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Abstract

In this paper we give a sufficient condition for the exact controllability of the following model of the suspension bridge equation proposed by Lazer and McKenna in [7]

$$\begin{cases} w_{tt} + cw_t + dw_{xxxx} + kw^+ = p(t,x) + u(t,x) + f(t,w,u(t,x)), & 0 < x < 1 \\ w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0, & t \in I\!R \end{cases}$$

where $t \geq 0$, $d > 0$, $c > 0$, $k > 0$, the distributed control $u \in L^2(0,t_1; L^2(0,1))$, $p : I\!R \times [0,1] \to I\!R$ is continuous and bounded, and the non-linear term
$f : [0,t_1] \times I\!R \times I\!R \to I\!R$ is a continuous function on t and globally Lipschitz in the other variables. i.e., there exists a constant $l > 0$ such that for all $x_1, x_2, u_1, u_2 \in I\!R$ we have

$$\|f(t,x_2,u_2) - f(t,x_1,u_1)\| \leq l\left\{\|x_2 - x_1\| + \|u_2 - u_1\|\right\}, \quad t \in [0,t_1].$$

To this end, we prove that the linear part of the system is exactly controllable on $[0,t_1]$. Then, we prove that the non-linear system is exactly controllable on $[0,t_1]$ for t_1 small enough. That is to say, the controllability of the linear system is preserved under the non-linear perturbation $-kw^+ + p(t,x) + f(t,w,u(t,x))$.

Key words. suspension bridge equation, strongly continuous groups, exact controllability.

AMS(MOS) subject classifications. primary: 34G10; secondary: 37B37.

Running Title: Exact Controllability of the suspension B.Eq.

1 Introduction

After The Tacoma Narrows Bridge collapsed on November 7, 1940 a lot of work have been done in the study of suspension bridge models. An important contribution is the work done by A.C. Lazer and P.J. McKenna in [7] and J. Glover, A.C. Lazer and P.J. McKenna in [6] who proposed the following mathematical model for suspension bridges

$$\begin{cases} w_{tt} + cw_t + dw_{xxxx} + kw^+ = p(t,x), & 0 < x < 1, \quad t \in I\!R, \\ w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0, & t \in I\!R \end{cases} \tag{1.1}$$

where $d > 0$, $c > 0$, $k > 0$ and $p : I\!R \times [0,1] \to I\!R$ is continuous and bounded function acting as an external force.

The existence of bounded solutions of this model (1.1) and other similar equations has been carried out recently in [2], [3], [1], [8], [9] and [5]. To our knowledge, the exact controllability of this model under non-linear action of the control has not been studied before. So, in this paper we give a sufficient condition for the exact controllability of the following controlled suspension bridge equation

$$\begin{cases} w_{tt} + cw_t + dw_{xxxx} + kw^+ = p(t,x) + u(t,x) + f(t,w,u(t,x)), 0 < x < 1 \\ w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0, \quad t \in I\!R \end{cases} \tag{1.2}$$

where the distributed control u belong to $L^2(0,t_1;L^2(0,1))$ and $f : [0,t_1] \times I\!R \times I\!R \to I\!R$ is a continuous function on t and globally Lipschitz in the other variables. i.e., there exists a constant $l > 0$ such that for all $x_1, x_2, u_1, u_2 \in I\!R$ we have

$$\|f(t,x_2,u_2) - f(t,x_1,u_1)\| \leq l\left\{\|x_2 - x_1\| + \|u_2 - u_1\|\right\}, \quad t \in [0,t_1]. \tag{1.3}$$

To this end, we prove that the linear part of this system

$$\begin{cases} w_{tt} + cw_t + dw_{xxxx} + kw^+ = u(t,x), 0 < x < 1 \\ w(t,0) = w(t,1) = w_{xx}(t,0) = w_{xx}(t,1) = 0, \quad t \in I\!R \end{cases} \tag{1.4}$$

is exactly controllable on $[0,t_1]$ for all $t_1 > 0$; moreover, we find the formula(4.31) to compute explicitly the control $u \in L^2(0,t_1;L^2(0,1))$ steering an initial state $z_0 = [w_0,v_0]^T$ to a final state $z_1 = [w_1,v_1]^T$ in time $t_1 > 0$ for the the linear system (1.4). Then, we use this formula to construct a sequence of controls u_n that converges to a control u that steers an initial state z_0 to a final state z_1 for the non-linear system (1.2), which proves the exact controllability of this system. That is to say, the controllability of the linear system (1.4) is preserved under the non-linear perturbation $-kw^+ + p(t,x) + f(t,w,u(t,x))$.

