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INTRODUCTION TO CLASSICAL DESCRIPTIVE SET THEORY

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PRESENTATION

These notes are part of a series of lectures "On Descriptive Set Theory" given by professor Joseph Diestel, from Kent State University, during his visit to our Department on January 1993. Before finishing this presentation we would like to thank professor Diestel for sharing his mathematical knowledge and valuable time with our comunity.

1 A THEOREM OF TALAGRAND

In the spring of the 1979, Michel Talagrand proved the following:

Theorem 1.1 [*Talagrand*] *If X is a separable Banach space which contains an isomorphic copy of every separable Banach space with the Radon-Nikodym property, then X does not have the Radon-Nikodym property.*

His proof is, so far as I know, unpublished to this very day and since it hints at the variety of open problems in measure theory and Banach spaces theory that are derived from techniques peculiar to descriptive set theory, it seems an apt-take-off point for these lectures.

Let X be a **separable** Banach space, C be a non-empty closed bounded convex subset of X and $\varepsilon > 0$.

We define $T(C,\varepsilon)$ to be *the subset of C that remains after we remove all open slices of C having $norm-diameter \leq \varepsilon$.* More precisely, an **open slice** of C is a set of the form

$$S(C;x^*,a) = \{x \in C \mid x^*(x) > a\}$$

where $x^* \in X^*$ and $a \in R$; we assume, too, that we are working only in **real** Banach spaces. So

$$T(C,\varepsilon) = \bigcap_{diam\, S(C,x^*,a) \leq \varepsilon} \{x \in C \mid x^*(x) \leq a\}.$$

It follows that $T(C,\varepsilon)$ is a **closed bounded convex subset of** C .

Now we start our task throught the ordinals:

$$\begin{aligned} T_1(C,\varepsilon) &= T(C,\varepsilon) \\ T_2(C,\varepsilon) &= T(T_1(C,\varepsilon),\varepsilon) \\ &\vdots \\ T_{\beta+1}(C,\varepsilon) &= T(T_\beta(C,\varepsilon),\varepsilon) \end{aligned}$$

and should α be a limit ordinal, then

$$T_\alpha(C,\varepsilon) = \bigcap_{\beta<\alpha} T_\beta(C,\varepsilon).$$

Naturally, certain stability properties are almost transparent.

Proposition 1.2 *Let C and D be closed bounded convex subsets of X.*

i) If $C \subseteq D$, then $T_\alpha(C,\varepsilon) \subseteq T_\alpha(D,\varepsilon)$.

ii) For $\lambda > 0$, $T_\alpha(\lambda C, \lambda\varepsilon) = \lambda T_\alpha(C,\varepsilon)$.

iii) For $x \in X$, $T_\alpha(x + C, \varepsilon) = x + T_\alpha(C,\varepsilon)$.

Since X is separable and $(T_\alpha(C,\varepsilon))_\alpha$ is decreasing in α, there is a **first ordinal** $\tau(C,\varepsilon) < \omega_1$ ($= 1^{st}$ uncountable ordinal) such that for any $\alpha \geq \tau(C,\varepsilon)$,

$$T_\alpha(C,\varepsilon) = T_{\tau(C,\varepsilon)}(C,\varepsilon).$$

Denote by $\tau_n(X) := \tau(B_X, 1/n)$ and let $\tau(X) = \sup_n \tau_n(X)$. Notice that $\tau(X) < \omega_1$.

Lemma 1.3 (a) *If Y is a closed linear subspace of X, then*

$$\tau_n(Y) \leq \tau_n(X) \quad \textit{for each } n \textit{ and so} \quad \tau(Y) \leq \tau(X).$$

(b) *If X and Y are isomorphic, then $\tau(X) = \tau(Y)$.*

Proof. (a) follows easily from the fact that for any $\varepsilon > 0$, $T(B_Y,\varepsilon) \subseteq T(B_X,\varepsilon)$.
(b) takes a bit more thought about what's going on. To start notice that whenever $V : X \to Y$ is a bounded linear operator, then for any $C \subset X$,

$$diam\, V(C) \leq ||V||\, diam\,(C).$$

Next, suppose $V : X \to Y$ is an isomorphism of X on Y and suppose $p \in N$ satisfies

$$||V|| \leq p \quad \text{and} \quad ||V^{-1}|| \leq p.$$

If C is any non-empty closed bounded convex subset of X, then $V(C)$ is a non-empty closed bounded convex subset of Y; what's more,

$$S := S(V(C), y^*, a) = V\left(S(C, V^* y^*, a)\right).$$

It is now easy to justify the following chain of inclusions:

$$
\begin{aligned}
T(V(C),\varepsilon) &= \bigcap_{diam S(V(C),y^*,a)\leq\varepsilon} V(C)\setminus S \\
&= \bigcap_{diam V(S(C,V^* y^*,a))\leq\varepsilon} V(C)\setminus V(S)
\end{aligned}
$$

(which, since V is an isomorphism, is)

$$
= V\left(\bigcap_{diam S(C,V^* y^*,a)\leq\varepsilon} C\setminus S\right)
$$

(which because $diam V(S)\leq\varepsilon$ implies $diam(S)\leq p\varepsilon$)

$$
\begin{aligned}
&\supseteq V\left(\bigcap_{diam S(C,x^*,a)\leq p\varepsilon} C\setminus S\right) \\
&= V(T(C,p\varepsilon)).
\end{aligned}
$$

Summarily we have

$$
V(T(C,p\varepsilon))\subseteq T(V(C),\varepsilon).
$$

Start with $C=B_X$. Then

$$
\begin{aligned}
V(T(B_X,\varepsilon)) &\subseteq T(V(B_X),\varepsilon/p) \\
&\subseteq T(pB_Y,\varepsilon/p) \\
&= pT(B_Y,\varepsilon/p^2).
\end{aligned}
$$

So,

$$
\begin{aligned}
V(T_2(B_X,\varepsilon)) &= V(T(T(B_X,\varepsilon),\varepsilon)) \\
&\subseteq T(V(T(B_X,\varepsilon)),\varepsilon/p) \\
&= T(T(V(B_X),\varepsilon/p)),\varepsilon/p) \\
&= T_2(V(B_X),\varepsilon/p)
\end{aligned}
$$

$$\subseteq T_2(pB_X, \varepsilon/p)$$

$$= pT_2(B_Y, \varepsilon/p^2),$$

trala, trala, trala,... For any α !

$$V(T_\alpha(B_X, \varepsilon) \subseteq pT_\alpha(B_Y, \varepsilon/p^2).$$

Consequently,

$$\tau_n(X) \leq \tau_{p^2n}(Y),$$

and so

$$\tau(X) \leq \tau(Y).$$

Turn about is fair play and so (b) is proved. ♠

All this is a general abstract nonsense of the finest sort which gains considerable import if we assume, as we will, that X has the Radon- Nikodym property. For our present purposes, **a Banach space has the Radon-Nikodym property if every non-empty closed bounded convex subset of X has non-empty open slices of arbitrarily small diameter**. It is a deep and beautiful theorem of *Messrs* Bill Davis, Bob Huff, Hugh Maynard, Bob Phelps and Mark Rieffel that relates the property so defined with the classical Radon-Nikodym theorem in basic measure theory.

Of course, in the present context, *X having the Radon-Nikodym property is tantamount to saying that for a closed bounded convex set $C \subsetneq X$ that's non-empty, $T(C, \varepsilon) \subsetneq C$.* It follow that if C is a non-empty closed bounded convex subset of X with the Radon-Nikodym property, then

$$T_{\tau(C,\varepsilon)}(C, \varepsilon) = \emptyset.$$

Now we're in plain to prove Talagrand's theorem.

First, we need to recall a few "*facts-of-the-life*" regarding spaces with the Radon-Nikodym property. These facts are not difficult to prove if one has the measure theoretic formulation of the Radon-Nikodym property well-in-hand. They are typical of the kinds of stability one needs to apply the technique present by being employed. The facts are these:

(I) If $X_1, X_2, \ldots, X_n$ are Banach spaces with the Radon-Nikodym property and $1 \leq p \leq \infty$, then

$$\left(\sum_{k=1}^{n} \oplus X_k\right)_{\ell_p}$$

has the Radon-Nikodym property, too.
(II) If $(X_\alpha)_\alpha$ is a family of Banach spaces with the Radon-Nikodym property and $1 \leq p \leq \infty$, then

$$\left(\sum_{\alpha} \oplus X_\alpha\right)_{\ell_p}$$

has the Radon-Nikodym property, too.

