sequence of elliptic problems at successive time steps. Let t_n denotes the nth time level and set $\Delta t= t_n - t_{n-1}$. Then (14) can be approximated by

$$Lu \equiv (ku')' - \rho c\frac{u}{\Delta t} = -\rho c\frac{u_{n-1}}{\Delta t} + f(x,t_n) \qquad (15.a)$$

$$u(0) = \Gamma u'(0) + \alpha(t_n) \qquad (15.b)$$

$$u(s) = 0 \qquad (15.c)$$

$$k(s)u'(s) = -\lambda\rho\frac{s-s_{n-1}}{\Delta t} + \mu(s,t_n) \qquad (15.d)$$

and now we applied the following algorithm for n=1,2,...:

Step1. We take s_{n-1} as initial approximations for s_n.

Step2. We solve the BVP (15.a)-(15.c) using the last approximation for s_n in place of it.

Step3. Solving the scalar equation (15.d) for s we find a new approximation for s, and if the imposed convergence condition is not fulfilled, we go back to step number 2.

The numerical results for $k=\rho=c=\lambda=1$, $f=\mu=\Gamma=0$, $\alpha=-1$, whose exact solution is

$$s(t)= 1.2404\sqrt{t}$$

$$u(x,t)= -1 + \Phi(x/\sqrt{t})/\Phi(0.6202), \text{ for } 0\le x\le s(t)$$

where

$$\Phi(x)= 2/\sqrt{\pi} \int_0^x e^{-\tau^2} d\tau$$

taking $\Delta t=\Delta x=0.01$, and $t=0.5$ are summarized in the following table

Exact Boundary		Approximate Boundary	
0.8770		0.8947	
x	Uex	Uap	Error
0	-1	-1	0
0.2	-0.7441	-0.7421	0.002
0.4	-0.4982	-0.4945	0.004
0.6	-0.2712	-0.2664	0.048
0.8	-0.0698	-0.0876	0.028
s(t)	0.0153	0	0.015

5.3 BVP in an unbounded interval

We have also applied the ODM to solve the Holt's problem

$$y'' - (x^2+R)y = 0$$
$$y(0) = \beta$$
$$\lim_{x\to\infty} y(x) = 0$$

where $R= 2m+1$, $m \in \mathbb{N}$. This problem is considered a classical test problem.

In this case the ODM is simpler to apply since the system (7) results

$$\sqrt{\varepsilon}\, z' -q(x)z= \sqrt{\varepsilon}\, q'(x)y \tag{16.a}$$

$$\sqrt{\varepsilon}\, y' +q(x)y= z(x) \tag{16.b}$$

where $\varepsilon=1/R$ and $q(x)=\sqrt{1+x^2/R}$. So taking the boundary conditions $z(\infty) = 0$ and $y(0) = \beta$, we guarantee that the solution y of the above system is the solution of our original problem. Thus the algorithm for this case is:

Step1. Given an approximation for y, we solve the problem (16.a).

Step2. Given z we solve (16.b).

Step3. If the imposed convergence condition is not fulfilled, we go back to step number 1.

There exist some techniques which change the second boundary condition for one equivalent in a finite point, [14], but for this problem was enough to take the condition $y(10) = 10^{-30}$. In the following table appear the results for differents values of R.

R	Iterations number	Num. Error	$\lvert y_{num} - y_{rep}\rvert$
1	15	10^{-12}	10^{-5}
5	11	10^{-15}	10^{-5}
10	10	10^{-17}	?
10^2	8	10^{-22}	?
10^3	6	10^{-24}	?

Remark: Here y_{rep} denotes the numerical solution reported in [14].

5.4 Orr-Sommerfeld equation

This equation is a typical problem that appears in fluid mechanics.

The following eigenvalue problem was solved using the ODM

$$y^{iv} - 2\alpha y'' + \alpha^4 y = i\alpha R[(U-c)(y''-\alpha^2 y)-U''y],\ x\in[0,1]$$

$$y'(0)=y''(0)=0$$

$$y(1)=y'(1)=0$$

where $\alpha= 1$, $R=10^4$ (Reynold's number), $U= 1-x^2$ and $c= c_r+ ic_i$ is the eigenvalue.