2 Abstract Formulation of the Problem

The system(1.2) can be written as an abstract second order equation on the Hilbert Space $X = L^2(0,1)$ as follows:

$$\ddot{w} + c\dot{w} + dAw + kw^{+} = P(t) + u(t) + f(t, w, u(t)), t \in I\!R, \tag{2.5}$$

where the unbounded operator A is given by $A\phi = \phi_{xxxx}$ with domain

$D(A) = \{\phi \in X : \phi, \phi_x, \phi_{xx}, \phi_{xxx}$ are absolutely continuous, $\phi_{xxxx} \in X; \quad \phi(0) = \phi(1) = \phi_{xx}(0) = \phi_{xx}(1) = 1\}$, and has the following spectral decomposition:

a) For all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n < x, \phi_n > \phi_n = \sum_{n=1}^{\infty} \lambda_n E_n x, \tag{2.6}$$

where $\lambda_n = n^4\pi^4$, $\phi_n(x) = \sin n\pi x$, $< \cdot, \cdot >$ is the inner product in X and

$$E_n x =< x, \phi_n > \phi_n. \tag{2.7}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in X and

$x = \sum_{n=1}^{\infty} E_n x, \quad x \in X.$

b) $-A$ generates an analytic semigroup $\{e^{-At}\}$ given by

$$e^{-At}x = \sum_{n=1}^{\infty} e^{-\lambda_n t} E_n x. \tag{2.8}$$

c) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

and

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{2.9}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space with norm given by:

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|w\|_r^2 + \|v\|^2.$$

Using the change of variables $w' = v$, the second order equation (2.5) can be written as a first order system of ordinary differential equations in the Hilbert space

$Z_{1/2} = D(A^{1/2}) \times X = X^{1/2} \times X$ as:

$$z' = \mathcal{A}z + Bu + F(t, z, u(t)) \quad z \in Z_{1/2}, \quad t \geq 0, \tag{2.10}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -dA & -cI_X \end{bmatrix}, \tag{2.11}$$

$\mathcal{A}$ is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times X$, $P(t)(x) = p(t,x)$, $x \in [0,1]$ and the function $F : [0, t_1] \times Z_{1/2} \times X \to Z_{1/2}$ is given by

$$F(t, z, u) = \begin{bmatrix} 0 \\ P(t) - kw^+ + f(t, w, u) \end{bmatrix}. \tag{2.12}$$

Since $X^{1/2}$ is continuously included in X, we obtain (for all $z_1, z_2 \in Z_{1/2}$ and $u_1, u_2 \in X$) that

$$\|F(t, z_2, u_2) - F(t, z_1, u_1)\|_{Z_{1/2}} \leq L \left\{ \|z_2 - z_1\|_{1/2} + \|u_2 - u_1\| \right\}, \quad t \in [0, t_1], \tag{2.13}$$

where $L = k + l$. Throughout this paper, without lost of generality we will assume that,

$$c^2 < 4d\lambda_1.$$

3 The Uncontrolled Linear Equation

In this section we shall study the well-posedness of the following abstract linear Cauchy initial value problem

$$z' = \mathcal{A}z, \quad (t \in \mathbb{R}) \quad z(0) = z_0 \in D(\mathcal{A}), \tag{3.14}$$

which is equivalent to prove that the operator $\mathcal{A}$ generates a strongly continuous group. To this end, we shall use the following Lemma from [10].

Lemma 3.1 *Let Z be a separable Hilbert space and $\{A_n\}_{n\geq 1}$, $\{P_n\}_{n\geq 1}$ two families of bounded linear operators in Z with $\{P_n\}_{n\geq 1}$ being a complete family of orthogonal projections such that*

$$A_n P_n = P_n A_n, \quad n = 1, 2, 3, \ldots \tag{3.15}$$

Define the following family of linear operators

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \qquad t \geq 0. \tag{3.16}$$

Then:

(a) $T(t)$ is a linear bounded operator if

$$\|e^{A_n t}\| \leq g(t), \quad n = 1, 2, 3, \ldots \tag{3.17}$$

for some continuous real-valued function $g(t)$.

(b) under the condition (3.17) $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup in the Hilbert space Z whose infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_n P_n z, \qquad z \in D(\mathcal{A}) \tag{3.18}$$

with

$$D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty\} \tag{3.19}$$

(c) the spectrum $\sigma(\mathcal{A})$ of $\mathcal{A}$ is given by

$$\sigma(\mathcal{A}) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{A}_n)}, \tag{3.20}$$

where $\bar{A}_n = A_n P_n$.

