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CENTRAL LIMIT THEOREMS FOR NON-LINEAR FUNCTIONALS

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Central Limit Theorems for Non-Linear Functionals of *d* Gaussian Processes

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1 Introduction

Let $\mathbf{X}_n = (X_n^1, \ldots, X_n^d)$ be a d-dimensional stationary Gaussian normal processes such that

$$\mathrm{E}\left(X_n^i\right) = 0, \text{ and } E\left(X_n^i\right)^2 = 1$$

Let H be a real valued function such that

$$\mathrm{E}\left(H(\mathbf{X}_n)\right) = 0 \text{ and } \mathrm{E}\left(H(\mathbf{X}_n)\right)^2 < \infty \tag{1.1}$$

Suppose that $X_n^1, \ldots, X_n^d$ are d orthogonal nomal random variables for all n. If it does not occur, we can construct $\delta \leq d$ orthorgonal normal random variables, using the Gram-Schmidt method:

$$Y_n^i = \sum_{j=1}^{\delta} a_{ij} X_n^j$$

such that $\mathrm{E}\left(Y_n^i\right) = 0$, and $E\left(Y_n^i\right)^2 = 1$ and we would consider

$$\widetilde{H}(Y_n^1, \ldots, Y_n^\delta) = H(X_n^1, \ldots, X_n^d)$$

If H satisfies 1.1 then it can be expanded in the form

$$H(X_n^1, \ldots, X_n^d) = \sum_{r=k}^{\infty} \sum_{\mathbf{g}_i \in \mathsf{A}_r} c_{\mathbf{g}_i} \prod_{i=1}^{d} H_{g_i(l)}\left(X_n^i\right)$$

where $\mathbf{g}_i = (g_i(1), \ldots, g_i(d))$, $\mathsf{A}_r = \{\mathbf{g}_i / \ g_i(1) + \cdots + g_i(d) = r\}$, H_l is the lth Hermite polynomial and

$$\sum_{r=k}^{\infty} \sum_{\mathbf{g}_i \in \mathsf{A}_r} c_{\mathbf{g}_i}^2 \prod_{l=1}^{d} g_i(l)! < \infty \tag{1.2}$$

(see, e.g. [9]) We say that H has Hermite rank k if

$$k = \min\{\sum_{l=1}^{d} g_i(l) / c_{\mathbf{g}_i} \neq 0\}$$

We define the random processes

SUMMARY

Let $\{X_n^1\}, \ldots, \{X_n^k\}$ be k stationary Gaussian processes. Suppose H is a functional of k variables. This paper proves the normality asymptotic of the sequence:

$$\frac{1}{A_N} \sum_{j=nN}^{(n+1)N} H(X_j^1, \ldots, X_j^k)$$

where A_N are appropriate norming constants and assuming that the correlation matrix tends fastly to 0. The method is the same used by Breuer and Major

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$$Z_n^N = Z_n^N(H) = A_N^{-1} \sum_{j \in B(n,N)} H(\mathbf{X}_j) \tag{1.3}$$

where A_N are appropiate norming constants and

$$B(n,N) = \{j / \; nN \leq j < (n+1)N\}$$

We are interested in the assymptotic distribution of Z_n^N

Breuer and Major [1] studied the problem when $d = 1$. Suppose that H has Hermite rank k. They showed that if

$$\sum_{n=0}^{\infty} |r(n)|^k < \infty$$

where $r(n) = \mathrm{E}(X_0^1 X_n^1)$, then the Z_n^N sequence is asymtotically normally distributed as $N \to \infty$ with $A_N = N^{1/2}$

In the case that

$$(Y_{n+t_1}, \ldots, Y_{n+t_d}) = \left(X_n^1, \ldots, X_n^d\right)$$

where Y_n is a normal stationary process such that satisfies our conditions. Ho-Sun showed in [5] that if

$$\sum_{n=0}^{\infty} |r_y(n)|^k < \infty$$

then Z_n^N is also asymtotically normal with $A_N = (N)^{1/2}$

A Central limit Theorems holds for Z_n^N under similar conditions of the correlation functions $r_{i\bar{i}}(n)$. The proof works with the same techniques of [5] and [1]. We have modificatied the original proofs for the d^2 correlations functions differents. We prove that the moments of Z_n^N tend to the moments of the normal randon variable using induction.

2 Central Limit Theorems

In orden to formulate our results, we introduce the following notations and definitions. The last can be found in [5].

