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Cardinal topologies and strict topologies

Carlos E. Uzcátegui A. and Jorge Vielma *

Abstract

The cardinal topologies Ψ_α^X are introduced in the space of bounded continuous functions on a completely regular Hausdorff space X. If X is a discrete space it is shown that $|X| \leq \aleph_\alpha$ if and only if the unit ball $B_1(X)$ in $C_b(X)$ is Ψ_α^X-compact, and also, if and only if Ψ_α^X coincides with the topology of pointwise convergence. Also we prove that if X is discrete then β_0 and the Ψ_α^X's can be compared always. We present a characterization of real and Ulam measurable cardinals in terms of the compactness of the unit ball with respect to some known strict topologies.

1 Introduction

Wheeler in [11] characterized a discrete space X as the one for which the unit ball $B_1(X)$ in $C_b(X)$ is β_0-compact, where β_0 is the strict topology introduced by Buck in [1]. Since on a discrete space the only significant property is its cardinality, then it seems natural to ask whether there are topologies on $C_b(X)$ which characterizes the cardinality of X via the compactness of the unit ball. We introduce a family of topologies Ψ_α^X on $C_b(X)$ (that we call *cardinal topologies*) and give a definite answer to that question. We will show that the cardinal topologies we define are always comparable with the strict topology β_0.

There are some characterization of Real and Ulam measurable cardinals in terms of properties of measure spaces (see [3], [5], [6] and [7]). We will show that similar results can be proved looking at the compactness of the unit ball in $C_b(X)$ with respect to the strict topologies β_p and β_σ.

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2 Preliminaries and notation

Let X be a completely regular Hausdorff space. $B_1(X)$ will denote the closed unit ball, i.e. the set $\{f \in C_b(X) : \|f\| \leq 1\}$, $| X |$ will denote the cardinality of X. For each $Y \subseteq X$ with $| Y | < | X |$, let T_Y be the linear map $T_Y : C_b(X) \to C_b(Y)$ defined by $T_T(f) = f \mid Y$, i.e. the restriction of f to Y.

If (E, τ) is a Hausdorff locally convex topological vector space and E' is its topological dual then $\sigma(E, E')$ and $\tau(E, E')$ denotes the weak and Mackey topologies of the duality $< E, E' >$, respectively (see [8]). As it is customary, any locally convex topology β on E such that $\sigma(E, E') \leq \beta \leq \tau(E, E')$ is said to be consistent with the duality and in this case the dual of (E, β) is E'.

If X is a completely regular Hausdorff space then the topology β_0 is the finest locally convex topology on $C_b(X)$ which coincides on the norm-bounded sets with the compact open topology. The dual of $(C_b(X), \beta_0)$ is the space $M_t(X)$ of tight measures on X (see [12]). If X is locally compact, β_0 coincide with the strict topology of Buck [1], that is to say, β_0 is determined by the seminorms $\| . \|_h$

$$\| f \|_h = Sup\{| f(x)h(x) | : x \in X\}$$

where h is a bounded real valued function defined on X, such that $\{x : | h(x) | \geq \varepsilon\}$ is relatively compact for every $\varepsilon > 0$, i.e. h is a bounded continuous function vanishing at infinity.

When $C_b(X)$ is given the supremum norm $\| . \|$, we know (by the Alexandroff representation theorem) that its dual is given by the space $M(X)$ of all finite, finitely additive Baire measures on X (see e.g. [12]). βX denotes the Stone-Cech compactification of X. For every set $K \subseteq \beta X - X$ the spaces $C_b(X)$ and $C_b(\beta X - K)$ are isomorphic. Then the topology β_0 on $C_b(\beta X - K)$ induces a topology β_K on $C_b(X)$ which makes this two spaces homeomorphic. If we consider on $C_b(X)$ the inductive topology induced by $(C_b(X), \beta_K)$ and the identity maps when K runs on a family of subsets of $\beta X - X$, the topology obtained is often called a strict topology.