Magical step ! **We claim that for every ordinal $\alpha < \omega_1$ there is a separable Banach space X_α having the Radon-Nikodym property such that $\tau_3(X_\alpha) \geq \alpha$.**

To start, let $X_1 = \mathbb{R}$. Then $B_{\mathbb{R}} = [-1, 1]$. Open slices of $B_{\mathbb{R}}$ of diameter $\leq 1/3$ are $[-1, -2/3)$ and $(2/3, 1]$. Hence, $T(B_{\mathbb{R}}, 1/3)$ is just $[-2/3, 2/3]$ and $\tau_3(\mathbb{R}) \geq 1$.

Let's see how to continue. Let $\gamma \leq \omega_1$. Suppose we're taken each magical step $\tau_3(X_\alpha) \geq \alpha$ for each $\alpha \leq \gamma$.

If γ is a limit ordinal, then let

$$X_\gamma = \left(\sum_{\alpha<\gamma} \oplus X_\alpha\right)_{\ell_2}.$$

Since each X_α is a separable Banach space, **(II)** assures us X_γ has the Radon-Nikodym property. Since each of the X_α's satisfies $\tau_3(X_\alpha) \geq \alpha$, then $\tau_3(X_\gamma) \geq \tau_3(X_\alpha) \geq \alpha$, too; but this is so for every $\alpha < \gamma$, so

$$\tau_3(X_\gamma) \geq \gamma.$$

What if $\gamma = \beta + 1$ for some β ?. Then we let

$$X_\gamma = (X_\beta \oplus \mathbb{R})_{\ell_\infty}.$$

By **(I)** X_γ has the Radon-Nikodym property since X_β is supposed to; X_γ is also separable, simile X_β is. Of course, we have inductively hipothesized that $\tau_3(X_\beta) \geq \beta$ so, for any $\delta \leq \beta$ one has

$$T_\delta(B_{X_\beta}, 1/3) \neq \emptyset.$$

May be β is itself a limit ordinal. In this case we know $0 \in T_\delta(B_{X_\beta}, 1/3)$ for each $\delta < \beta$ and so

$$0 \in \bigcap_{\delta<\beta} T_\delta(B_{X_\beta}, 1/3) = T_\beta(B_{X_\beta}, 1/3).$$

But $\jmath : X_\beta \to X_\gamma$ via $x \mapsto (x, 0)$ is an isometric inclusion, so

$$(0,0) \in T_\beta(B_{X_\gamma}, 1/3) \quad \text{and} \quad \tau_3(X_\gamma) \geq \beta + 1 = \gamma.$$

Alas! we're left with the possibility that β is itself $= \delta+1$. But our inductive posturing ensures us that $T_\delta(B_{X_\delta}, 1/3) \neq \emptyset$. Take $t \in B_{\mathbb{R}} = [-1, 1]$. Then

$$\begin{aligned} T_\delta(B_{X_\delta}, 1/3) \times \{t\} &= T_\delta(B_{X_\delta} \times \{0\}, 1/3) + (0, t) \\ &= T_\delta(B_{X_\delta} \times \{0\} + (0, t), 1/3) \\ &= T_\delta(B_{X_\delta} \times \{t\}, 1/3) \\ &\subseteq T_\delta(B_{X_\gamma}, 1/3). \end{aligned}$$

But $0 \in T_\delta(B_{X_\delta}, 1/3)$, so $(0, t) \in T_\delta(B_{X_\gamma}, 1/3)$ for every $t \in [-1, 1]$; that is,

$$\{0\} \times [-1, 1] \subseteq T_\delta(B_{X_\gamma}, 1/3).$$

Hence

$$\begin{aligned} (0,0) \in T(\{0\} \times [-1,1], 1/3) &\subseteq T(T_\delta(B_{X_\gamma}, 1/3)) \\ &= T_{\delta+1}(B_{X_\gamma}, 1/3). \end{aligned}$$

Hence $\tau_3(X_\gamma) \geq \delta + 2 = \gamma$. ♠

2 POLISH SPACES

A **Polish** space is a topological space that's homeomorphic to a complete separable metric space.

EXAMPLES:

1.) Separable Banach spaces and their non-empty **closed subsets** are Polish.

2.) Compact meric spaces are Polish.

3.) If S is a compact metric space and $\mathcal{K}(S)$ **denotes the collection of non-empty closed (hence compact) subsets of** S, then $\mathcal{K}(S)$ can be realized as a Polish space.

Here's how. Let S be a compact metric space and let

$$\mathcal{K}(S) = \{K \subseteq S : K \text{ is compact, } K \neq \emptyset\}.$$

Equip $\mathcal{K}(S)$ with the following Hausdorff topology: basis for this topology is generated by open sets $U, V_1, \ldots, V_n$ in S as follow:

$$\mathcal{O}(U, V_1, \ldots, V_n) = \{K \in \mathcal{K}(S) : K \subseteq U, \ K \cap V_1 \neq \emptyset, \ldots, K \cap V_n \neq \emptyset\}.$$

Denote by τ the topology generated by the $\mathcal{O}(U, V_1, \ldots, V_n)$'s. $(\mathcal{K}(S), \tau)$ is plainly a Hausdorff space. In fact we have

Theorem 2.1 *$(\mathcal{K}(S), \tau)$ is a compact metric space. Denoting by d the metric of S, the metric generating τ is given by*

$$D(K, L) = \sup_{x \in S} |d(x, K) - d(x, L)|.$$

Proof. For $K \in \mathcal{K}(S)$, denote by d_K the function $d_K(x) = d(x, K)$ for all $x \in S$. Since $|d_K(x) - d_K(y)| \leq d(x, y)$, $\{d_K : K \in \mathcal{K}(S)\}$ is bounded and equicontinuous set in $C(S)$. By the Arzela-Ascoli theorem, we see that $\{d_K : K \in \mathcal{K}(S)\}$ is relatively compact in $C(S)$.

More is so. If $(K_n)_n$ is a sequence in $\mathcal{K}(S)$ and $d_{K_n} \to f$ in the norm of $C(S)$, then $f = d_K$ where $K = \{x \in S : f(x) = 0\}$. In fact, if $x \in S$ is fixed then for every y,

$$|f(x) - f(y)| \leq d(x, y),$$

so, if $y \in K$ we have

$$|f(x)| = |f(x) - f(y)| \leq d(x,y)$$

ensuring $|f(x)| \leq d_K(x)$, - of course $f \geq 0$ so this is just saying $f(x) \leq d_K(x)$. On the other hand, if from each K_n we pick a y_n so that $d(x, y_n) = d(x, K_n)$ then, by passing to a subsequence if necessary, we can suppose $y_n \to y$ for some $y \in S$. It's plain (*or close to it*) that $d(x,y) = f(x)$ and this is so for *every* $x \in S$!. In particular, $f(y) = d(y,y) = 0$, so $y \in K$. Finally, $f(x) = d(x,y) \geq d_K(x)$ and $f = d_K$! In this way, we see that $\mathcal{K}(S)$ is a compact subset of $C(S)$; when we realize that the metric D is what it is, we see that $(\mathcal{K}(S), D)$ is isometrically inside $C(S)$, too.

Alas notice that (with proper ID's in hand)

$$\{K : K \subseteq L\} = \{d_K : d_K \geq d_L\}$$

and

$$\{K : K \cap L \neq \emptyset\} = \{d_K : d_K \wedge d_L = 0\}.$$

Then we see that D generates a bigger topology than τ, hence D generates τ. ♠

4.) Closed subspaces of Polish spaces are Polish.

5.) Open subspaces of Polish spaces are Polish.

Proof. Let X be a Polish space and d be a complete metric on X that generates its separable topology. Let U a non-empty open subset of X, with $U \neq X$. Define the distance D on U by

$$D(u,v) = d(u,v) + \left| \frac{1}{d(u,U^c)} - \frac{1}{d(v,U^c)} \right|.$$

It is easy to see that D and d enjoy the same sets of convergent sequences (with limits) in U. Indeed, $D \geq d$ so should $u_n \xrightarrow{D} u$, $u_n \xrightarrow{d} u$, too. If $u_n \xrightarrow{d} u$, then, as $u_n, u \in U$, $1/d(u_n, U^c)$ and $1/d(u, U^c)$ are real and $d(u_n, U^c) \to d(u, U^c)$ so $d(u_n, U^c)^{-1} \to d(u, U^c)^{-1}$, too. It follows that $u_n \xrightarrow{D} u$.