This example has a turning point for $c_i=0$ and $c_r\approx 1-x^2$. The obtained value of c was (0.237501 + i 0.0037378).

Here it was necessary to take a non uniform grid given by

$$x_j= \frac{1}{2}\, w_j \exp\left[\rho(w_j^2-1)\right],\ j=1,\ldots,N$$

with $\rho=$ logR. The number of points was N=500 and the relative error was 10^{-7}. The execution time was about 15 minutes because of the second iterative process that was necessary to implement for the eigenvalue. We used for that the inverse iteration method. More details of the use of the ODM for this equation can be seen in [15].

6. Summary and Conclusions

The ODM is an iterative method for singularly perturbed ordinary second order linear BVP. This method based on the characteristic roots reduces the original problem to a sequence of Cauchy's problems, that are easier to solve.

We prove that, under certain conditions the stiffer the original BVP the faster the algorithm converges.

The big variety of numerical experiments and applications of this method show its efficiency.

The method can not be applied in some cases as for example: when the real part of the characteristic roots changes its sign in the interval or when the derivative of the roots are not bounded. It is our opinion that the multiple shooting ideas might be helpful in these cases.

Finally we want to say that the ODM can beapplied if the boundary conditions are nonlinear and in this case we only have to solve a nonlinear algebraic system instead of a linear one.

Aknowledgements

We want to thank Dr. A. A. Abramov from the Computer Center of Moscow, who supplied the original ideas of the ODM. The authors would also like to thank Dr. J. L. Estrada from the UNAM for his useful comments and to Dr. E. Cozlova for her contribution in the solution of the Stefan problem.

References

[1] Kreiss H. O., Nichols N. K., Brown D. L., Numerical methods for stiff two-point boundary value problems,Siam J.Num. Anal. Vol. 23, No.2, April 1986.

[2] Ascher U. M., Two families of Symmetric Difference Schemes for singular perturbation problems, Progress in Scientific Computing, Vol.5, Numerical Boundary Value ODEs, 1985.

[3] Brown D. L., A Numerical Method for Singular Perturbation Problems with Turnig Points,

[4] Ascher U. M., Robert M. M., Rusell R. D., Numerical Solution of boundary-Value Problems for Ordinary Differential Equations, Englewood Cliffs, New Jerry, Prentice-Hall, Inc. 1988.

[5] Hoog F., Mattheij R., The Role of Conditioning in Shooting Techniques, Progress in Scientific Computing, Vol.5, Numerical Boundary Value ODEs, 1985.

[6] Loon P., Riccatti Transformation: when and how to use?, Progress in Scientific Computing, Vol.5, Numerical Boundary Value ODEs, 1985.

[7] Abramov A. A., Alvarez L. M., A Method of Solving Boundary-Value Problems Based on Operator Splitting, U.S.S.R. Comput. Maths. Math. Phys., Vol.29, No.6, pp.139-146, 1989.

[8] Alvarez L. M., Rodríguez A. R., Armas M. A., Revista Ciencias Físico Técnicas y Matemáticas (to be published).

[9] Hemker P. W., A Numerical Study of Stiff two-point Boundary Problem, Mathematisch Centrum, Amterdam 1977.

[10] Meyer G. H., Continuos Orthornormalization for Boundary-Value Problems, Journal of Computational Physics 62, pp.248-

262, 1986.

[11] Press W. H., Flannery B. P., et. al., Numerical Recipes, Cambridge, University Press, 1986.

[12] Alvarez L. M., Rodríguez A. R., Viada C. E., STIFF-PC un programa eficiente para la solución de una clase de problemas de perturbación singular, Revista Ciencias Físico Técnicas y Matemáticas (to be published).

[13] Meyer G. H., Front Traking for the Conductive Stefan Problem with Surface Tension, International Series of Numerical Mathematics, Vol.86, 1988.