Theorem 3.1 *The operator $\mathcal{A}$ given by (2.11), is the infinitesimal generator of a strongly continuous group $\{T(t)\}_{t\in I\!R}$ given by*

$$T(t)z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z_{1/2}, \quad t \geq 0 \tag{3.21}$$

where $\{P_n\}_{n\geq 0}$ is a complete family of orthogonal projections in the Hilbert space $Z_{1/2}$ given by

$$P_n = diag\,[E_n, E_n]\ \ , n \geq 1\ , \tag{3.22}$$

and

$$A_n = B_n P_n, \quad B_n = \begin{bmatrix} 0 & 1 \\ -d\lambda_n & -c \end{bmatrix}, \ n \geq 1. \tag{3.23}$$

This group $\{T(t)\}_{t\in I\!R}$ decays exponentially to zero. In fact, we have the following estimate

$$\|T(t)\| \leq M(c,d)e^{-\frac{c}{2}t}, \quad t \geq 0, \tag{3.24}$$

where

$$\frac{M(c,d)}{2\sqrt{2}} = \sup_{n\geq 1}\left\{ 2\left|\frac{c\pm\sqrt{4d\lambda_n - c^2}}{\sqrt{c^2 - 4d\lambda_n}}\right|, \left|(2+d)\sqrt{\frac{\lambda_n}{4d\lambda_n - c^2}}\right| \right\}.$$

Proof Computing $\mathcal{A}z$ yields,

$$\begin{aligned}
\mathcal{A}z &= \begin{bmatrix} 0 & I \\ -dA & -c \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} \\
&= \begin{bmatrix} v \\ -dAw - cv \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=1}^{\infty} E_n v \\ -d\sum_{n=1}^{\infty} \lambda_n E_n w - c\sum_{n=1}^{\infty} E_n v \end{bmatrix} \\
&= \sum_{n=1}^{\infty} \begin{bmatrix} E_n v \\ -d\lambda_n E_n w - cE_n v \end{bmatrix} \\
&= \sum_{n=1}^{\infty} \begin{bmatrix} 0 & 1 \\ -d\lambda_n & -c \end{bmatrix}\begin{bmatrix} E_n & 0 \\ 0 & E_n \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} \\
&= \sum_{n=1}^{\infty} A_n P_n z.
\end{aligned}$$

It is clear that $A_n P_n = P_n A_n$. Now, we need to check condition (3.17) from Lemma 3.1. To this end, we compute the spectrum of the matrix B_n. The characteristic equation of B_n is given by

$$\lambda^2 + c\lambda + d\lambda_n = 0,$$

and the eigenvalues $\sigma_1(n)$, $\sigma_2(n)$ of the matrix B_n are given by

$$\sigma_1(n) = -\mu + il_n, \quad \sigma_2(n) = -\mu - il_n,$$

where,

$$\mu = \frac{c}{2} \ \text{ and } \ l_n = \frac{1}{2}\sqrt{4d\lambda_n - c^2}.$$

Therefore,

$$\begin{aligned} e^{B_n t} &= e^{-\mu t}\left\{\cos l_n t I + \frac{1}{l_n}(B_n + \mu I)\right\} \\ &= e^{-\mu t}\begin{bmatrix} \cos l_n t + \frac{c}{2l_n}\sin l_n t & \frac{\sin l_n t}{l_n} \\ -dS(n)\lambda_n^{1/2}\sin l_n t & \cos l_n t - \frac{c}{2l_n}\sin l_n t \end{bmatrix}, \end{aligned}$$

From the above formulas we obtain that

$$e^{B_n t} = e^{-\mu t}\begin{bmatrix} a(n) & \frac{b(n)}{l_n} \\ -dS(n)\lambda_n^{1/2}c(n) & d(n) \end{bmatrix}$$

where

$$a(n) = \cos l_n t + \frac{c}{2l_n}\sin l_n t, \quad b(n) = \sin l_n t,$$

$$c(n) = \sin l_n t, \quad d(n) = \cos l_n t - \frac{c}{2l_n}\sin l_n t,$$

and

$$S(n) = \sqrt{\frac{\lambda_n}{4d\lambda_n - c^2}}.$$

Now, consider $z = (z_1, z_2)^T \in Z_{1/2}$ such that $\|z\|_{Z_{1/2}} = 1$. Then,

$$\|z_1\|_{1/2}^2 = \sum_{j=1}^{\infty}\lambda_j\|E_j z_1\|^2 \leq 1 \ \text{ and } \ \|z_2\|_X^2 = \sum_{j=1}^{\infty}\|E_j z_2\|^2 \leq 1.$$

Therefore, $\lambda_j^{1/2}\|E_j z_1\| \leq 1$, $\|E_j z_2\| \leq 1, \quad j = 1, 2, \ldots.$