The issue, of course, is that D is a <u>*complete*</u> metric on U. If $(u_n)_n$ is D-Cauchy, then $(u_n)_n$ is d-Cauchy and so converges to some $x \in X$ in the d-metric. The point is $x \in U$.

In fact, if $x \in U^c$ then $d(x, U^c) = 0$. So $\lim_n d(u_n, U^c) = 0$. It follows that we can find a subsequence $(u_{n_k})_k$ of $(u_n)_n$ such that

$$\left| \frac{1}{d(u_{n_k}, U^c)} - \frac{1}{d(u_{n_j}, U^c)} \right| \geq k, \qquad j = 1, 2, \ldots, k-1.$$

From this $\overline{\lim}_{j,k} D(u_{nk}, u_{nj}) = \infty$, contradicting $(u_{n_k})_k$'s D-Cauchyness. ♠

6.) Countable disjoint sums of Polish spaces are Polish.

Here if $(X_n)_n$ is a countable family of metric spaces (assumed to be disjoint) we give $\bigoplus_{n=1}^{\infty} X_n$ the metric

$$d(x,y) = \begin{cases} d_n(x,y) & \text{if } x, y \in X_n \\ 1 & \text{if } x, y \text{ are in different } X_n\text{'s.} \end{cases}$$

7.) Countable products of Polish spaces are Polish.

8.) Theorem (Alexandrov). Let X be a Polish space. $S \subseteq X$ is Polish if and only if S is a G_δ in X.

Proof. Suppose $S = \cap_n U_n$ where U_n is open in X for all $n \in \mathbb{N}$. By **5.)**, each U_n is Polish. Hence $\prod_n U_n$ is Polish. Look to

$$\Delta = \left\{ (u_n)_n \in \prod_{n=1}^{\infty} U_n : u_j = u_k \text{ for all } j, k \in \mathbb{N} \right\}.$$

Δ is a closed subspace of the Polish space $\prod_n U_n$, so Δ is a Polish space. But S is homeomorphic to Δ, so S is Polish, too.

Now suppose S is a Polsih subspace of the complete separable metric space (X, d). Let D be a complete metric that generates S's topology. For each n let

$$V_n = \bigcap \{ W \subseteq X : W \text{ is open, } W \cap S \neq \emptyset,\ D - \operatorname{diam}(W \cap S) \leq 1/n \}.$$

We **claim** that

$$S = \overline{S} \cap \left(\bigcap_{n=1}^{\infty} V_n \right).$$

Take $s \in S$. Look at $U = \{z \in S : D(z,s) < 1/3n\}$. U is d-open in S (D and d generate the same topology of S) and $D - diam(U) \leq 2/3n$. Now $U = W \cap S$ for some open set $W \subseteq X$ so $s \in V_n$. It follows that

$$S \subseteq \overline{S} \cap \left(\bigcap_{n=1}^{\infty} V_n \right).$$

Now suppose

$$x \in \overline{S} \cap \left(\bigcap_{n=1}^{\infty} V_n \right).$$

Since $x \in \cap_n V_n$ there is for each n a d-open W_n in X that contains x, intersects S and with $D - diam(W_n \cap S) \leq 1/n$. Replacing W_n with $W_1 \cap \cdots \cap W_n$, if necesary, we may suppose W_n's are decreasing. Further, by looking to d-open balls centered at x we may assume that $d - diam(W_n) < 1/n$, too. Keep in mind that $x \in \overline{S}$ so $x \in W_n$ still ensures $W_n \cap S \neq \emptyset$. Cantor's nested interval theorem tell us there is a unique $s \in S$ common to all $\overline{W_n \cap S}^{D}$. But x is the only element common to $\cap_n \overline{W_n}^{(X,d)}$.

Ah ha! Notice

$$\begin{aligned} \overline{W_n \cap S}^{(S,D)} &= \overline{W_n \cap S}^{(S,d)} \\ &\subseteq \overline{W_n}^{(X,d)}, \end{aligned}$$

so $s = x$ and $x \in S$!. For this

$$S = \overline{S} \cap \left(\bigcap_{n=1}^{\infty} V_n \right).$$

But $\overline{S}$ is closed in the metrizable space X so $\overline{S}$ is a G_δ; it soon follows that S as the intersection of two G_δ's is a G_δ. ♠

9.) Locally compact Hausdorff spaces that satisfy the second axiom of countability are Polish.

Indeed, such spaces have compact metrizable one point compactification and are open, therein:

10.) $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$ is Polish.

11.) $\mathbb{I}$, the set of irrational numbers, is Polish.

12.) $\mathbb{Q}$, the set of rational numbers, is <u>not</u> Polish.

13.) Let C be a non-empty complete separable metrizable convex subset of a locally convex space E. Then $ext\, C$, the set of **extreme points** of C, is Polish.

Indeed, a point x of C is extreme if and only if whenever $x = \frac{c_1 + c_2}{2}$ with $c_1, c_2 \in C$, $x = c_1 = c_2$.
Hence,

$$(ext\, C)^c = \bigcup_{n=1}^{\infty} \left\{ \frac{c_1 + c_2}{2} : c_1, c_2 \in C,\ d(c_1, c_2) \geq 1/n \right\}$$

making $(ext\, C)^c$ an F_σ-set and $ext\, C$ a G_δ.

NOTE: This last example is of some importance in abstract analysis. A famous theorem of Choquet says that if K is a compact metrizable convex subset of a locally convex space E, then *every point of K is the **baricenter** of a regular Borel probability measure on K that's supported by the extreme points of K.* In efforts to generalize Choquet's theorem, the Polish nature of extreme points plays a central role. Another key role is the following famous theorem of K. Kuratowski and Cz. Ryll-Nardzewski.

Theorem 2.2 [*Kuratowski -- Ryll - Nardzewski*]. *Let (Ω, Σ) be a measurable space and S be a Polish space. Let $\mathcal{F}(S)$ be the collection of non-empty closed subsets of S, and suppose $F : \Omega \to \mathcal{F}(S)$ is such that for each open set V in S,*

$$\{\omega \in \Omega : F(\omega) \cap V \neq \emptyset\} \in \Sigma.$$

Then there exists an $f : \Omega \to S$ such that

i) $f^{-1}(B) \in \Sigma$ for each Borel set $B \subseteq S$,

and

ii) $f(\omega) \in F(\omega)$ for each $\omega \in \Omega$.

Proof. Suppose d is a complete metric on S with $d - diam(S) < 1$ such that d generates S's separable Polish topology. We plan to construct a sequence $(f_n)_{n=0}^{\infty}$ of functions from Ω such that

(a) each f_n is $\Sigma - Bo(S)$ measurable;

(b) $d(f_n(\omega), F(\omega)) < 2^{-n}$ for each n and each ω;

and

(c) $d(f_n(\omega), f_{n-1}(\omega)) < 2^{-n+1}$ for each $n \geq 1$ and each ω.

Once done, (c) ensures us that $(f_n(\omega))_{n\geq 0}$ is pointwise d-Cauchy in S. Hence $f(\omega) := \lim_n f_n(\omega)$ exist in S and is $\Sigma - Bo(S)$ measurable by (a). (b) kicks in to assure that $f(\omega) \in \overline{F(\omega)}^d = F(\omega)$ and we're done.

Let $(s_n)_{n\geq 1}$ be dense in S. Dedine $f_0 : \Omega \to S$ by $f_0(\omega) \equiv s_1$. (a) through (c) follow by default.

To get f_1 we proceed as follows: for $j \geq 1$ set

$$C_j = \left\{\omega \in \Omega : d(s_j, F(\omega)) < \frac{1}{2}\right\},$$

$$D_j = \{\omega \in \Omega : d(s_j, f_0(\omega)) < 1\}.$$

Let $U_r(s) = \{x \in S : d(s,x) < r\}$ be the open r-ball centered at s. Then

$$C_j = \{\omega \in \Omega : F(\omega) \cap U_{1/2}(s_j) \neq \emptyset\} \quad \text{and} \quad D_j = f_0^{-1}(U_1(s_j))$$

are each in Σ, C_j by hypothesis and D_j by design.

