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Existence and Partial Characterization of the Global Attractor of the Equation $\dot{x}(t) = -kx(t) + \beta tanh(x(t-r))$.

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1 Introduction

In this paper we are going to consider the following delay differential equation

$$\dot{x}(t) = -kx(t) + \beta tanh(x(t-r)), \tag{1.1}$$

where $k > 0$ and $\beta \neq 0$; and we will prove that the global atractor of the equation (1.1) is an equilibrium point for any β such that $|\beta| < k$.

In order to accomplish our goal, we use the technique developed in [4], which basically states a connection between the continuous semi-dynamical system induced by (1.1) and certain discrete dynamical systems.

We point out that the equation (1.1) arises in many applications. For instance in a simplified neural network in which each neuron is represented by a linear circuit with a linear resistor and a linear capacitor. Introducing a nonlinear feedback term, we arrive to the equation (1.1). See Leslie Shayer and Sue Ann Campbell [1] and the literature cited therein for more details.

2 Preliminaries

In this section we will show that the equation (1.1) admits a global atractor for any k and β, and we will summarize some known results about the location of the roots of certain trascendental equation. It is worth noting that the only equilibrium point of the equation (1.1) for any $\beta < k$ is the trivial one.

Let us first introduce a few notations that we will need in the sequel. For a given $\sigma \in \mathbb{R}$ and any function $x : [\sigma - r, \infty) \longrightarrow \mathbb{R}^n$, let us define the function $x_t : [-r, 0] \longrightarrow \mathbb{R}^n$ by $x_t(\theta) = x(t+\theta)$, for any $\theta \in [-r, 0]$ and $t \geq \sigma$. By using the step by step method, we obtain that for any $\phi \in C = C([-r, 0], \mathbb{R})$ the equation (1.1) has a unique solution $x(t, \phi)$ defined for any $t \geq 0$, which depends continuously on the initial data and parameters. Let us set $T(t)\phi = x_t(\phi)$ for $t \geq 0$. It is well known that $\{T(t)\}_{t\geq 0}$ is a semigroup of strongly continuous operators on C.

Let $x(t, \phi)$ be the solution of (1.1). Integrating this equation we obtain that

$$x(t, \phi) = x(0, \phi)e^{-kt} + \int_0^t \beta e^{-k(t-s)} tanh(x(s - r, \phi))ds,$$

which implies that

$$|x(t, \phi)| \leq |\phi(0)|e^{-kt} + \tfrac{|\beta|}{k}\left[1 - e^{-kt}\right] \ , \ t \geq 0.$$

The last inequality immediately proves the dissipativeness of the equation (1.1). Furthermore, a straightforward application of Arzela-Ascoli's Lemma gives us that the operator $T(t)$ is completely continuous for any $t \geq r$.

Thus, the existence of the global attractor of the equation (1.1) is an immediate consequence of theorem 3.4.8 in page 40, in [5].

Now, we are going to study the local stability of trivial solution of (1.1).

Linearizing (1.1) about $x = 0$, we obtain the equation

$$\dot{x}(t) = -kx(t) + \beta x(t - r), \tag{2.1}$$

which characteristic equation is given by

$$P(\lambda) = \lambda + k - \beta e^{-\lambda r} = 0 \tag{2.2}$$

By using the theorem A.5 pag. 416 in [6], we get the following result:

Theorem 1 *All roots of the equation* (2.2) *have negative real part if and only if*

(i) $0 < -\beta < \sqrt{\xi^2 r^{-2} + k^2}$, *where* ξ *is the unique root of the equation* $\xi = -krtan(\xi)$, $\pi/2 < \xi < \pi$, *if* $\beta < 0$; *or*

(ii) $k > \beta$, *if* $\beta > 0$.

3 Global stability of the equilibrium point.

In this section, we will show that for $-k < \beta < k$ the trivial solution of (1.1) is global asymptotically stable.

In order to carry out this study we follow basically the main ideas developed in [4]. Which states a connection between the continuous semi-dynamical system induced by (1.1) and certain discrete dynamical systems. More concretely, we will associate to the semi-flow generated by the solution of (1.1) a map $g : I \longrightarrow I$, where I is an interval, which is given by $g(x) = \beta\, tanh(x)/k$. Then, the global stability of the equilibrium point of (1.1) is obtained from the global stability of the fixed point of g.

