Universidad de los Andes
Facultad de Ciencias
Departamento de Matemática

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F. Rivero

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Abstract

The Golden Number $\varphi = (1 + \sqrt{5})/2$ appears in many fields of mathematics like proportions, continuous fractions and Fibonacci Sequences. We give a sequence of quadratic irrational numbers φ_n, such that $\varphi_1 = \varphi$ and φ_n tends to 1 as n increases. This yields a generalization of the golden number. We give an interpretation of φ_n, for each n, in terms of equations, proportions, continuous fractions, and Fibonacci Sequences.

1 Introduction

We start by considering those pairs of rational numbers (x,y) which satisfy the relation

$$\frac{x+y}{y} = x \tag{1}$$

This equation can be interpreted as a " wrong method " for simplification of fraction, usually found in freshmen students. The method fails for many numbers, for instance

$$\frac{3+2}{2} = \frac{5}{3} \neq 3$$

Let us look at the set of all pairs (x, y) satisfying (1), such that the second component y is a natural number.If we solve (1), for x we obtain

$$x = \frac{y}{y-1}$$

As y runs over the set $2, 3, 4, 5, \ldots$ etc the corresponding values of x, gives rise to the sequence

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$$

So equation (1) has infinitely many solutions and this shows that it is possible for student to get right answers infinitely many times, using a wrong method.

For the decreasing rational sequence $x_n = \dfrac{n}{n-1}$ we have

$$\lim_{n\to\infty} x_n = 1$$

Consider now a slightly different equation given by

$$\frac{x+y}{y} = x^2 \tag{2}$$

Solving for x produces

$$x = \frac{1 \pm \sqrt{1+4y^2}}{2y}$$

We are interested again in considering those pairs of solutions (x, y) with y a natural number, and $x > 0$. This gives us the following sequence of cuadratic irrationals

$$\varphi_n = \frac{1 \pm \sqrt{1+4n^2}}{2n} \qquad n = 1, 2, \ldots \tag{3}$$

We observe that $\varphi_1 = (1+\sqrt{5})/2 = \varphi$ is the golden number. Moreover, $\{\varphi_n\}$ is a decreacing sequence approaching to 1. So again we have

$$\lim_{n\to\infty} \varphi_n = 1$$

Let's look at the first ten values of φ_n

n	φ_n	φ_n
1	$\frac{1+\sqrt{5}}{2}$	1.61803399
2	$\frac{1+\sqrt{17}}{4}$	1.28077641
3	$\frac{1+\sqrt{37}}{6}$	1.18046042
4	$\frac{1+\sqrt{65}}{8}$	1.13278222
5	$\frac{1+\sqrt{101}}{10}$	1.10498756
6	$\frac{1+\sqrt{145}}{12}$	1.08679955
7	$\frac{1+\sqrt{197}}{14}$	1.07397635
8	$\frac{1+\sqrt{257}}{16}$	1.06445122
9	$\frac{1+\sqrt{325}}{18}$	1.05709758
10	$\frac{1+\sqrt{401}}{20}$	1.05124922

By looking at this table, one concludes that φ_n converges to 1 very fast.

Now we may ask. What kind of interesting properties does φ_n have, for each n?. To answer that question we recall some of the nice properties of the golden number, and somehow we will try to reproduce them for these numbers. We will see that for every n, φ_n is a very special number, so we call φ_n, **the n- Golden Number**.

To begin with, notice that the n- Golden Number satisfies a quadratic equation, very similar to the equation for the Golden Number, namely

$$n(\varphi_n)^2 - \varphi_n - n = 0$$

2 Proportions and The Golden Rectangle

If we construct a rectangle,whose sides are in proportion X:Y, then it is called a Golden Rectangle if $\frac{X}{Y} = \varphi$. It is possible to divide a Golden Rectangle by a vertical line in two adjacent subrectangles, such that something fairly good happens {see the picture}

The rectangle $\square\ ABCD$ is a Golden Rectangle since its sides are in proportion 1: $\varphi - 1$, and it follows

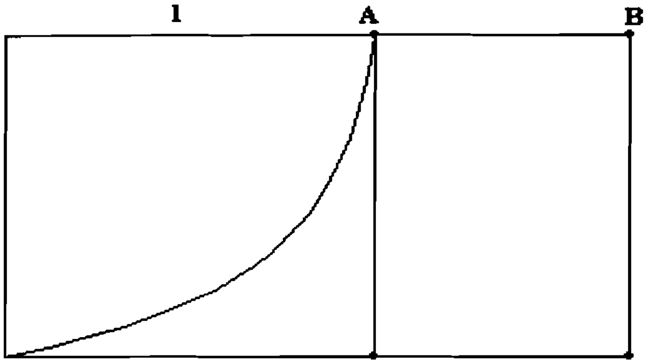

Figure 1:

$$\frac{1}{\varphi - 1} = \varphi$$

It is possible to carry out a similar analysis for the n- Golden Numbers. Suppose $n \geq 2$, and draw a rectangle whose sides are in proportion X: Y, such that $\frac{X}{Y} = \varphi_n$. Then we call that rectangle a n- Golden Rectangle. We will see that that rectangle can be subdivided into n squares and another rectangle with the same proportion φ_n.