Let $A_j = C_j \cap D_j$. Then

$$A_j \in \Omega \quad \text{and} \quad \Omega = \bigcup_{j=1}^{\infty} A_j.$$

Define $f_1 : \Omega \to S$ by $f_1(\omega) = s_{k(\omega)}$ where is the first positive integer such that $\omega \in A_{k(\omega)} \setminus \bigcup_{j<k(\omega)} A_j$.

If B is a Borel set in S, then

$$\begin{aligned} f_1^{-1}(B) &= \bigcup_{s_k \in B} f_1^{-1}(\{s_k\}) \\ &= \bigcup_{s_k \in B} \left(A_k \setminus \bigcup_{j<k} A_j\right) \in \Sigma. \end{aligned}$$

(a) follows from this. (b) and (c) hold simply because $\omega \in A_{k(\omega)} = C_{k(\omega)} \cap D_{k(\omega)}$.

Now to build $f_2 : \Omega \to S$ set

$$\begin{aligned} C_j^2 &= \{\omega \in \Omega : d(s_j, F(\omega)) < 1/2^2\} \\ &= \{\omega \in \Omega : F(\omega) \cap U_{1/2^2}(s_j) \neq \emptyset\} \end{aligned}$$

and

$$D_j^2 \;=\; f_1^{-1}(U_{1/2}(s_j)).$$

Notice $C_j^2, D_j^2 \in \Sigma$ and set $A_j^2 = C_j^2 \cap D_j^2$. So $A_j^2 \in \Sigma$, too. Note once again that $\Omega = \cup_j A_j^2$. In fact, if $\omega \in \Omega$ and $s \in F(\omega) \cap U_{1/2}(f_1(\omega))$, then the open set U

$$U = U_{1/2^2}(s) \cap U_{1/2}(f_1(\omega))$$

must contain a point s_k; let $k(\omega)$ be the first k so that $s_k \in U$ and notice that $\omega \in A^2_{k(\omega)}$.

As before, if $\omega \in \Omega,\ \omega \in A^2_{k(\omega)} \setminus \bigcup_{j<k(\omega)} A_j^2$; set $f_2(\omega) = s_{k(\omega)}$ and continue.

♠

Theorem 2.3 *Let (X, d) be a metric space and μ be a Borel probability measure on X. Then for any Borel set $B \subseteq X$ and any $\varepsilon > 0$ there is a closed set $F \subseteq B$ and a open set $G \supseteq B$ for which $\mu(G \setminus F) < \varepsilon$.*

If (X, d) is a Polish space, then each Borel probability measure μ on X satisfies: given any Borel set $B \subseteq X$ and any $\varepsilon > 0$ there is a compact set $K \subseteq B$ for which $\mu(B \setminus K) < \varepsilon$.

Proof. Let $\mathcal{B}$ denote the collection of μ-**approximable** Borel subsets of X; i.e., $B \in \mathcal{B}$ if given $\varepsilon > 0$ there's a closed set $F \subseteq B$ and an open set $G \supseteq B$ with $\mu(G \setminus F) < \varepsilon$.

Closed sets belong to $\mathcal{B}$. In fact, if C is closed and we let $f(x) = d(x, C)$ for all $x \in X$, then f is a continuous real-valued function for which $C = Z(f) := \{x \in X : f(x) = 0\} = \cap_n \{x \in X : f(x) < 1/n\}$.

$\mathcal{B}$ is plainly permanent under the taking of complements.

Finally, $\cap_n B_n$ belongs to $\mathcal{B}$ whenever B_n does for each $n \geq 1$. To see this, let $\varepsilon > 0$ be given and choose closed sets $F_n \subseteq B_n$ and open sets $G_n \supseteq B_n$ so that $\mu(G_n \setminus F_n) < \varepsilon/2^{n+1}$. Noticing that $\cap_{n=1}^{\infty} F_n = \cup_{n=1}^{\infty} \cup_{k=1}^{n} F_k$ we can find an n_0 for which

$$\mu\left(\bigcup_{n=1}^{\infty} F_n \setminus \bigcup_{k=1}^{n_0} F_k\right) < \varepsilon/2.$$

Let $F = \bigcup_{k=1}^{n_0} F_k$. Then F is a closed set, $F \subseteq \bigcup_{n=1}^{\infty} B_n \subseteq \bigcup_{n=1}^{\infty} G_n = G$ (= an open set). Of course, $\mu(G \setminus F) < \varepsilon$.

This proves the first assertion.

To prove the second we need only show that given an $\varepsilon > 0$ there is a compact set $K \subseteq X$ such that $\mu(K) > 1 - \varepsilon$; this is a benefit drawn from the first assertion.

Polish spaces are homeomorphic to complete separable metric spaces and so we may just as well assume (X, d) is a complete separable metric space. Separability ensures the existence for each n of a sequence $\{B_k(n) : k \geq 1\}$ of open $1/n - balls$ that covers X. We can (and do) assume that the center of $(B_k(n))_{k\geq 1}$ and those of $(B_k(m))_{k\geq 1}$ coincide for any m, n.

Now there are only so many $B_k(1)$'s needed to eat up all but $\varepsilon/2$ of X; say

$$\mu\left(X \setminus \bigcup_{i=1}^{k(1)} B_i(1)\right) < \varepsilon/2.$$

There are only so many $B_k(2)$'s needed to eat up all but $\varepsilon/4$ of X; say $B_1(2)$, $\ldots$, $B_{k(1)}(2)$, $\ldots$, $B_{k(2)}(2)$ are all that's needed. Generally, only so many $B_k(n)$'s (say $B_1(n)$, $\ldots$, $B_{k(1)}(n)$, $\ldots$, $B_{k(2)}(n)$, $\ldots$, $B_{k(n)}(n)$) are needed to eat up all but $\varepsilon/2^n$ of X.

Two conclusions can be drawn: **first** we conclude that

$$\bigcap_{n=1}^{\infty} \left(B_1(n) \cup \cdots \cup B_{k(n)}(n)\right)$$

is totally bounded, leading us to **second** conclusion that the closure K of

$$\bigcap_{n=1}^{\infty} \left(\bigcup_{i=1}^{k(n)} B_i(n)\right)$$

is a compact set for which $\mu(K) > 1 - \varepsilon$. ♠

3 ANALYTIC SETS

Let X be a Polish space. A subset A of X is **analytic** if there is a Polish space Z and a continuous function $f : Z \to X$ such that $f(Z) = A$.

EXAMPLES:

1.) Every **closed** subset of a Polish space is analytic.

2.) Every **open** subset of a Polish space is analytic.

3.) The **union** of a countable collection of analytic subsets of a Polish space is analytic.

Indeed, suppose $(A_n)_n$ is a countable collection of analytic subsets of a Polish space. For each n, let Z_n be a Polish space and $f_n : Z_n \to X$ be a continuous function such that $f_n(Z_n) = A_n$. Let $Z = \bigoplus_n Z_n$ be the disjoint sum of Z_n's. Z is Polish. Define $f : Z \to X$ by taking $z \in Z$, searching to locate $z \in Z_n$ and letting $f(z) = f_n(z)$. Plainly f is continuous and $f(Z) = \bigcup\limits_n A_n$.

4.) The **intersection** of a countable collection of analytic subsets of a Polish space is analytic.

Again, suppose $(A_n)_n$ is a countable collection of analytic subsets of a Polish space X, Z_n are Polish spaces accompanied by continuous functions $f_n : Z_n \to X$ such that $f_n(Z_n) = A_n$. Then $\prod_n Z_n$ is Polish and $\Delta = \{(z_k) \in \prod_k Z_k : f_i(z_i) = f_k(z_k) \text{ for each } i, k\}$ is closed in $\prod_n Z_n$, hence Polish. If we define $f : \Delta \to X$ by $f((z_k)) = f(z_1)$, then f is continuous and $f(\Delta) = \bigcap\limits_n A_n$.

5.) Suppose X is a Hausdorff space. Then the Borel σ-algebra, $Bo(X)$, of X is the smallest family $\mathcal{T}$ of subsets of X such that
(a) Every open set is in $\mathcal{T}$.
(b) Every closed set is in $\mathcal{T}$.
(c) If $(F_n)_n$ is a countable collection of members of $\mathcal{T}$, then $\bigcap\limits_n F_n \in \mathcal{T}$.
(d) If $(F_n)_n$ is a countable collection of disjoint members of $\mathcal{T}$, then $\bigcup\limits_n F_n \in \mathcal{T}$.