and so,

$$
\begin{aligned}
\|e^{A_n t}z\|^2_{Z_{1/2}} &= e^{-2\mu t}\left\|\left[\begin{array}{c} a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2 \\ -dS(n)c(n)\lambda_n^{\frac{1}{2}}E_nz_1 + d(n)E_nz_2 \end{array}\right]\right\|^2_{Z_{1/2}} \\
&= e^{-2\mu t}\|a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2\|^2_{\frac{1}{2}} + e^{-2\mu t}\| \\
&- dS(n)c(n)\lambda_n^{\frac{1}{2}}E_nz_1 + d(n)E_nz_2\|^2_X \\
&= e^{-2\mu t}\sum_{j=1}^{\infty}\lambda_j\|E_j\left(a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2\right)\|^2 \\
&+ e^{-2\mu t}\sum_{j=1}^{\infty}\|E_j\left(-dS(n)c(n)\lambda_n^{\frac{1}{2}}E_nz_1 + d(n)E_nz_2\right)\|^2 \\
&= e^{-2\mu t}\lambda_n\|a(n)E_nz_1 + \frac{b(n)}{l_n}E_nz_2\|^2 + e^{-2\mu t}\| \\
&- dS(n)c(n)\lambda_n^{\frac{1}{2}}E_nz_1 + d(n)E_nz_2\|^2 \\
&\le e^{-2\mu t}(|a(n)| + |\frac{\lambda^{\frac{1}{2}}}{l_n}b(n)|)^2 + e^{-2\mu t}(|dS(n)c(n)| + |d(n)|)^2,
\end{aligned}
$$

where

$$
|\frac{\lambda^{\frac{1}{2}}}{l_n}b(n)| = \left|\sqrt{\frac{\lambda_n}{c^2 - 4d\lambda_n}}\right|.
$$

If we set,

$$
\frac{M(c,d)}{2\sqrt{2}} = \sup_{n\ge1}\left\{2\left|\frac{c\pm\sqrt{4d\lambda_n - c^2}}{\sqrt{c^2 - 4d\lambda_n}}\right|, \left|(2+d)\sqrt{\frac{\lambda_n}{4d\lambda_n - c^2}}\right|\right\},
$$

we have,

$$
\|e^{A_n t}\| \le M(c,d)e^{-\mu t}, \quad t \ge 0 \quad n = 1, 2, \ldots.
$$

Hence, applying Lemma 3.1 we obtain that $\mathcal{A}$ generates a strongly continuous group given by (3.21). Next, we will prove this group decays exponentially to zero. In fact,

$$
\begin{aligned}
\|T(t)z\|^2 &\le \sum_{n=1}^{\infty}\|e^{A_n t}P_nz\|^2 \\
&\le \sum_{n=1}^{\infty}\|e^{A_n t}\|^2\|P_nz\|^2 \\
&\le M^2(c,d)e^{-2\mu t}\sum_{n=1}^{\infty}\|P_nz\|^2 \\
&= M^2(c,d)e^{-2\mu t}\|z\|^2.
\end{aligned}
$$

Therefore,

$$\|T(t)\| \leq M(c,d)e^{-\mu t}, \quad t \geq 0.$$

□

4 Exact Controllability of the Linear System

Now, we shall give the definition of controllability in terms of the linear system

$$z' = \mathcal{A}z + Bu \quad z \in Z_{1/2}, \quad t \geq 0, \tag{4.25}$$

where

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -dA & -cI_X \end{bmatrix}. \tag{4.26}$$

For all $z_0 \in Z_{1/2}$ equation (4.25) has a unique mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad 0 \leq t \leq t_1. \tag{4.27}$$

Definition 4.1 *(***Exact Controllability***) We say that system (4.25) is exactly controllable on* $[0, t_1], \quad t_1 > 0$*, if for all* $z_0, z_1 \in Z_r$ *there exists a control* $u \in L^2(0, t_1; X)$ *such that the solution* $z(t)$ *of (4.27) corresponding to* u*, verifies:* $z(t_1) = z_1$*.*

Consider the following bounded linear operator

$$G : L^2(0, t_1; U) \to Z_{1/2}, \quad Gu = \int_0^{t_1} T(-s)B(s)u(s)ds. \tag{4.28}$$

Then, the following proposition is a characterization of the exact controllability of the system (4.25).

Proposition 4.1 *The system (4.25) is exactly controllable on* $[0, t_1]$ *if and only if, the operator* G *is surjective, that is to say*

$$G(L^2(0, t_1; X)) = Range(G) = Z_{1/2}.$$

Now, consider the following family of finite dimensional systems

$$y' = A_j P_j y + P_j B u, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \dots, \infty, \tag{4.29}$$

where $\mathcal{R}(P_j) = \text{Range}(P_j)$.

Then, the following proposition can be shown the same way as Lemma 1 from [11].

Proposition 4.2 *The following statements are equivalent:*

(a) System (4.29) is controllable on $[0, t_1]$.

(b) $B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow y = 0,$

(c) Rank $\left[P_j B \vdots A_j P_j B \right] = 2$

(d) The operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ *given by:*

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} B B^* e^{-A_j^* s} ds, \tag{4.30}$$

is invertible.

Now, we are ready to formulate the main result on exact controllability of the linear system (4.25).