Let $p \in \mathrm{N}$ and $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_p)$ with $\mathbf{g}_i = (g_i(1), \ldots, g_i(d))$

We say a diagram $G \in \Gamma(p, d, \mathbf{g})$ if G has the propierties:

1. G has $\sum_{i=1}^{p} \sum_{l=1}^{d} g_i(l)$ vertices, wich we shall denote by $\{i, l, t\}$ for $1 \leq i \leq p$, $1 \leq l \leq d$ and $1 \leq t \leq g_i(l)$. The set of all vertices $v_{i,l,t}$ with the same index i is called the i block, i.e. there are p blocks and the set of all vertices with the same indices i and l is called the (i, l)-th level.
2. Each vertex is of degree 1.

3. Edges may pass only between vertices from different blocks, e.g. for each edges $w = (v_{i_1,l_1,t_1}, v_{i_2,l_2,t_2})$ in G we must have $i_1 \neq i_2$, we shall always use the convention that $i_1 < i_2$. We shall denote by $G(V)$ the set of all edges in G. We shall write $d_1(w) = i_1$, $d_2(w) = i_2$, $L_1(w) = l_1$ and $L_2(w) = l_2$.

Now we formulate the following

Theorem 1 *Suppose that H has the Hermite rank k and the correlation functions r_{ij} satisfy:*

$$\sum_{n=-\infty}^{\infty} \left(\sum_{i,\bar{i}}^{d} \left| r_{i,\bar{i}}(n) \right| \right)^{k} < \infty \tag{2.1}$$

Put $A_N = N^{1/2}$. Then

$$\begin{aligned} \sigma_r^2 &= \lim_{N\to\infty} E\left(Z_0\left(H_r\right)\right)^2 \\ &= \lim_{N\to\infty} \frac{1}{A_N^2} \sum_{u=0}^{N-1} \sum_{v=0}^{N-1} \sum_{\mathbf{g}_i, \mathbf{g}_j \in \mathsf{A}_r} c_{\mathbf{g}_i} c_{\mathbf{g}_j} \sum_{G\in\Gamma} \prod_{w\in G(V)} r_{L_1(w)L_2(w)}(u-v) \end{aligned}$$

exist for all $r \geq k$, where $\Gamma = \Gamma\left(2, d, (\mathbf{g}_i, \mathbf{g}_j)\right)$ and

$$H_r(X_1, \ldots, X_d) = \sum_{\mathbf{g}\in\mathsf{A}_r} c_{\mathbf{g}} \prod_{l=1}^{d} H_{g(l)}(X_l) \tag{2.2}$$

and the infinite sum

$$\sigma^2 = \sum_{r=k}^{\infty} \sigma_r^2 < \infty$$

The finite dimensional distributions of processes $Z_n^N(H)$ defined in formula 1.3 tend, as $N \to \infty$, to the finite dimensional distributions of the process σZ_n^*, $n \in \mathbf{Z}$, where Z_n^* are independent standard normal variable.

3 Proof

In orden to prove Theorem 1 we need the following lemmas:

Lemma 1 (Diagram Formula) *Let $\left(X_{m_h}^1, \ldots, X_{m_h}^d\right)$, $h = 1, \ldots, p$ be a Gaussian vectors. $E\left(X_{m_k}^i\right) = 0$, $E\left(X_{m_l}^i X_{m_k}^j\right) = r_{ij}(m_l - m_k)$; $r_{ij}(0) = \delta_{ij}$; $l, k = 1, \ldots, p$. Then*

$$E\left(\prod_{i=1}^{p} \prod_{l=1}^{d} H_{g_i(l)}(X_{m_i}^l) \right) = \sum_{G\in\Gamma} J_G^{\mathfrak{m}},$$

where $\Gamma = \Gamma(p, d, \mathbf{g})$, $\mathbf{g} = (\mathbf{g}_1, \dots, \mathbf{g}_p)$, $\mathbf{g}_i = (g_i(1), \dots, g_i(d))$, $\mathsf{m} = (m_1, \dots, m_p)$ *and*

$$J_G^{\mathsf{m}} = \prod_{w \in G(V)} r_{L_1(w)L_2(w)}(m_{d_1(w)} - m_{d_2(w)})$$

This is a particular case of lemma given in [7]. The following is other version of Diagram Formula, it can be found in [8], see also e.g. [10,6]

Lemma 2 *Let* $(X_1, \dots, X_d, X_{d+1}, \dots, X_{2d})$ *be a Gaussian random vector with zero means and correlation matrix*

$$\Sigma = \begin{pmatrix} I_{d\times d} & A \\ A^t & I_{d\times d} \end{pmatrix}; \text{ where } A = \begin{pmatrix} \rho_{1,d+1} & \cdots & \rho_{1,2d} \\ \vdots & \ddots & \vdots \\ \rho_{d,d+1} & \cdots & \rho_{d,2d} \end{pmatrix}$$