Let $\mathcal{S}$ be the smallest collection of subsets of X that satisfy (a) through (d). Let $\mathcal{S}_0 = \{S \in \mathcal{S} : S^c \in \mathcal{S}\}$. Plainly $\mathcal{S}_0 \subseteq \mathcal{S} \subseteq Bo(X)$. Further, (a) and (b) combine to ensure $\mathcal{S}_0$ contains every open set of X; $\mathcal{S}_0$ is closed under taking complements.

Suppose $(E_n)_n$ is a sequence of members of $\mathcal{S}_0$. Then

$$\bigcup_{n=1}^{\infty} E_n = E_1 \cup (E_2 \cap E_1^c) \cup (E_3 \cap E_1^c \cap E_2^c) \cup \cdots \quad \in \mathcal{S}$$

and

$$\left(\bigcup_{n=1}^{\infty} E_n\right)^c = \bigcap_{n=1}^{\infty} E_n^c \quad \in \mathcal{S},$$

so $\bigcup_n E_n \in \mathcal{S}_0$. $\mathcal{S}_0$ is a σ- field of subsets of X that contains all the open sets, hence $\mathcal{S}_0 \supseteq Bo(X)$! and $\mathcal{S}_0 \subseteq \mathcal{S} \subseteq Bo(X) \subseteq \mathcal{S}_0$.

6.) Corollary. Every Borel subset of a Polish space is analytic.

7.) Let $(X_n)_n$ be a countable collection of Polish spaces. Suppose A_n is an analytic subset of X_n for each n. Then $\prod_n A_n$ is an analytic subset of $\prod_n X_n$.

Of course, $\prod_n X_n$ is Polish and so we're in position to test $\prod_n A_n$ for analyticity. For each n pick down a Polish space Z_n and a continuous function $f_n : Z_n \to X_n$ such that $f_n(Z_n) = A_n$. $\prod_n Z_n$ is Polish. Define $f : \prod_n Z_n \to \prod_n X_n$ by $f((z_n)) = (f_n(z_n))$. Plainly f is continuous and $f(\prod_n Z_n) = \prod_n A_n$.

8.) Theorem. Every Polish space is the continuous image of $\mathcal{N} = \mathbb{N}^{\mathbb{N}}$.

Proof. Let X be a non-empty Polish space and suppose d is a complete metric on X that generates the Polish topology of X.

We plan to construct a family of non-empty closed subsets of X,

$$\{C(n_1, \ldots, n_k)\},$$

indexed by the set of all finite tuples $(n_1, \ldots, n_k)$ of positive integers, in such a way that

(a) $d - diam(C(n_1, \ldots, n_k)) \leq 1/k$;

(b) $C(n_1, \ldots, n_{k-1}) = \bigcup_{n_k} C(n_1, \ldots, n_{k-1}, n_k)$;

and

(c) $X = \bigcup_{n_1} C(n_1) \left(= \bigcup_{n_1, n_2} C(n_1, n_2) = \cdots = \bigcup_{n_1, \ldots, n_k} C(n_1, \ldots, n_k) = \cdots \right)$

Start with $k = 1$. Let (x_{n_1}) be a countable dense set in X. For each n_1, let $C(n_1)$ be the closed ball centered at x_{n_1} with radius $1/2$. *(It may be* *that (x_{n_1}) is but a finite collection. Not to worry; repeat the C's - we never claimed they'd be disjoint !).*

Next, for each $C(n_1)$ locate a countable dense set, place the points of this set at the center of closed balls of radius 1/4 and list those closed balls as $(C(n_1, n_2))_{n_2}$. The procedure should be clear.

Now to define $f : \mathcal{N} \to X$. Take $n = (n_i) \in \mathcal{N}$. The sequence $(C(n_1, \ldots, n_k))$ is a decreasing sequence of non-empty closed subsets of the complete metric space (X, d) such that $d - diam(C(n_1, \ldots, n_k)) \to 0$ as $k \to 0$. Cantor's nested interval theorem assures us that there is a unique x_n common to the sets $C(n_1, \ldots, n_k)$. Set

$$f(n) = x_n.$$

If $m, n \in \mathcal{N}$ and $m_i = n_i$ for $i = 1, \ldots, k$ then

$$C(m_1, \ldots, m_k) = C(n_1, \ldots, n_k) \quad \text{and so} \quad f(m), f(n) \in C(n_1, \ldots, n_k).$$

It follows that $d(f(m), f(n)) \leq 1/k$; but this just says close points $m, n \in \mathcal{N}$ have close images $f(m), f(n)$ in X. So f is continuous.

Now if $x \in X$, then $x \in C(n_1)$ for some n_1 (by (c)) and so $x \in C(n_1, n_2)$ for some n_1, n_2 and so $x \in C(n_1, n_2, n_3)$ for some n_1, n_2, n_3, etc., etc., etc. by (b) (and (c)). In this way we find an $n = (n_i) \in \mathcal{N}$ such that

$$x \in \bigcap_n C(n_1, \ldots, n_k) \quad \text{and so} \quad f(n) = x.$$

♠

9.) Corollary. Every non-empty analytic subset of a Polish space is the continuous image of $\mathcal{N}$.

We're ready to the *fundamental separation theorem* for analytic sets.

10.) Fundamental Separation Theorem. Let X be a Polish space and A_1 and A_2 be disjoint analytic subsets of X. Then there exist **disjoint Borel** sets B_1 and B_2 in X such that $A_1 \subseteq B_1$ and $A_2 \subseteq B_2$.

For convenience we say the disjoint sets A_1 and A_2 are **separated** if there are disjoint Borel sets B_1 and B_2 such that $A_i \subseteq B_i$ for $i = 1, 2$.
(I) If $C_1, C_2, \ldots$ *and* D are subsets of X such that for each n, C_n and D are separated, then $\bigcup_n C_n$ and D are separated.

In fact, for each n there is a Borel set B_n such that $C_n \subseteq B_n$ and $D \subseteq B_n^c$. Now $\bigcup_n B_n$ is a Borel set, $\bigcup_n C_n \subseteq \bigcup_n B_n$ and $D \subseteq \bigcap_n (B_n)^c = (\bigcup_n B_n)^c$ a Borel set disjoint from $\bigcup_n B_n$.

(II) If $E_1, E_2, \ldots$ and $F_1, F_2, \ldots$ are subsets of X such that for each m and n, E_m and F_n are separated, then $\bigcup_n E_n$ and $\bigcup_n F_n$ are separated, too.

In fact, by repeated application of (I) we see that for each m, E_m and $\bigcup_n F_n$ are separated. Apply (I) once again to get $\bigcup_n E_n$ and $\bigcup_n F_n$ separated.

Preliminaries are out of the way.

Proof of the Fundamental Separation Theorem. Let A_1 and A_2 be non-empty analytic subsets of X with $A_1 \cap A_2 =\neq \emptyset$. By **Corollary 9.**, there are continuous functions $f, g : \mathcal{N} \to X$ such that $f(\mathcal{N}) = A_1$ and $g(\mathcal{N}) = A_2$.

We will suppose contrarily that A_1 and A_2 cannot be separated. For each $k \in \mathbb{N}$ and $n_1, \ldots, n_k \in \mathbb{N}$, set

$$\mathcal{N}(n_1, \ldots, n_k) = \{(m_i) \in \mathcal{N} : m_1 = n_1, \ldots, m_k = n_n\}.$$

Notice

$$A_1 = \bigcup_{m_1} f(\mathcal{N}(m_1)) \quad \text{and} \quad A_2 = \bigcup_{n_1} g(\mathcal{N}(n_1)).$$

By (II), there are m_1, n_1 so $f(\mathcal{N}(m_1))$ and $g(\mathcal{N}(n_1))$ cannot be separated. But

$$f(\mathcal{N}(m_1)) = \bigcup_{m_2} f(\mathcal{N}(m_1, m_2)) \quad \text{and} \quad g(\mathcal{N}(n_1)) = \bigcup_{n_2} g(\mathcal{N}(n_1, n_2))$$

so there are m_2, n_2 so that $f(\mathcal{N}(m_1, m_2))$ and $g(\mathcal{N}(n_1, n_2))$ cannot be separated.

In this manner we trace out $m = (m_i), n = (n_i) \in \mathcal{N}$ such that for each k,

$$f(\mathcal{N}(m_1, \ldots, m_k)) \quad \text{and} \quad g(\mathcal{N}(n_1, \ldots, n_k))$$

cannot be separated.