The strict topologies we are going to use in this paper are the following:

(1) If $\mathcal{K} = \{K \subseteq \beta X - X : K \text{ is compact }\}$ the strict topology obtained is denoted by β_τ and the dual of $(C_b(X), \beta_\tau)$ is known to be the space $M_\tau(X)$ of all Baire τ-additive measure over X (see [5]).

(2) If $\mathcal{K} = \{Z \subseteq \beta X - X : Z \text{ is a zero set }\}$ we get the topology β_σ which gives as dual the space $M_\sigma(X)$ of all Baire σ-additive measure over X (see [5]).

(3) If $\mathcal{K} = \{D \subseteq \beta X - X : D \text{ is a distinguished set }\}$ the strict topology we obtain is β_p and the corresponding dual space is $M_p(X)$ of all Baire perfect measure on X (see [5]).

(4) Finally, if $\mathcal{K} = \{C \subseteq \beta X - X :$ There is partition of unity $(f_\alpha)_{\alpha \in A}$ for

X such that $f_\alpha \mid C = 0$ for all $\alpha \in A\}$ we obtain the topology β_μ which deals as dual the space $M_\mu(X)$ of all μ-additive Baire measure over X (see [5]).

We will be using β_z as a generic symbol for the various strict topologies used in this paper and $M_z(X)$ will denote its corresponding dual space. Let us recall that on a discrete space X we have that $M_t(X) = M_\tau(X) = M_\mu(X)$. If τ and τ^* are topologies on some space, $\tau \leq \tau^*$ will denote that τ^* is finer than τ. All topologies on $C_b(X)$ use in this paper will be finer than the pointwise topology, which will be denoted by t_p. In fact we have that $t_p \leq \beta_0 \leq \beta_z$. Our set theoretic notation is standard as in [4].

3 Cardinal topologies and main result

Now we introduce the cardinal topologies on the space $C_b(X)$.

Definition 3.1 *The topology Ψ_0^X on $C_b(X)$ is defined as the projective topology induced by the spaces $(C_b(Y), \|.\|)$ and the restriction maps T_Y where $Y \subseteq X$ and $\mid Y \mid < \mid X \mid$, i.e. the smallest topology for which the maps T_Y's are continuous.*

By transfinite induction we define the topologies Ψ_α^X on $C_b(X)$ for every ordinal α. If $\alpha = \beta + 1$, then Ψ_α^X is the projective topology induced by the spaces $(C_b(Y), \Psi_\beta^Y)$ and the restriction maps T_Y, where $Y \subseteq X$ and $\mid Y \mid < \mid X \mid$. Finally, if α is a limit ordinal, then Ψ_α^X is the projective topology induced by $(C_b(Y), \bigcap_{\beta<\alpha} \Psi_\beta^Y)$ and the restriction maps T_Y, where $Y \subseteq X$ and $\mid Y \mid < \mid X \mid$.

The main theorem is the following:

Theorem 3.2 *The following statements are equivalent:*
(i) $\mid X \mid \leq \aleph_\alpha$
(ii) $\Psi_\alpha^X = t_p$
(iii) $B_1(X)$ is Ψ_α^X-compact.

Before we give the proof we will show some lemmas. The following result is well known and it is the prototype of our result.

Lemma 3.3 *X is finite if and only if $B_1(X)$ is $\|.\|$-compact.* □

Now we will show some basic facts about the topologies Ψ_α^X.

Lemma 3.4 *(i) For every α, $\Psi_{\alpha+1}^X \leq \Psi_\alpha^X$. Moreover, if α and β are ordinals with $\alpha < \beta$ then $\Psi_\beta^X \leq \Psi_\alpha^X$.*

(ii) For every ordinal α, $t_p \leq \Psi_\alpha^X$, where t_p denotes the topology of pointwise convergence.

(iii) $(C_b(X), \Psi_\alpha^X)$ is a locally convex topological vector space.