Theorem 4.1 *The system (4.25) is exactly controllable on* $[0, t_1]$. *Moreover, the control* $u \in L^2(0, t_1; X)$ *that steers an initial state* z_0 *to a final state* z_1 *at time* $t_1 > 0$ *is given by the following formula:*

$$u(t) = B^* T^*(-t) \sum_{j=1}^{\infty} W_j^{-1}(t_1) P_j (T(-t_1) z_1 - z_0). \tag{4.31}$$

Proof . First, we shall prove that each of the following finite dimensional systems is controllable on $[0, t_1]$

$$y' = A_j P_j y + P_j B u, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \dots, \infty. \tag{4.32}$$

In fact, we can check the condition for controllability of the systems

$$B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow y = 0.$$

In this case the operators $A_j = B_j P_j$ and $\mathcal{A}$ are given by

$$B_j = \begin{bmatrix} 0 & 1 \\ -d\lambda_j & -c \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -dA & -cI \end{bmatrix},$$

and the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$ of the matrix B_j are given by

$$\sigma_1(j) = -\mu + il_j, \quad \sigma_2(j) = -\mu - il_j,$$

where,

$$\mu = \frac{c}{2} \text{ and } l_j = \frac{1}{2}\sqrt{4d\lambda_j - c^2}.$$

Therefore, $A_j^* = B_j^* P_j$ with

$$B_j^* = \begin{bmatrix} 0 & -1 \\ d\lambda_j & -c \end{bmatrix},$$

and

$$\begin{aligned} e^{B_j t} &= e^{-\mu t}\left\{\cos l_j t I + \frac{1}{l_j}\left(B_j + cI\right)\right\} \\ &= e^{-\mu t}\begin{bmatrix} \cos l_j t + \frac{c}{2l_j}\sin l_j t & \frac{\sin l_j t}{l_j} \\ -dS(j)\lambda_j^{1/2}\sin l_j t & \cos l_j t - \frac{c}{2l_j}\sin l_j t \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} e^{B_j^* t} &= e^{-\mu t}\left\{\cos l_j t I + \frac{1}{l_j}\left(B_j^* + \mu I\right)\right\} \\ &= e^{-\mu t}\begin{bmatrix} \cos l_j t + \frac{c}{2l_j}\sin l_j t & -\frac{\sin l_j t}{l_j} \\ dS(j)\lambda_j^{1/2}\sin l_j t & \cos l_j t - \frac{c}{2l_j}\sin l_j t \end{bmatrix}, \end{aligned}$$

$$B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad B^* = [0, I_X] \text{ and } BB^* = \begin{bmatrix} 0 & 0 \\ 0 & I_X \end{bmatrix}.$$

Now, let $y = (y_1, y_2)^T \in \mathcal{R}(P_j)$ such that

$$B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1].$$

Then,

$$e^{-\mu t}\left[dS(j)\lambda_j^{1/2}\sin l_j t y_1 + \left(\cos l_j t - \frac{c}{2l_j}\sin l_j t\right) y_2\right] = 0, \quad \forall t \in [0, t_1],$$

which implies that $y = 0$.

From Proposition 4.2 the operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ given by:

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} BB^* e^{-A_j^* s} ds = P_j \int_0^{t_1} e^{-B_j s} BB^* e^{-B_j^* s} ds P_j = P_j \overline{W}_j(t_1) P_j$$

is invertible.

Since

$$\|e^{-A_j t}\| \leq M(c,d) e^{\mu t}, \quad \|e^{-A_j^* t}\| \leq M(c,d) e^{\mu t},$$

$$\|e^{-A_j t} BB^* e^{-A_j^* t}\| \leq M^2(c,d) \|BB^*\| e^{2\mu t},$$

we have

$$\|W_j(t_1)\| \leq M^2(c,d) \|BB^*\| e^{2\mu t_1} \leq L(c,d), \quad j = 1, 2, \ldots.$$

Now, we shall prove that the family of linear operators,

$$W_j^{-1}(t_1) = \overline{W}_j^{-1}(t_1) P_j : Z_{1/2} \to Z_{1/2}$$

is bounded and $\|W_j^{-1}(t_1)\|$ is uniformly bounded. To this end, we shall compute explicitly the matrix $\overline{W}_j^{-1}(t_1)$. From the above formulas we obtain that

$$e^{B_j t} = e^{-\mu t} \begin{bmatrix} a(j) & b(j) \\ -a(j) & c(j) \end{bmatrix}, \quad e^{B_j^* t} = e^{-\mu t} \begin{bmatrix} a(j) & -b(j) \\ d(j) & c(j) \end{bmatrix},$$

where

$$a(j) = \cos l_j t + \frac{c}{2l_j} \sin l_j t, \quad b(j) = \frac{\sin l_j t}{l_j},$$

$$c(j) = dS(j) \lambda_j^{1/2} \sin l_j t, \quad d(j) = \cos l_j t - \frac{c}{2l_j} \sin l_j t,$$

and

$$S(j) = \sqrt{\frac{\lambda_j}{4d\lambda_j - c^2}}.$$

Then

$$e^{-B_j s} BB^* e^{-B_j^* s} = \begin{bmatrix} b(j)c(j)\lambda_j^{1/2} I & -b(j)d(j) I \\ -d(j)c(j)\lambda_j^{1/2} I & d^2(j) I \end{bmatrix}.$$