[14] Balla K., Viczek M. On the Reduction of Holt's Problem to a Finite Interval, Numer. Math. 51, 1987, pp.291-302.

[15] Alvarez L. M., Ditkin V. V.
U.S.S.R. Comp. Maths. Math. Phys, Vol.30, No.4, pp.611-615, 1990.

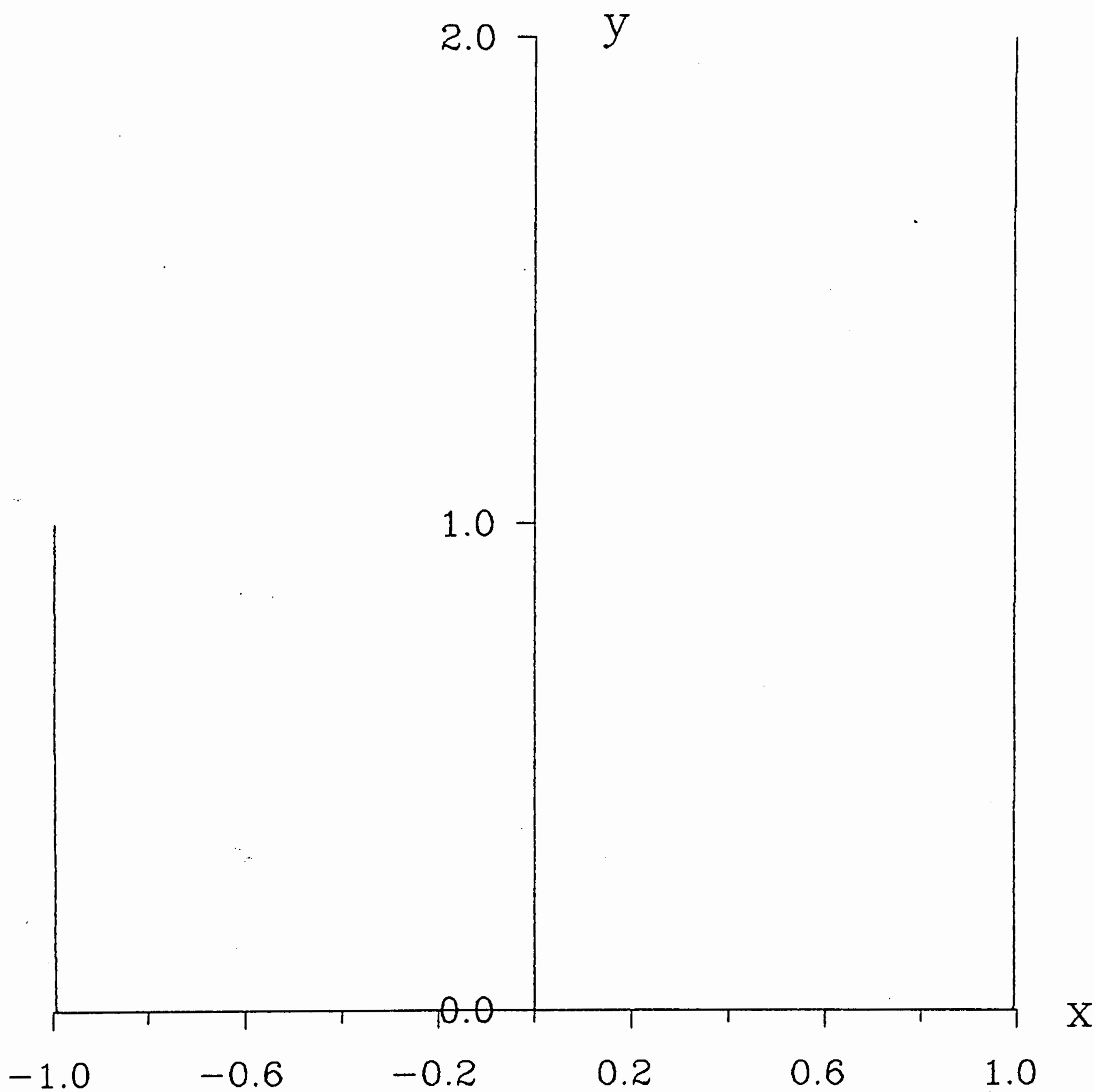

Fig. 1