It follows that $f(m) = g(n)$! In fact, if $f(m) \neq g(n)$ then there'd be disjoint open sets in X one containing $f(m)$, the other $g(n)$. The preimages of these open sets would be disjoint open sets in $\mathcal{N}$ one containing m the other n. But the sequence $(\mathcal{N}(m_1, \ldots, m_k))$ of open sets about m is a neighborhood basis in $\mathcal{N}$ of m as is the sequence $(\mathcal{N}(n_1, \ldots, n_k))$ a neighborhood basis in $\mathcal{N}$ of n. Hence we eventually will find $\mathcal{N}(m_1, \ldots, m_k)$ inside the open set about m and $\mathcal{N}(n_1, \ldots, n_k)$ inside that about n. It follows that

$$f(\mathcal{N}(m_1, \ldots, m_k)) \quad \text{and} \quad g(\mathcal{N}(n_1, \ldots, n_k))$$

are separated by open sets !. OOPS.

Now we're really in a quandary. Our hypothesis that A_1 and A_2 can not be separated has lead us to conclude that $f(m) = g(n)$, where $f(m) \in A_1$ and $g(n) \in A_2$. Something smells in Denmark. ♠

11.) Corollary. Let X be a Polish space. If A and A^c are both analytic, then A is Borel.

Theorem 3.1 [*Mauldin*] *In* $C[0,1]$*, the collection of nowhere differentiable functions is analytic.*

Proof. $f \in C[0,1]$ is differentiable at some point x of $[0,1]$ precisely when for all n there is m such that:

$$(*)\begin{cases} \text{if} & 0 < |h_1|, |h_2| < 1/m \quad \text{and} \quad x + h_1, x + h_2 \in [0,1] \\ \text{then} & \left| \dfrac{f(x+h_1) - f(x)}{h_1} - \dfrac{f(x+h_2) - f(x)}{h_2} \right| \leq 1/n. \end{cases}$$

For $m, n \in \mathbb{N}$, let

$$E_{m,n} = \{(f, x) \in (C[0,1]) \times [0,1] : (*) \; holds\}.$$

The somewhere differentiable functions of $C[0,1]$ are, then, precisely those in

$$\mathcal{P}_{C[0,1]} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} E_{m,n}.$$

But $E_{m,n}$ is closed in $(C[0,1]) \times [0,1]$ for each m, n: if $(f_i, x_i) \in E_{m,n}$ and $(f_i, x_i) \to (f, x)$, then $||f_i - f||_\infty \to 0$ and $|x_i - x| \to 0$; hence, $f_i(x_i) \to f(x)$

and should $0 < |h_1|, |h_2| < 1/m$ and $x + h_1, x + h_2 \in [0, 1]$

$$\begin{aligned}
& \left|\frac{f(x+h_1)-f(x)}{h_1} - \frac{f(x+h_2)-f(x)}{h_2}\right| \\
= \quad & \lim_i \left|\frac{f_i(x+h_1)-f_i(x)}{h_1} - \frac{f_i(x+h_2)-f_i(x)}{h_2}\right| \\
= \quad & \lim_i \left|\frac{f_i(x_i+h_1)-f(x_i)}{h_1} - \frac{f_i(x_i+h_2)-f(x_i)}{h_2}\right| \\
\leq \quad & 1/n.
\end{aligned}$$

♠

4 THE KURATOWSKI-TARSKI SYMBOLIC DESCRIPTION CLASSES. A Brief Introduction

Let X be a variable symbol designating a property of a class of sets. For example, $X = G$ refers to open sets, $X = F$ to closed sets, $X = K$ compact sets, $X = A$ analytic sets, ... X_σ denote sets that are countable union of sets having X; X_δ refers to sets that are countable intersection of sets having X. CX denotes sets whose complements have X while PX refers to sets that are projections of sets with X. $X \cup Y$ consist of sets that are the union of a set with X a set with Y; $X \cap Y$ consist of sets that are the intersection of a set with X and a set with Y. (*Be careful in this regard:* $X \cup Y$ *is* **not** *the class of sets that are in either* X *or* Y*. Likewise* $X \cap Y$ *is* **not** *the class of sets enjoiying both* X *and* Y.)

One more thing. We only deal with properties X such that if M has property X in S, then $M \times T$ does too in $S \times T$.

RULES

Suppose $\alpha(t), \beta(s,t), \dots$ are propositional functions where $t \in T, s \in S, \dots$.

(I) If $\{t : \alpha(t)\}$ has X, then $\{t : \sim \alpha(t)\}$ has CX.
Next, to clarify our notation $X \cup Y$ and $X \cap Y$ we have

(II) If $\{t : \alpha(t)\}$ has X and $\{t : \alpha'(t)\}$ has Y, then $\{t : \alpha(t) \text{ } or \text{ } \alpha'(t)\}$ has $X \cup Y$ and $\{t : \alpha(t) \text{ } and \text{ } \alpha'\}$ has $X \cap Y$.

(III) If for every $n \in \mathbb{N}$, $\{t : \alpha_n(t)\}$ has X, then $\{t : \exists n, \ \alpha_n(t)\}$ has X_σ and $\{t : \forall n, \ \alpha_n(t)\}$ has X_δ.

(IV) If $\{t : \alpha(t)\}$ has X, then $\{(t,s) : \alpha(t)\}$ has X.

(V) If $\{(t,s) : \beta(t,s)\}$ has X, then $\{t : \exists s, \ \beta(t,s)\}$ is PX and $\{t : \forall s, \ \beta(t,s)\}$ is $CPCX$.

There are many other **RULES** that can be formulated and indeed facility at deriving such is what makes the Kuratowski-Tarski calculus useful.

To highlight the calculus, we recall the Hausdorff metric on $\mathcal{K}(S)$. S is a non-empty compact metric space, $\mathcal{K}(S)$ is the space of all non-empty compact subsets of S and $d(K,J) = ||d(\cdot\ ,K) - d(\cdot\ ,J)||_\infty$ where $d(\cdot\ ,K) \in C(S)$ is the function $d(x,K)\ \ (x \in S)$.

Here is a list of useful information involving $\mathcal{K}(S)$.

1. The following sets are compact:
   - $\{(s,K) \in S \times \mathcal{K}(S) : s \in S\}$
   - $\{(K,L) \in \mathcal{K}(S) \times \mathcal{K}(S) : K \subseteq L\}$
   - $\{(K,L) \in \mathcal{K}(S) \times \mathcal{K}(S) : K \cap L \neq \emptyset\}$

2. The following are continuous functions:
   - $S \to \mathcal{K}(S)$
$s \to \{s\}$.
   - $\mathcal{K}(S) \times \mathcal{K}(S) \to \mathcal{K}(S)$
$(K,L) \to K \cap L$
   - $\mathcal{K}(S \times T) \to \mathcal{K}(S)$
$K \to \prod_S(K)$
   - $\mathcal{K}(S) \to [0,\infty)$
$K \to diam\ K$

Now we'll give an example of how a clever description of a particular property leads to a precise determination of the character of this property via Kuratowski-Tarski calculus.

Theorem 4.1 (Banach) *The non-empty **perfect** subsets of a non-empty compact metric space S constitute a G_δ-subset of $\mathcal{K}(S)$.*

Proof. For a solid basis of comparison, fix a countable open basis $\{U_n : n \in \mathbb{N}\}$ for the topology of S. Let $K \in \mathcal{K}(S)$.

$$\begin{aligned} K \quad \text{is } \textbf{not} \text{ perfect} \quad &\iff K \quad \text{has isolated points} \\ &\iff \exists n\ \exists x\ :\ x \in K,\ x \in U_n\ :\ K \subseteq \{x\} \cup U_n^c \end{aligned}$$

Now $\{(x,K) \ : \ x \in K, \ K \subseteq \{x\} \cup U_n^c\}$ is a compact set in $S \times \mathcal{K}(S)$ so here's the Kuratowski-Tarski depiction of what's going on:

$$K \quad \text{is **not** perfect} \iff \exists n \, \exists x \ : \ x \in K, \ x \in U_n \ : \ K \subseteq \{x\} \cup U_n^c$$

DECODING

$$\begin{array}{lll} [\qquad\qquad\quad]_\sigma & \exists n & \\ [P \qquad\qquad\quad]_\sigma & \exists n & \exists x \\ [P(K \cap G \cap K)]_\sigma & \exists n & \exists x, \ x \in K, \ x \in U_n, \ K \subseteq \{x\} \cup U_n^c \end{array}$$

So

$$\begin{aligned} [P(K \cap G \cap K)]_\sigma &= [P(K \cap K_\sigma \cap K)]_\sigma \\ &= [P(K_\sigma)]_\sigma \\ &= (K_\sigma)_\sigma = K_\sigma \end{aligned}$$

♠

Corollary 4.2 (Hurewicz) *The collection of uncountable compact subsets of an uncountable compact metric space S is an analytic subset of $\mathcal{K}(S)$.*

Proof. Let $\mathcal{U}$ denote the collection of compact subsets of S that are uncountable and let $\mathcal{P}$ denote the G_δ-subsets of $\mathcal{K}(S)$ consisting of non-empty perfect subsets of S. The classical Cantor-Bendixon theorem warns us that

$$K \in \mathcal{U} \iff \exists L \in \mathcal{P} \ : \ L \subseteq K.$$

Hence $\mathcal{U}$ is $P(G_\delta \cap K) = P(G_\delta) = A$. ♠

REMARK. Hurewicz also showed the much more difficult and subtle fact $\mathcal{U}$ *is **not** a Borel set* . This is of real importance in our next example's application.