Where $\rho_{ij} = E(X_i X_j)$ *and* $I_{d\times d}$ *is identical matrix. Then*

1.

$$E\left(\prod_{l=1}^{d} H_{q_i(l)}(X_l) H_{q_j(l)}(X_{l+d})\right) = 0, \quad \text{if } \sum_l q_i(l) \neq \sum_l q_j(l)$$

2.

$$r! \sum_{q_i, q_j \in \mathsf{A}_r} E\left(\prod_{l=1}^{d} \frac{H_{q_i(l)}(X_l) H_{q_j(l)}(X_{l+d})}{q_i(l)! q_j(l)!}\right) = \left(\sum_{j=d+1}^{2d} \sum_{i=1}^{d} \rho_{ij}\right)^r$$

3.

$$r! \sum_{q_i, q_j \in \mathsf{A}_r} \left| E\left(\prod_{l=1}^{d} \frac{H_{q_i(l)}(X_l) H_{q_j(l)}(X_{l+d})}{q_i(l)! q_j(l)!}\right)\right| \leq \left(\sum_{j=d+1}^{2d} \sum_{i=1}^{d} |\rho_{ij}|\right)^r$$

4. *If* $X_i = X_{i+d}$ *for* $i \leq d$ *then:*

$$E\left(\prod_{l=1}^{d} H_{q_i(l)}(X_i) H_{q_j(l)}(X_{i+d})\right) = \begin{cases} q_i(1)! \cdots q_i(d)! & q_i(l) = q_j(l) \quad \forall l = 1, \dots, d \\ 0, & \text{otherwise} \end{cases}$$

where $\mathsf{A}_r = \{\mathbf{q} / \ r = q(1) + \cdots + q(d)\}$

We shall call a diagram regular if its blocks can be paired in such a way that no edge passes between blocks in different pairs.

Lemma 3 *If $G \in \Gamma(p, d, \mathbf{g})$ is not regular diagram, then for every $\mathbf{j} = (j_1, \ldots, j_p)$ occurs that*

$$\lim_{N \to \infty} N^{-p/2} \sum_{\mathbf{m} \in M_p(j,N)} \sum_{w \in G(V)} r_{L_1(w)L_2(w)}(m_{d_1(w)} - m_{d_2(w)}) = 0$$

where $\mathbf{m} = (m_1, \ldots, m_p)$ and

$$M_p = M_p(\mathbf{j}, N) = \{\mathbf{m} /\ n_{j_i} N \leq m_i \leq (n_{j_i} + 1)N - 1, \ \ \textit{for all } 1 \leq i \leq p\}$$

proof Using

$$\prod_{w \in G(V)/\ d_1(w) = i} \left| r_{L_1(w)L_2(w)}(m) \right| \leq \left(\sum_{l, \bar{l} = 1}^{d} |r_{l\bar{l}}(m)| \right)^{K_G(i)}$$

where $K_G(i)$ is the cardinality of the edges $w \in G(V)$ such that $d_1(w) = i$ and calling $r(m) = \sum_{l\bar{l}=1}^{d} r_{l,\bar{l}}(m)$, then the lemma can be proved in the same way as in [1].

Let us return to the proof of the Theorem 1.

Choose any $0 < \lambda < 1$ and choose a constant Q such that for all n, $|n| > Q$

$$\sum_{i,\bar{i}}^{d} |r_{i\bar{i}}(n)| \leq \lambda < 1$$

Let H_r defined in (2.2). Then by lemma 1

$$\mathrm{E}\left(Z_n^N(H_r)\right)^2 =$$

$$\frac{1}{A_N^2} \sum_{(j_1,j_2) \in B_Q(0,N)} \mathrm{E}\left(\sum_{\mathbf{g}_1 \in \mathsf{A}_r} c_{\mathbf{g}_1} \prod_{l=1}^{d} H_{g_1(l)}(X_{j_1}^l) \sum_{\mathbf{g}_2 \in \mathsf{A}_r} c_{\mathbf{g}_2} \prod_{l=1}^{d} H_{g_2(l)}(X_{j_2}^l) \right)$$

$$+ \frac{1}{A_N^2} \sum_{(j_1,j_2) \in B_Q^c(0,N)} \sum_{\mathbf{g}_1, \mathbf{g}_2 \in \mathsf{A}_r} c_{\mathbf{g}_1} c_{\mathbf{g}_2} \mathrm{E}\left(\prod_{l=1}^{d} H_{g_1(l)}(X_{j_1}^l) H_{g_2(l)}(X_{j_2}^l) \right)$$