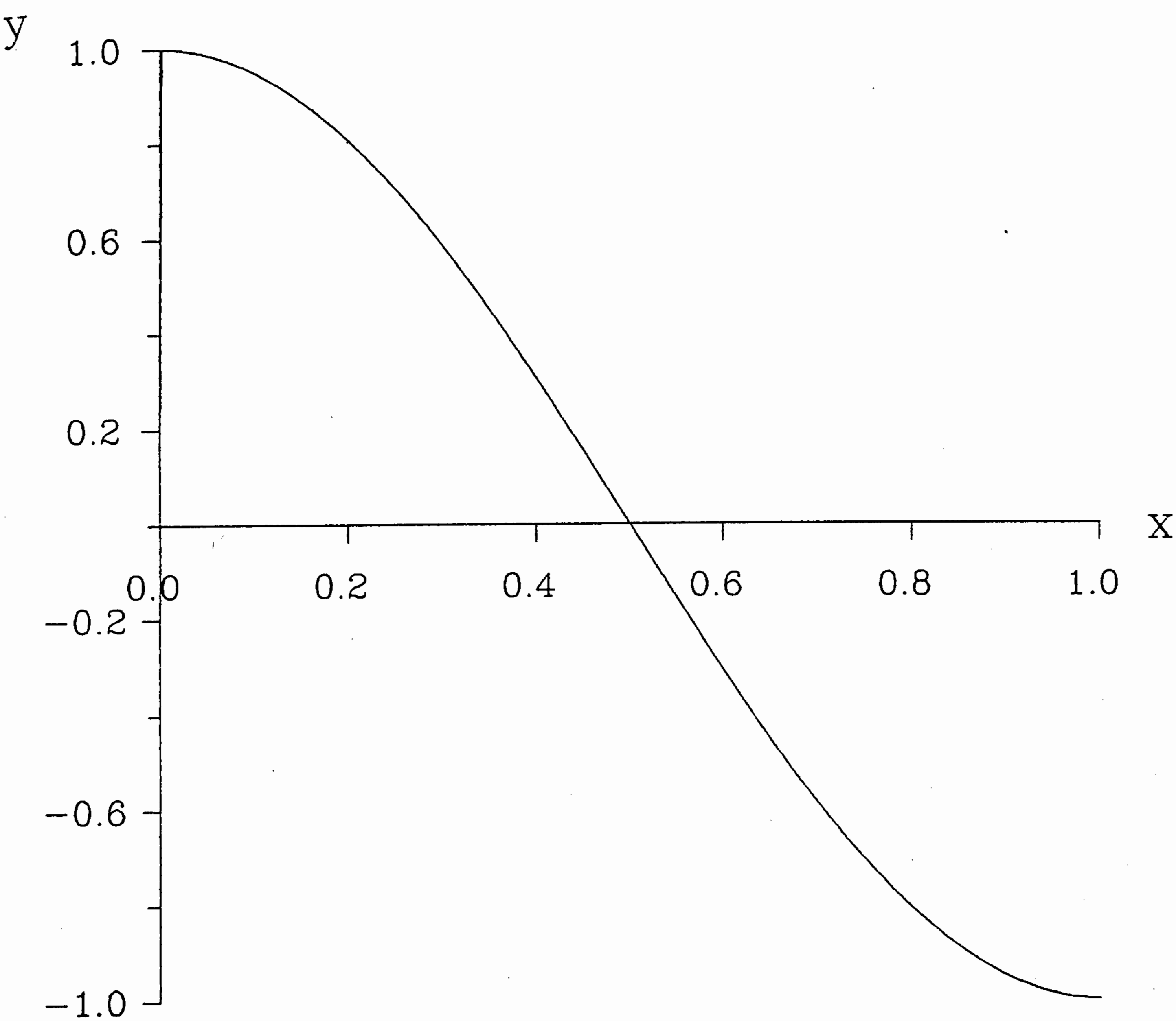

Fig.2

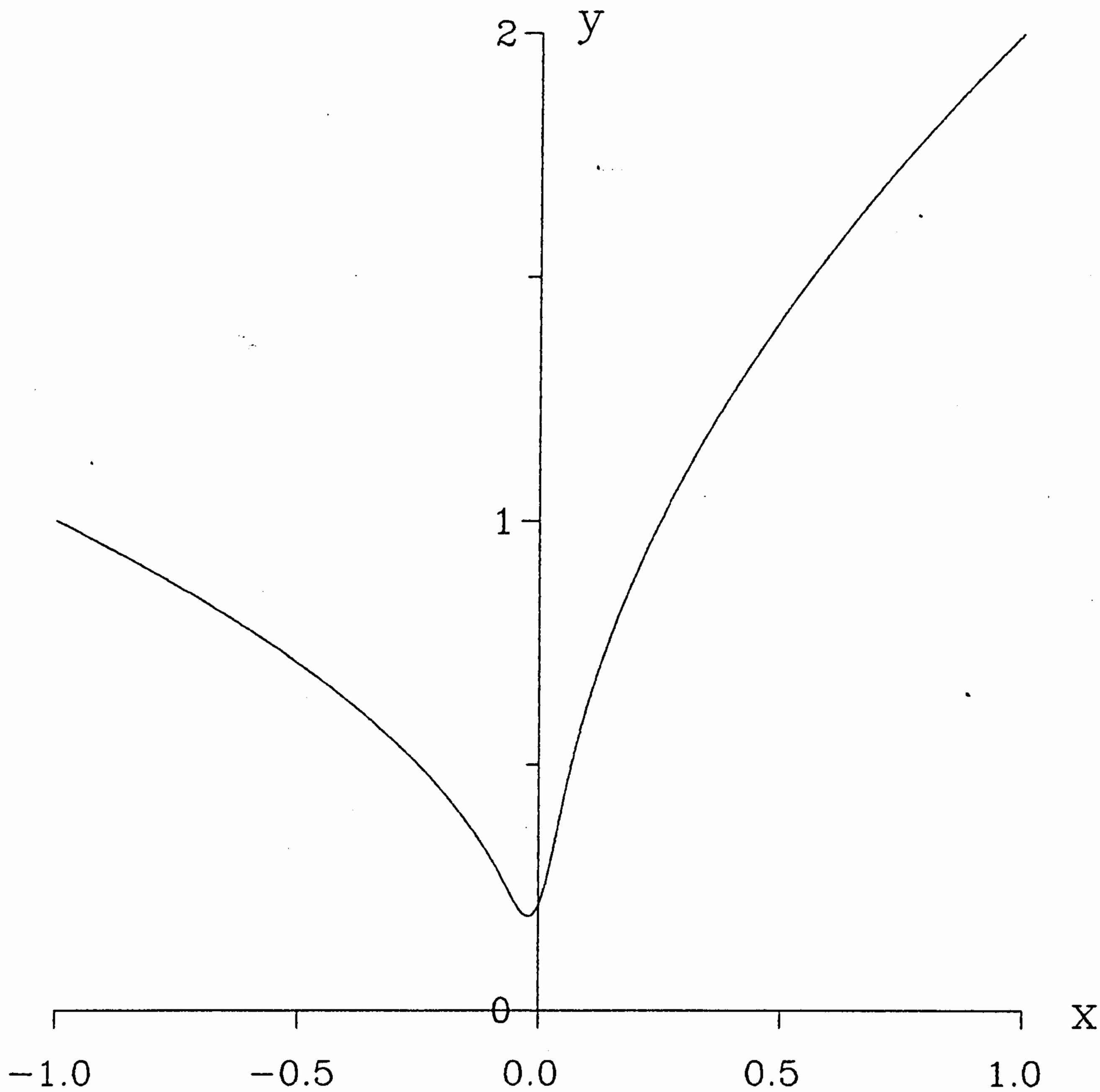

Fig.3