Let Δ denote the Cantor set and $\mathcal{K}(\Delta)$ denote the space of non-empty compact subsets of Δ. Let X be a separable Banach space and suppose some $C(K)$ embeds in X wiht $K \in \mathcal{K}(\Delta)$. If

$$\mathcal{K}(X) := \{K \in \mathcal{K}(\Delta) \ : \ C(K) \quad \text{embeds isomorphically in} \quad X\},$$

then

Theorem 4.3 (Bourgain) $\mathcal{K}(X)$ *is analytic.*

Proof. Our first step is to position ourselves to apply our calculus. Here is how a $K \in \mathcal{K}(\Delta)$ gets to be in $\mathcal{K}(X)$.

$$(*)\begin{cases} K \in \mathcal{K}(X) \iff & \exists\, \delta > 0\ \exists\, T : C(\Delta) \to X,\ \|T\| \leq 1 \textbf{ such that} \\ & \|Tf\| \geq \delta\ \|f_{|K}\|\ \forall f \in C(\Delta) \end{cases}$$

Let us suppose $(*)$ to be so. Then take D to be a countable dense subset of the closed unit ball of $C(\Delta)$. Notice that $B_{L(C(\Delta),X)}$, closed unit ball of $L(C(\Delta), X)$, when equipped with the (strong) topology of pointwise convergence is homeomorphic to a closed subset of X^D. Hence $(B_{L(C(\Delta),X)},\ strong)$ is Polish (hence G_δ). Let

$$Z = \mathcal{K}(\Delta) \times (B_{L(C(\Delta),X)},\ strong) \times (0,1].$$

Z is $K \times G_\delta \times G_\delta$ hence G_δ. Look to

$$Q = \{(K,T,\delta) \in Z\ :\ \|Tf\| \geq \delta\ \|f_{|K}\|,\ \forall f \in C(\Delta)\}.$$

Q is closed in Z, so G_δ and

$$\mathcal{K}(X) = P_{\mathcal{K}(\Delta)}(Q) = P(G_\delta) = A.$$

So it remains to show $(*)$ characterizes those $K \in \mathcal{K}(\Delta)$ that belong to $\mathcal{K}(X)$. If $K \in \mathcal{K}(X)$, then there is an isomorphic embedding $u : C(K) \hookrightarrow X$; we can scale u to suppose $\|u\| = 1$. $T : C(\Delta) \to X$ can now be defined by $T(f) = u(f_{|K})$; because u an isomorphism, $\|ug\| \geq \|u^{-1}\|^{-1}\ \|g\|$ for each $g \in C(K)$ and so $\delta = \|u^{-1}\|^{-1}$. On the other hand, if $T : C(\Delta) \to X$ satisfies $\|T\| \leq 1$ and $\|Tf\| \geq \delta\ \|f_{|K}\|$ for each $f \in C(\Delta)$, then an appeal to the Borsuk-Dugundji extension theorem is in order; recall that if M and N are metric spaces with $M \subseteq N$, then there is a bounded linear operator $E : C_b(M) \to C_b(N)$ between the spaces of bounded continuous real-valued functions on M and N respectively such that $\|E\| = 1$, $Ef \geq 0$, if $f \geq 0$ and, most strinkingly, $Ef_{|M} = f$, for each $f \in C_b(M)$. Let E be the Borsuk-Dugundji extension operator from $C(K)$ to $C(\Delta)$. Then $T \circ E : C(K) \to X$ is an isomorphic embedding.

Comment. Now let's suppose we have the full force of Hurewicz's descriptions in hand; that is, that **the collection of non-empty countable compact subsets of Δ is a non-Borel, co-analytic subset of $\mathcal{K}(\Delta)$.**

Suppose X is a separable Banach space such that $C(K) \hookrightarrow X$ for every countable compact set K. Then, of course, $C(K) \hookrightarrow X$ for every countable compact subset K of Δ. Hence $\mathcal{K}(X)$, a known analytic set, contain every countable compact subset of Δ. It must contain at least one uncountable compact set, too. If not, $\mathcal{K}(X)$ would be an analytic and co-analytic subset of $\mathcal{K}(\Delta)$ making it a Borel set which thanks to Heruwicz we know not to be the case.

Now here's the kicker: a famous theorem of Milutin states that **any $C(K)$, K an uncountable compact metric space, is isomorphic to $C(\Delta)$**. Hence, if the separable Banach space X has the property that $C(K) \hookrightarrow X$ for every countable compact metric space K, then $C(\Delta) \hookrightarrow X$, too. But $C(\Delta)$ is itself universal for all separable Banach spaces so X must be, too. ♠

5 UNIVERSAL MEASURABILITY

Recall that if (Ω, Σ, μ) is a probability space and $A \subseteq \Omega$ we define μ^*, **the outer measure generated by** μ, via the formula

$$\mu^*(A) = \inf\{\mu(B) \ : \ A \subseteq B \in \Sigma\};$$

further, we define μ_*, **the inner measure generated by** μ, via the formula

$$\mu_*(A) = \sup\{\mu(B) \ : \ A \supseteq B \in \Sigma\}.$$

The σ-field Σ_μ of **μ-measurable sets** then turns out be determined by:

$$E \in \Sigma_\mu \text{ precesily when } \mu_*(E) = \mu^*(E).$$

So, Σ_μ consists of those $E \subseteq \Omega$ such that there exist $S\ , B \in \Sigma,\ \ S \subseteq E \subseteq B$ such that $\mu(B \setminus S) = 0$.

While neither μ^* nor μ_* are typically measures they do exhibit some measure-like properties. Here's one we'll need.

Lemma 5.1 *Let (Ω, Σ, μ) be a probability space and $(A_n)_n$ be an increasing sequence of subsets of Ω. Then*

$$\mu^*(\cup_{n=1}^\infty A_n) = \lim_{n\to\infty} \mu^*(A_n).$$

Proof. μ^* is monotone so bigger sets are assigned bigger values by μ^*. It follows that in our present set-up, $\lim_n \mu^*(A_n)$ exists and is $\leq \mu^*(\cup_n A_n)$. Epsilonies to yhe rescue. Let $\varepsilon > 0$ be given. For each n, choose $B_n \in \Sigma,\ A_n \subseteq B_n$ so $\mu(B_n) \leq \mu^*(A_n) + \varepsilon$. If we replace B_n by $B_n \cap B_{n+1} \cap \cdots$, then we still have a member of Σ that contains A_n and is assigned by μ a value $\leq \mu^*(A_n) + \varepsilon$; what's more, the new sequence is increasing, too. Let it be done. Now μ's countable additivity kicks in to give us $\mu(\cup_n B_n) = \lim_n \mu(B_n)$. So

$$\begin{aligned}
\mu^*(\cup_{n=1}^\infty A_n) &\leq \mu(\cup_{n=1}^\infty B_n) \\
&= \lim_{n\to\infty} \mu(B_n) \\
&\leq \lim_{n\to\infty} (\mu^*(A_n) + \varepsilon) \\
&= \lim_{n\to\infty} \mu(A_n) + \varepsilon.
\end{aligned}$$

Done. ♠

Theorem 5.2 *Let X be a Polish space and μ be a (regular) Borel probability on X. Then every analytic subset of X is μ-measurable.*

Proof. Let A be an analytic subset of X. We will show that regardless of $\varepsilon > 0$ there is a compact $K \subseteq A$ such that $\mu(K) \geq \mu^*(A) - \varepsilon$. From this it is easy to conclude that $\mu_*(A) = \mu^*(A)$.