where $\Gamma = \Gamma(2, d, (\mathbf{g}_1, \mathbf{g}_2))$ and

$$B_Q = \{(j_1, j_2) /\ |j_1 - j_2| \leq Q \text{ and } j_1, j_2 \in B(0, N)\}$$

By Cauchy-Schwarz and lemma 2 we obtain

$$\mathrm{E}\left(Z_n^N(H_r)\right)^2 \leq$$

$$
\begin{aligned}
&\leq \frac{1}{A_N^2} \sum_{|n| \leq Q} \sum_{\mathbf{g}_1 \in \mathsf{A}_r} |c_{\mathbf{g}_1}|^2 \prod_{l=1}^{d} g_1(l)! \\
&+ \frac{K}{A_N^2} \sum_{|n|>Q} (N - |n|) \left(\sum_{i\bar{i}=1}^{d} |r_{i\bar{i}}(n)| \right)^r \left(\sum_{\mathbf{g} \in \mathsf{A}_r} c_{\mathbf{g}}^2 \prod_{l=1}^{d} g(l)! \right) \\
&\leq \frac{NK}{A_N^2} \sum_{\mathbf{g} \in \mathsf{A}_r} c_{\mathbf{g}}^2 \prod_{l=1}^{d} g(l)! + \frac{NK}{A_N^2} \sum_{|n|>Q} \left(\sum_{i\bar{i}=1}^{d} |r_{i\bar{i}}(n)| \right)^r
\end{aligned}
$$

(2.1) and (1.2) imply that $\mathrm{E}\left(Z_n^N(H_r)\right)^2 < \infty$

We can restrict ourselve to the case when H is a polynomial orden T, since

$$
\begin{aligned}
&\mathrm{E}\left|Z_n^N(H_T) - Z_n^N(H)\right|^2 \leq \\
&\frac{NK}{A_N^2} \sum_{r=T}^{\infty} \sum_{\mathbf{g} \in \mathsf{A}_r} c_{\mathbf{g}}^2 \prod_{l=1}^{d} g(l)! + \frac{NK}{A_N^2} \sum_{r=T}^{\infty} \lambda^{r-k} \sum_{|n|>Q} \left(\sum_{i\bar{i}=1}^{d} |r_{i\bar{i}}(n)| \right)^k \\
&\longrightarrow 0, \qquad \text{as } T \to \infty
\end{aligned}
$$

uniformly in N and n, where

$$
H_T(X_1, \ldots, X_d) = \sum_{r=k}^{T} \sum_{\mathbf{g} \in \mathsf{A}_r} c_{\mathbf{g}} \prod_{l=1}^{d} H_{g(l)}(X_l)
$$

be a truncated Hermite expansion of H.

$$
\text{Let } \Sigma_N = \sum_{s=1}^{u} b_s Z_{n_s}^N(H), \text{ where } n_s \in \mathbf{Z} \text{ and } b_s \in \mathbf{R},\ s = 1, \ldots, u
$$

In orden to prove our Theorem we can see that

$$
\begin{aligned}
\lim_{N \to \infty} \mathrm{E}\,(\Sigma_N)^p &= (p-1)!! \left(\sum_{s=1}^{u} b_s^2 \sigma^2 \right)^{p/2} \quad \text{if } p = 2q \\
&= 0 \qquad \text{if } p = 2q+1
\end{aligned}
\tag{3.1}
$$

The proof is by induction under q. We will introduce a similar notation to the used in [5]

Let $\mathbf{j} = (j_1, \ldots, j_p)$, $\mathbf{g} = (\mathbf{g}_1, \ldots, \mathbf{g}_p)$ with $\mathbf{g}_i = (g_i(1), \ldots, g_i(d))$

$$
\begin{aligned}
J_p \;=\; &\{(\mathbf{j}, \mathbf{g}) /\ 1 \leq j_i \leq u \text{ for all } 1 \leq i \leq p \text{ and } 0 \leq g_i(l) \leq T \text{ for all } 1 \leq i \leq p \\
&\text{and } 1 \leq l \leq d \text{ and } k \leq g_i(1) + \cdots + g_i(d) \leq T \text{ for all } 1 \leq i \leq p\}
\end{aligned}
$$

$$\mathsf{b}_j = \prod_{i=1}^{p} b_{j_i}; \quad c_{\mathbf{g}} = \prod_{i=1}^{p} c_{\mathbf{g}_i}$$