Now, we will outline some facts from the theory of one-dimensional maps. (see [2] and [7])

Definition 1 *Let g be a real function having at least three continuous derivatives. The Schwarzian derivative of g at point x, denoted by $(Sg)(x)$, is defined by*

$$(Sg)(x) = \frac{g'''(x)}{g'(x)} - \frac{3}{2}\left[\frac{g''(x)}{g'(x)}\right]^2 .$$

Theorem 2 *Let I be an interval and $g, h \in C^3(I, I)$. If $Sg < 0$ and $Sh < 0$, then:*

(a) $S(g \circ h) < 0$.

(b) $g'(x)$ *cannot have a positive local minimum or a negative local maximum.*

(c) The function $g(x)$ have at most one inflection point.

Theorem 3 *Let $g \in C^3(I, I)$ be a strictly monotone function, such that $Sg(x) < 0$. If g has a unique fixed point x^* which is a local attractor, then x^* is a global attractor.*

Proof. Let us first assume that g is a strictly decreasing function. It is obvious that $f(x) = g^2(x)$ is a strictly increasing. From theorem 2 and the fact that $Sf(x) < 0$, it follows that x^* is the unique fixed point of f.

Let us assume that $x \in I$ and $x < x^*$. Thus, the sequence $\{f^n(x)\}_{n\geq 1}$ is monotone increasing and bounded from above by x^*. Therefore, $f^n(x) \longrightarrow f(x^*) = x^*$, due to the continuity and the uniqueness of the fixed point of the function f. The case when $x \in I$ and $x > x^*$, can be treated by using an analogous reasoning. This leads to that x^* is a global attractor.

If g is a monotone increasing function, the proof follows as above, just setting $f = g$. ■

Now, having in mind that

$$g(x) = \frac{\beta}{k} tanh(x),$$

a straightforward computation leads to:

$$g'(0) = \frac{\beta}{k} \quad , \quad (Sg)(x) = -2 < 0.$$

Taking into account that, the Schwarzian derivative of g is always negative, we obtain that the fixed point of the map g is a global attractor if and only if $-1 < g'(0) < 1$; i.e. if and only if $-k < \beta < k$.

The following theorem is the main result of this paper.

Theorem 4 *If $-k < \beta < k$, then the trivial equilibrium of the equation (1.1) is global asymptotically stable.*

Proof. From Theorem 1, it follows that the trivial equilibrium of the equation (1.1) is local asymptotically stable for any β such that $-k < \beta < k$.

Let us suppose first that $-k < \beta < 0$, in this case the function g is strictly decreasing.

Let $\phi \in A^*$, where A^* is the global attractor of the equation (1.1). From (1.1), we obtain that

$$x(t,\phi) = x(\tau,\phi)e^{-k(t-\tau)} + \beta \int_{\tau}^{t} e^{-k(t-s)} tanh(x_s(\phi)))ds$$

Since solutions on the global attractor are bounded and defined on the whole real line, letting $\tau \longrightarrow -\infty$ in the previous formulae, we get that any solution of (1.1) with initial data in A^* admits the following representation

$$x(t,\phi) = \beta \int_{-\infty}^{t} e^{-k(t-s)} tanh(x_s(\phi)))ds$$

Let us set

$$m = inf\{x(t,\phi) : t \in \mathbb{R}, \phi \in A^*\} \quad \text{and} \quad M = sup\{x(t,\phi) : t \in \mathbb{R}, \phi \in A^*\}.$$

It is obvious that $0 \in [m, M]$. Taking into account that A^* is a compact set, then there exist ϕ_1 and t_1 such that

$$m = x(t_1,\phi_1) = \beta \int_{-\infty}^{t_1} e^{-k(t_1-s)} tanh(x_s(\phi_1))ds \geq \beta \int_{-\infty}^{t_1} e^{-k(t_1-s)} tanh(M)ds = g(M)$$

Analogously, there exist ϕ_2 and t_2 such that

$$M = x(t_2,\phi_2) = \beta \int_{-\infty}^{t_2} e^{-k(t_2-s)} tanh(x_s(\phi_2))ds \leq \beta \int_{-\infty}^{t_2} e^{-k(t_2-s)} tanh(m)ds = g(m).$$

From the last two inequalities, we get

$$[m, M] \subset g([m, M]) \subset g^2([m, M]) \subset \ldots \subset g^n([m, M]) \subset \ldots \quad . \tag{3.1}$$

Let us suppose that the trivial solution of the equation (1.1) is not global asymptotically stable. Henceforth $m < M$ and from (3.1), it follows that the fixed point of the dynamical system induced by g cannot be global asymptotically stable. Which is a contradiction. Therefore $M = m$.

In the case that $0 < \beta < k$, the proof follows analogously to the former reasoning, except obvious modifications. This certainly proves our assertion. ■

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