We need only look to non-empty A. Since A is a non-empty analytic set inside X, there is a continuous function $f : \mathcal{N} \to X$ such that $f(\mathcal{N}) = A$.

Some notation: for $k, n_1, \ldots, n_k \in \mathbb{N}$, let

$$\mathcal{L}(n_1, \ldots, n_k) := \{m \in \mathcal{N} \ : \ m_1 \leq n_1, \ m_2 \leq n_2, \ldots, m_k \leq n_k\}.$$

Let $\varepsilon > 0$. We plan to construct $n \in \mathcal{N}$ such that

$$\mu^*(f(\mathcal{L}(n_1, \ldots, n_k)) > \mu^*(A) - \varepsilon$$

for each k. Here's how we construct n:

$(\mathcal{L}(n_1))_{n_1}$ is an increasing sequence of sets whose union is $\mathcal{N}$ and so $(f(\mathcal{L}(n_1)))_{n_1}$ is an increasing sequence whose union is A. Hence,

$$\mu^*(A) = \lim_{n_1} \mu^*(f\mathcal{L}(n_1))),$$

thanks to our Lemma. We can, therefore, pick n_1 so

$$\mu^*(f(\mathcal{L}(n_1))) > \mu^*(A) - \varepsilon.$$

But $(\mathcal{L}(n_1, n_2))_{n_2}$ is an increasing sequence of sets and

$$\mathcal{L}(n_1) = \bigcup_{n_2} \mathcal{L}(n_1, n_2) \quad ; \quad f(\mathcal{L}(n_1)) = \bigcup_{n_2} f(\mathcal{L}(n_1, n_2));$$

so as before we can pick n_2 so

$$\mu^*(f\mathcal{L}(n_1, n_2))$$

is close enough to $\mu^*(f(\mathcal{L}(n_1))) = \lim_{n_2} \mu^*(f(\mathcal{L}(n_1, n_2)))$ that

$$\mu^*(f(\mathcal{L}(n_1, n_2))) > \mu^*(A) - \varepsilon.$$

The way is clear. Follow the way. n soon emerges.

Let $L = \bigcap_k \mathcal{L}(n_1, \ldots, n_k)$. L is a compact subset of $\mathcal{N}$. $f(L) = K$ is a compact subset of A. We **claim** that

$$\mu(K) \geq \mu^*(A) - \varepsilon.$$

To establish our claim we start by showing that K is bigger than one might expect. After all

$$\begin{aligned} K = f(L) &= f\left(\bigcap_k \mathcal{L}(n_1, \ldots, n_k)\right) \\ &\subseteq \bigcap_k f(\mathcal{L}(n_1, \ldots, n_k)) \\ &\subseteq \bigcap_k \overline{f(\mathcal{L}(n_1, \ldots, n_k))}; \end{aligned}$$

actually,

$$K = \bigcap_k \overline{f(\mathcal{L}(n_1, \ldots, n_k))}.$$

Indeed, suppose d is a complete metric that generates the Polish toplogy of X. Take $x \in \bigcap_k \overline{f(\mathcal{L}(n_1, \ldots, n_k))}$. Then $x \in \overline{f(\mathcal{L}(n_1, \ldots, n_k))}$ for each k; so we can choose m_k from $\mathcal{L}(n_1, \ldots, n_k)$ such that $d(f(m_k), x) \leq 1/k$. A simple diagonal argument, avaible through the good graces of the form of $\mathcal{L}(n_1, \ldots, n_k)$, allows us to pass to a subsequence (m'_k) of (m_k) which converges to some $m \in \mathcal{N}$. m must be in $\bigcap_k f(\mathcal{L}(n_1, \ldots, n_k))$ and $f(m)$ must be $\lim_k f(m'_k) = x$. So

$$x = f(m) \in f\left(\bigcap_k \mathcal{L}(n_1, \ldots, n_k)\right) = f(L) = K.$$

Now we'll in business:

$\overline{f(\mathcal{L}(n_1, \ldots, n_k))}$ is a closed set containing $f(\mathcal{L}(n_1, \ldots, n_k))$ so

$$\begin{aligned} \mu\left(\overline{f(\mathcal{L}(n_1, \ldots, n_k))}\right) &\geq \mu^*(f(\mathcal{L}(n_1, \ldots, n_k))) \\ &\geq \mu^*(A) - \varepsilon \quad . \end{aligned}$$

for each k. Further, $(\overline{f(\mathcal{L}(n_1, \ldots, n_k))})_k$ is a decreasing sequence with intersection K. Hence

$$\mu(K) \geq \mu^*(A) - \varepsilon,$$

too. We're done. ♠

Let X be a Polish space and $Bo(X)$ denote the σ-field of Borel subsets of X. Let $\mathcal{U}(X)$ be defined by

$$\mathcal{U}(X) = \bigcap_{\mu \in M_1(X)} Bo(X)_\mu$$

where $M_1(X)$ are the probability measures on $Bo(X)$. Members of $\mathcal{U}(X)$ are called **universally measurable**.

Corollary 5.3 *If X is a Polish space and A is an anaytic subset of X, then A is universally measurable subset of X.*

More generally, if X is a topological space we define the σ-field of universally measurable sets with respect to **probabilities** on $Bo(X)$ or with respect to the **regular probabilities** on $Bo(X)$ $(= M_{reg}(X))$ by

$$\mathcal{U}(X) = \bigcap_{\mu \in M_1(X)} Bo(X)_\mu$$

or

$$\mathcal{U}_{reg}(X) = \bigcap_{\mu \in M_{reg}(X)} Bo(X)_\mu.$$

Here's a useful fact that says that in a sense "bigger is better".

Proposition 5.4 *Let X and Y be Polish spaces and $f : X \to Y$ be a Borel function. Then f is $\mathcal{U}(X) - \mathcal{U}(Y)$ measurable.*

Proof. Let $U \in \mathcal{U}(Y)$ and μ be a (regular) Borel probability on X. Look to the image measure $\mu \circ f^{-1}$ on Y,

$$\mu \circ f^{-1}(E) = \mu(f^{-1}(E)) \qquad E \in Bo(X).$$

$\mu \circ f^{-1}$ is a probability on $Bo(X)$ and $U \in \mathcal{U}(Y)$ so $U \in Bo(Y)_{\mu \circ f^{-1}}$. Hence there exist $S, B \in Bo(Y)$, $S \subseteq U \subseteq B$, so that $\mu \circ f^{-1}(B \setminus S) = 0$. But $f^{-1}(S)$, $f^{-1}(B)$ are Borel subsets of X, $f^{-1}(S) \subseteq f^{-1}(U) \subseteq f^{-1}(B)$ and

$$\begin{aligned} \mu(f^{-1}(B) \setminus f^{-1}(S)) &= \mu(f^{-1}(B \setminus S) \\ &= \mu \circ f^{-1}(B \setminus S) \\ &= 0. \end{aligned}$$

Hence $f^{-1}(U) \in Bo(X)_\mu$.

This is so for each probability μ on X so $U \in \mathcal{U}(X)$. ♠

This Proposition has many variations and ensures a rich collection of $\mathcal{U}(X) - \mathcal{U}(Y)$ measurable functions.

A brief foray into Banach spaces indicates some quicks in the study of universally measurable sets. First, a delicious tid bit about separable Banach spaces.

Theorem 5.5 *If X is a separable banach space, then*

$$Bo(X, ||\cdot||) = Bo(X, weak).$$

Proof. Since the weak toplogy of a Banach space is smaller than the norm topology, $Bo(X, weak) \subseteq Bo(X, ||\cdot||)$.

Mazur's theorem ensures us that closed balls are weakly closed hence weakly Borel. Open balls are countable unions of closed balls hence weakly Borel. In separable spaces, open sets are countable unions of open balls so weakly Borel. But now it follow that norm Borel sets (in separable spaces) are weakly Borel, too. ♠

Generally, $Bo(X, weak) \subseteq Bo(X, ||\cdot||)$ though there are lots of spaces that are non-separable in which $Bo(X, weak) = Bo(X, ||\cdot||)$, including the reflexive spaces. Nevertheless,

Theorem 5.6 (G. Edgar) $\mathcal{U}_{reg}(X, ||\cdot||) = \mathcal{U}_{reg}(X, weak)$ *for every Banach space X.*