Thus, by lemma (1)

$$\begin{aligned} \mathrm{E}(\Sigma_N)^p &= \sum_{(\mathbf{g}\mathsf{j})\in J_p} \mathsf{b}_{\mathsf{j}} c_{\mathbf{g}} A_N^{-p} \sum_{\mathsf{m}\in M_p} \sum_{G\in\Gamma} \prod_{w\in G(V)} r_{L_1(w)L_2(w)}(m_{d_1(w)} - m_{d_2(w)}) \\ &= \sum_{(\mathbf{g}\mathsf{j})\in J_p} \mathsf{b}_{\mathsf{j}} c_{\mathbf{g}} A_N^{-p} \sum_{\mathsf{m}\in M_p} \sum_{G\in\Gamma} \mathsf{J}_G^{\mathsf{m}} \end{aligned} \tag{3.2}$$

For $q = 1$, it is easy to see, taking into account the condition 2.1 that if $j_1 \neq j_2$ then:

$$A_N^{-2} \sum_{\mathsf{m}\in M_2} \sum_{G\in\Gamma} \prod_{w\in G(V)} r_{L_1(w)L_2(w)}(m_1 - m_2) \longrightarrow 0$$

where $M_2 = M_2((j_1, j_2), N)$

Thus, using lemma 3 and (3.2), we obtain

$$\lim_{N\to\infty} \mathrm{E}(\Sigma_N)^2 = \sum_{s=1}^{p} b_s^2 \sigma^2$$

Supose that 3.1 holds for $q - 1$. In the limit of (3.2), by lemma 3, only the regular diagrams count.

Let $G \in \Gamma(p, d, \mathbf{g})$ a regular diagrams, then we can see $G(V)$ as $G^{1i}(V) \cup G^{\widehat{1i}}(V)$, for a only $i > 1$ and uniques

$$G^{1i} \in \Gamma(p-2, d, (\mathbf{g}_1, \mathbf{g}_i)) = \Gamma^{1i}, \quad G^{\widehat{1i}} \in \Gamma(2, d, \mathbf{g}^{\widehat{1i}}) = \Gamma^{\widehat{1i}}$$

where $\mathbf{g}^{\widehat{1i}} = (\mathbf{g}_2, \ldots, \mathbf{g}_{i-1}, \mathbf{g}_{i+1}, \ldots, \mathbf{g}_p)$

Let $\mathsf{m}^{1i} = (m_1, m_i)$, $\mathsf{m}^{\widehat{1i}} = (m_2, \ldots, m_{i-1}, m_{i+1}, \ldots, m_p)$ Hence, by (3.2), we have

$$\begin{aligned} \lim_{N\to\infty} \mathrm{E}(\Sigma_N)^p &= \lim_{N\to\infty} A_N^{-p} \sum_{(\mathbf{g}\mathsf{j})\in J_p} \sum_{\mathsf{m}\in M_p} \sum_{i=2}^{p} \mathsf{b}_j c_{\mathbf{g}} \sum_{G^{1i}\in\Gamma^{1i}} \sum_{G^{\widehat{1i}}\in\Gamma^{\widehat{1i}}} J_{G^{1i}}^{\mathsf{m}^{1i}} J_{G^{\widehat{1i}}}^{\mathsf{m}^{\widehat{1i}}} \\ &= \lim_{N\to\infty} \sum_{i=2}^{p} \left(A_N^{-2} \sum_{((j_1,j_i),(g_1,g_i))\in J_2} \sum_{\mathsf{m}^{1i}\in M_2} b_{j_1} b_{j_i} c_{g_1} c_{g_i} \sum_{G^{1i}\in\Gamma^{1i}} J_{G^{1i}}^{\mathsf{m}^{1i}} \right) \\ &\quad \cdot A_N^{-(p-2)} \left(\sum_{(\mathbf{g}\mathsf{j})\in J_{p-2}} \sum_{\mathsf{m}\in M_{p-2}} \prod_{l=1}^{p-2} b_{j_l} c_{j_l} \sum_{G\in\Gamma(p-2,d,\mathbf{g})} J_G^{\mathsf{m}} \right) \end{aligned}$$

Using inductive hypothesis, for 1 and $q-1$ we finally obtain:

$$\begin{aligned}\lim_{N\to\infty} \mathrm{E}(\Sigma_N)^p &= \sum_{i=2}^{p}\left(\sum_{s=1}^{u} b_s\sigma^2\right)\left(\sum_{s=1}^{u} b_s\sigma^2\right)^{\frac{p-2}{2}} \\ &= (p-1)\left(\sum_{s=1}^{u} b_s\sigma^2\right)^{p/2}\end{aligned}$$

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