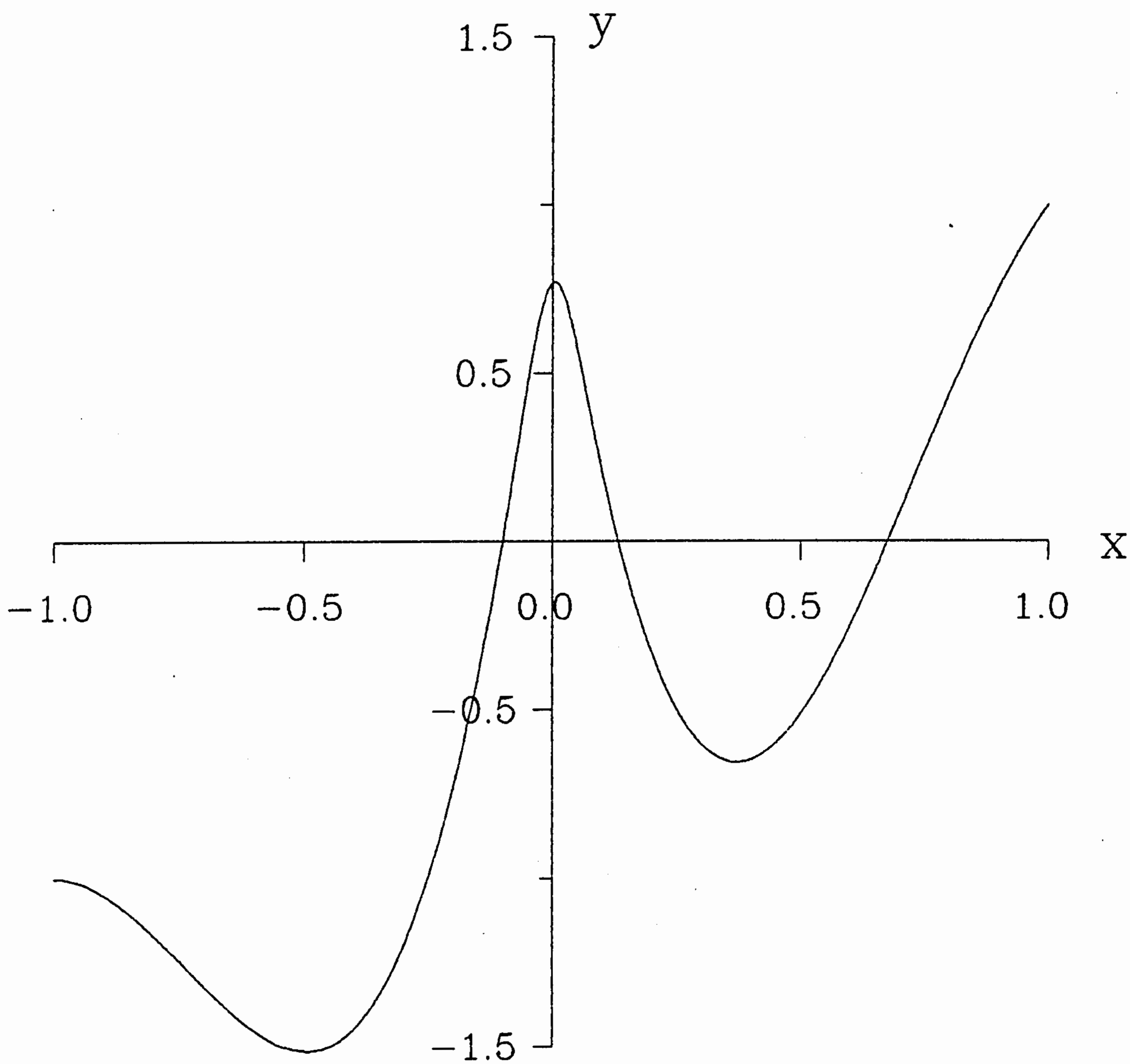

Fig.4

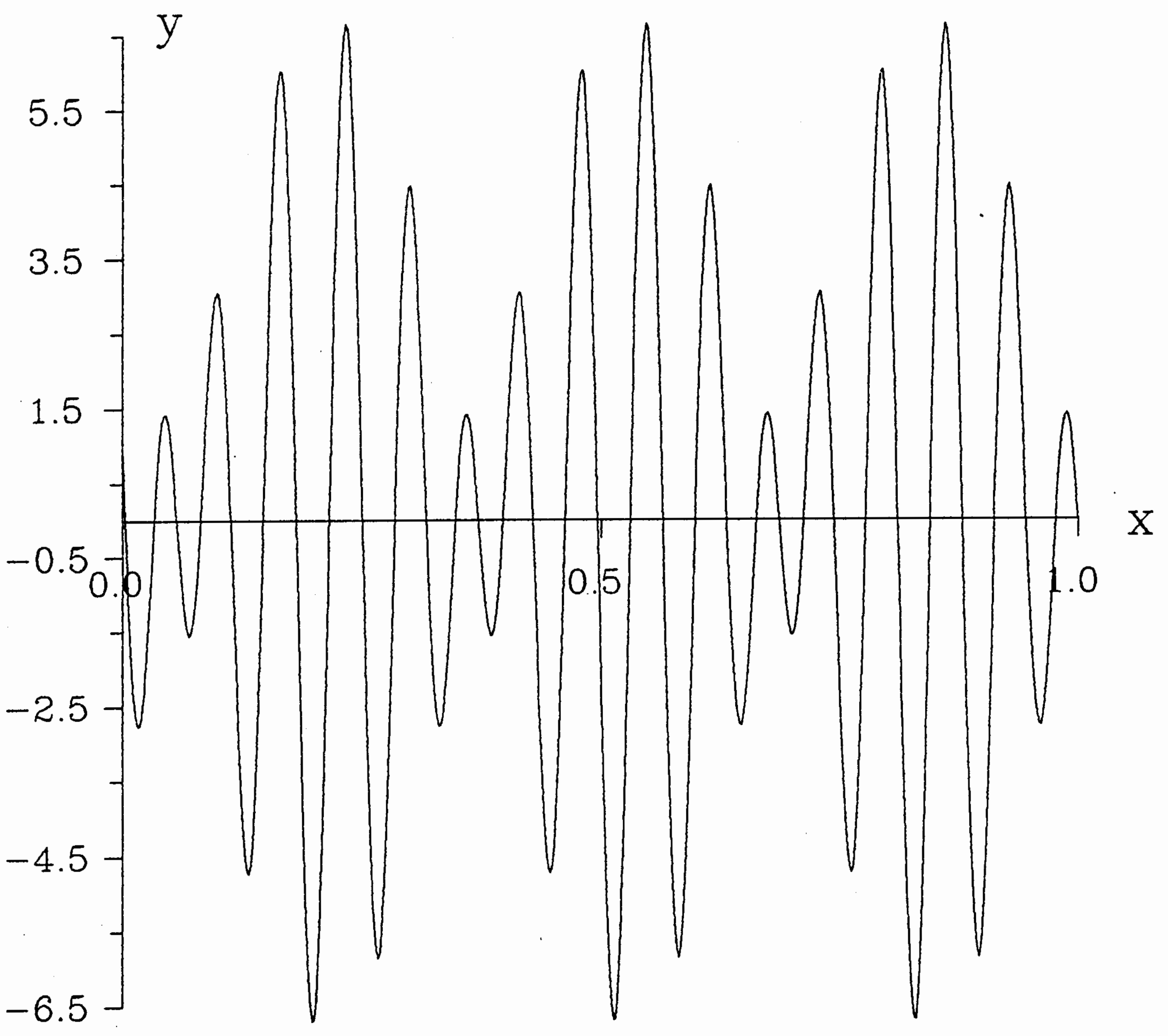

Fig.5