Therefore,

$$\overline{W}_j(t_1) = \left[\begin{array}{cc} \frac{dS(j)\lambda_j^{1/2}}{l_j} k_{11}(j) & \frac{1}{l_j} k_{12}(j) \\ -dS(j)\lambda_j^{1/2} k_{21}(j) & k_{22}(j) \end{array} \right],$$

where

$$\begin{aligned}
k_{11}(j) &= \int_0^{t_1} e^{2cs} \sin^2 l_j s ds \\
k_{12}(j) &= -\int_0^{t_1} e^{2cs} \left[\sin l_j s \cos l_j s - \frac{c \sin^2 l_j s}{2l_j} \right] ds \\
k_{21}(j) &= \int_0^{t_1} e^{2cs} \left[\sin l_j s \cos l_j s - \frac{c \sin^2 l_j s}{2l_j} \right] ds \\
k_{22}(j) &= \int_0^{t_1} e^{2cs} \left[\cos l_j s - \frac{c \sin l_j s}{2l_j} \right]^2 ds.
\end{aligned}$$

The determinant $\Delta(j)$ of the matrix $\overline{W}_j(t_1)$ is given by

$$\begin{aligned}
\Delta(j) &= \frac{dS(j)\lambda_j^{1/2}}{l_j} \left[k_{11}(j) k_{22}(j) - k_{12}(j) k_{21}(j) \right] \\
&= \frac{dS(j)\lambda_j^{1/2}}{l_j} \{ \left(\int_0^{t_1} e^{2\mu s} \sin^2 l_j s ds \right) \left(\int_0^{t_1} e^{2\mu s} \left[\cos l_j s - \frac{c \sin l_j s}{2l_j} \right]^2 ds \right) \\
&- \left(\int_0^{t_1} e^{2\mu s} \left[\sin l_j s \cos l_j s - \frac{c \sin^2 l_j s}{2l_j} \right] ds \right)^2 \}.
\end{aligned}$$

Passing to the limit as j goes to ∞, we obtain,

$$\lim_{j \to \infty} \Delta(j) = \frac{(e^{2\mu t_1} - 1)(1 - 2e^{\mu t_1} + e^{2\mu t_1})}{2^4 \mu^3}.$$

Therefore, there exist constants $R_1, R_2 > 0$ such that

$$0 < R_1 < |\Delta(j)| < R_2, \quad j = 1, 2, 3, \ldots$$

Hence,

$$\begin{aligned}
\overline{W}^{-1}(j) &= \frac{1}{\Delta(j)} \left[\begin{array}{cc} k_{22}(j) & -\frac{1}{l_j} k_{12}(j) \\ dS(j)\lambda_j^{1/2} k_{21}(j) & \frac{dS(j)\lambda_j^{1/2}}{l_j} k_{11}(j) \end{array} \right] \\
&= \left[\begin{array}{cc} b_{11}(j) & b_{12}(j) \\ b_{21}(j)\lambda_j^{1/2} & b_{22}(j) \end{array} \right],
\end{aligned}$$

where $b_{n,m}(j), \quad n = 1,2; m = 1,2; j = 1,2,\dots$ are bounded. Using the same computation as in Theorem 3.1 we can prove the existence of constant $L_2(c,d)$ such that

$$\|W_j^{-1}(t_1)\|_{Z_{1/2}} \leq L_2(c,d), \quad j = 1,2,\dots.$$

Now, we define the following linear bounded operators

$$W(t_1) : Z_{1/2} \to Z_{1/2}, \quad W^{-1}(t_1) : Z_{1/2} \to Z_{1/2},$$

by

$$W(t_1)z = \sum_{j=1}^{\infty} W_j(t_1)P_j z, \quad W^{-1}(t_1)z = \sum_{j=1}^{\infty} W_j^{-1}(t_1)P_j z.$$

Using the definition we see that, $W(t_1)W^{-1}(t_1)z = z$ and

$$W(t_1)z = \int_0^{t_1} T(-s)BB^*T^*(-s)z ds.$$

Next, we will show that given $z \in Z_{1/2}$ there exists a control $u \in L^2(0,t_1;X)$ such that $Gu = z$. In fact, let u be the following control

$$u(t) = B^*T^*(-t)W^{-1}(t_1)z, \quad t \in [0,t_1].$$

Then,

$$\begin{aligned} Gu &= \int_0^{t_1} T(-s)Bu(s)ds \\ &= \int_0^{t_1} T(-s)BB^*T^*(-s)W^{-1}(t_1)z ds \\ &= \left(\int_0^{t_1} T(-s)BB^*T^*(-s)ds\right) W^{-1}(t_1)z \\ &= W(t_1)W^{-1}(t_1)z = z. \end{aligned}$$

Then, the control steering an initial state z_0 to a final state z_1 in time $t_1 > 0$ is given by

$$\begin{aligned} u(t) &= B^*T^*(-t)W^{-1}(t_1)(T(-t_1)z_1 - z_0) \\ &= B^*T^*(-t)\sum_{j=1}^{\infty} W_j^{-1}(t_1)P_j(T(-t_1)z_1 - z_0). \end{aligned}$$

□

5 Exact Controllability of the Non-Linear System

Now, we shall give the definition of controllability in terms of the non-linear systems

$$\begin{cases} z' &= \mathcal{A}z + Bu + F(t, z, u(t)) \quad z \in Z_{1/2}, \quad t > 0, \\ z(0) &= z_0. \end{cases} \tag{5.33}$$

For all $z_0 \in Z_{1/2}$ equation (5.33) has a unique mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t)T(-s)[Bu(s) + F(s, z(s), u(s))]ds. \tag{5.34}$$

Definition 5.1 *(**Exact Controllability**) We say that system (5.33) is exactly controllable on* $[0, t_1], \quad t_1 > 0$*, if for all* $z_0, z_1 \in Z_{1/2}$ *there exists a control* $u \in L^2(0, t_1; X)$ *such that the solution* $z(t)$ *of (5.34) corresponding to* u*, verifies:* $z(t_1) = z_1$*.*

Consider the following non-linear operator

$$G_F : L^2(0, t_1; U) \to Z_{1/2}, \tag{5.35}$$

given by

$$G_F u = \int_0^{t_1} T(-s)B(s)u(s)ds + \int_0^{t_1} T(-s)F(s, z(s), u(s))ds, \tag{5.36}$$

where $z(t) = z(t; z_0, u)$ is the corresponding solution of (5.34).

Then, the following proposition is a characterization of the exact controllability of the non-linear system (5.33).

Proposition 5.1 *The system (5.33) is exactly controllable on* $[0, t_1]$ *if and only if, the operator* G_F *is surjective, that is to say*

$$G_F(L^2(0, t_1; X)) = Range(G_F) = Z_{1/2}.$$

Lemma 5.1 *Let* $u_1, u_2 \in L^2(0, t_1; X)$*,* $z_0 \in Z_{1/2}$ *and* $z_1(t; z_0, u_1), z_2(t; z_0, u_2)$ *the corresponding solutions of (5.34). Then the following estimate holds:*

$$\|z_1(t) - z_2(t)\|_{Z_{1/2}} \le M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_1 - u_2\|_{L^2(0,t_1;X)}, \tag{5.37}$$

where $0 \le t \le t_1$ *and*

$$M = \sup_{0 \le s \le t \le t_1} \{\|T(t)\|\|T(-s)\|\}. \tag{5.38}$$

Proof Let z_1, z_2 be solutions of (5.34) corresponding to u_1, u_2 respectively. Then

$$\begin{aligned}
\|z_1(t) - z_2(t)\| &\le \int_0^t \|T(t)\|\|T(-s)\|\|B\|\|u_1(s) - u_2(s)\| \\
&+ \int_0^t \|T(t)\|\|T(-s)\|\|F(s, z_1(s), u_1(s)) - F(s, z_2(s), u_2(s))\|ds \\
&\le M[\|B\| + L] \int_0^t \|u_1(s) - u_2(s)\| + ML \int_0^t \|z_1(s) - z_2(s)\|ds \\
&\le M[\|B\| + L]\sqrt{t_1}\|u_1 - u_2\| + ML \int_0^{t_1} \|z_1(s) - z_2(s)\|ds.
\end{aligned}$$

Using Gronwall's inequality we obtain

$$\|z_1(t) - z_2(t)\|_{Z_{1/2}} \le M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_1 - u_2\|_{L^2(0,t_1;X)}, \quad 0 \le t \le t_1.$$

□

Now, we are ready to formulate and prove the main Theorem of this section

Theorem 5.1 *If the following estimate holds*

$$\|B\|ML\|W^{-1}(t_1)\|H(t_1)t_1 < 1, \tag{5.39}$$

where $H(t_1) = M[\|B\| + L]e^{MLt_1}t_1 + 1$*, then the non-linear system (5.33) is exactly controllable on* $[0, t_1]$*.*

Proof Given the initial state z_0 and the final state z_1, and $u_1 \in L^2(0, t_1; X)$, there exists $u_2 \in L^2(0, t_1; X)$ such that

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z_1(s), u_1(s))ds - \int_0^{t_1} T(-s)Bu_2(s)ds,$$

where $z_1(t) = z(t; z_0, u_1)$ is the corresponding solution of (5.34).

Moreover, u_2 can be chosen as follows:

$$u_2(t) = B^*T^*(-t)W^{-1}(t_1)\left(z_1 - \int_0^{t_1} T(-s)F(s, z_1(s), u_1(s))ds\right).$$

For such u_2 there exists $u_3 \in L^2(0, t_1; X)$ such that

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z_2(s), u_2(s))ds - \int_0^{t_1} T(-s)Bu_3(s)ds,$$

where $z_2(t) = z(t; z_0, u_2)$ is the corresponding solution of (5.34), and u_3 can be taken as follows:

$$u_3(t) = B^*T^*(-t)W^{-1}(t_1)\left(z_1 - \int_0^{t_1} T(-s)F(s, z_2(s), u_2(s))ds\right).$$

Following this process we obtain two sequences

$$\{u_n\} \subset L^2(0, t_1; X), \quad \{z_n\} \subset L^2(0, t_1; Z_{1/2}), (z_n(t) = z(t; z_0, u_n)) \quad n = 1, 2, \ldots,$$

such that

$$u_{n+1}(t) = B^*T^*(-t)W^{-1}(t_1)\left(z_1 - \int_0^{t_1} T(-s)F(s, z_n(s), u_n(s))ds\right) \tag{5.40}$$

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z_n(s), u_n(s))ds - \int_0^{t_1} T(-s)Bu_{n+1}(s)ds. \tag{5.41}$$

Now, we shall prove that $\{z_n\}$ is a Cauchy sequence in $L^2(0, t_1; Z_{1/2})$. In fact, from formula (5.40) we obtain that

$$\begin{aligned} & u_{n+1}(t) - u_n(t) = \\ & B^*T^*(-t)W^{-1}(t_1)\left(\int_0^{t_1} T(-s)\left(F(s, z_{n-1}(s), u_{n-1}(s)) - F(s, z_n(s), u_n(s))\right) ds\right). \end{aligned}$$

Hence, using lemma 5.1 we obtain

$$\begin{aligned} & \|u_{n+1}(t) - u_n(t)\| \\ \leq\ & \|B\|ML\|W^{-1}(t_1)\| \int_0^{t_1} \left(\|z_n(s) - z_{n-1}(s)\| + \|u_n(s) - u_{n-1}(s)\|\right) ds \\ \leq\ & \|B\|ML\|W^{-1}(t_1)\| \int_0^{t_1} M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_n(s) - u_{n-1}(s)\|ds \\ +\ & \|B\|ML\|W^{-1}(t_1)\int_0^{t_1} \|u_n(s) - u_{n-1}(s)\|ds. \end{aligned}$$

Using Hóder's inequality we obtain

$$\|u_{n+1} - u_n\|_{L^2(0,t_1;X)} \leq \|B\|ML\|W^{-1}(t_1)\|H(t_1)t_1\|u_{n+1} - u_n\|_{L^2(0,t_1;X)}. \quad (5.42)$$

Since $\|B\|ML\|W^{-1}(t_1)\|H(t_1)t_1 < 1$, then $\{u_n\}$ is a Cauchy sequence in $L^2(0, t_1; X)$ and therefore there exists $u \in L^2(0, t_1; X)$ such that $\lim_{n\to\infty} u_n = u$ in $L^2(0, t_1; X)$.

Let $z(t) = z(t; z_0, u)$ be the corresponding solution of (5.34). Then we shall prove that

$$\lim_{n\to\infty} \int_0^{t_1} T(-s)F(s, z_n(s), u_n(s))ds = \int_0^{t_1} T(-s)F(s, z(s), u(s))ds.$$

In fact, using lemma 5.1 we obtain that

$$\begin{aligned}
&\left\| \int_0^{t_1} T(-s)[F(s, z_n(s), u_n(s)) - F(s, z(s), u(s))]ds \right\| \\
\leq\ & \int_0^{t_1} ML[\|z_n(s) - z(s)\| + \|u_n(s) - u(s)\|]ds \\
\leq\ & \int_0^{t_1} ML[M[\|B\| + L]e^{MLt_1}\sqrt{t_1}\|u_n - u\|_{L^2(0,t_1;X)} + \|u_n(s) - u(s)\|]ds \\
\leq\ & MLK(t_1)\sqrt{t_1}\|u_n - u\|_{L^2(0,t_1;X)}.
\end{aligned}$$

From here we obtain the result.

Finally, passing to the limit in (5.41) as n goes to ∞ we obtain that

$$0 = z_1 - \int_0^{t_1} T(-s)F(s, z(s), u(s))ds - \int_0^{t_1} T(-s)Bu(s)ds.$$

i.e.,

$$G_F u = z_1.$$

□

Remark 5.1 *a) The controllability of the system (1.2) is independent of the external force $P(t)$ since condition (5.39) does not depend on $P(t)$.*

b) If $f = 0$, the condition for the exact controllability of the system (1.2) can be expressed in terms of k. i.e.,

$$\|B\|Mk\|W^{-1}(t_1)\|H(t_1)t_1 < 1.$$

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