(see the picture)

So we start by subdividing rectangle $\square\, ABCD$, by a horizontal line and n-1 vertical lines in (n+1) subrectangles. It is clear that each of the n rectangles at the bottom of the big rectangle are congruent squares, having sides $\frac{X}{n}$.

But the remainig rectangle $\square\, EFCD$ is again a n-Golden Rectangle, since its sides are in proportion $X : Y - \frac{X}{n}$, and

$$\frac{X}{Y - \frac{X}{n}} = \frac{nx}{nY - X} = \frac{n}{n\varphi_n - 1} = \varphi_n \qquad (4)$$

Now we want to repeat this process over and over to obtain a double sequence of rectangles

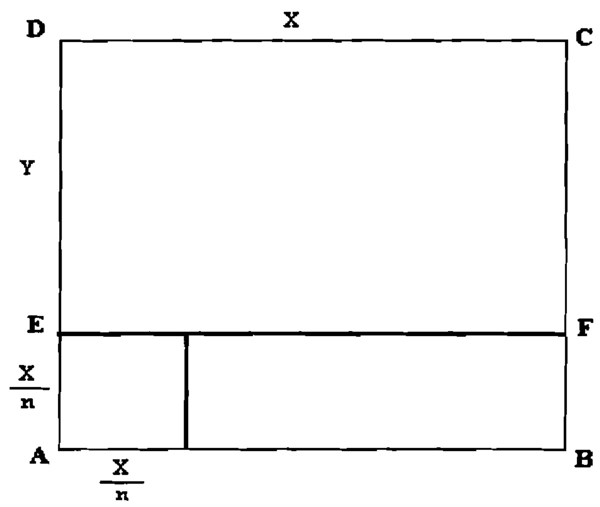

Figure 2:

$$\cdots R_k \supseteq R_{k+1} \cdots$$

Notice that each rectangle R_k contains a reduced copy of itself and n squares. Thus

$$R_k = R_{k+1} \bigcup G_k$$

where G_k is a union of n squares. Here G_k represents the difference between two succesive rectangles and is called the gnomom of R_k.

For the special case $k = 2$, we have the following picture of an infinite subdivision of a 2- Golden Rectangle in rectangles with the same proportions.

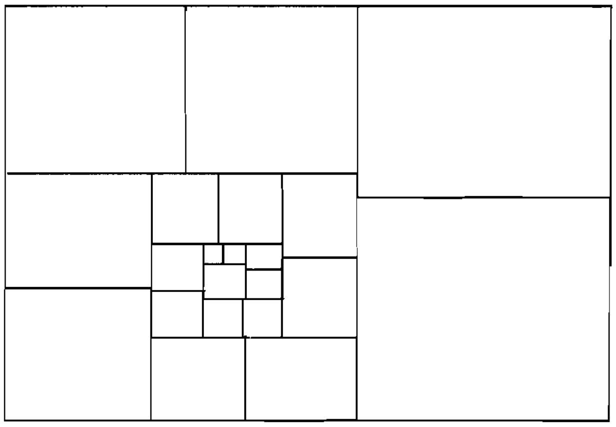

Figure 3:

We see that the sequence of the gnomon describes an infinite spiral on the plane, like stairs on a winding staircase.

We are interested now in obtaining a formula for the area of rectangle R_{n+1} in terms of the area of R_n. Lets call A_n = area of R_n = x.y, where $x > y$, and $\frac{x}{y} = \varphi_2$. Then we have

$$\begin{aligned} A_{n+1} &= (y + \tfrac{x}{2}) \cdot x \\ &= xy + \tfrac{x^2}{2} \\ &= A_n + \tfrac{x^2}{2} \\ &= A_n + \tfrac{\varphi_2 A_n}{2} \\ &= A_n \left(1 + \tfrac{\varphi_2}{2}\right) \\ &= A_n \varphi_2^2 \end{aligned}$$

Thus $A_{n+1} = A_n \varphi_2^2$. Therefore, each rectangle in the sequence increases its area by a factor φ_2^2. This is called the growing factor of the sequence. If we start the sequence taking a basic rectangle with sides φ and 1, we obtain the geometric series

$$\cdots \varphi_2 < \varphi_2^3 < \varphi_2^5 < \cdots$$

representing the areas of an increasing sequence of 2- Golden Rectangles.

3 Continuos Fractions

Let $n_1, n_2, n_3, \ldots$ be an infinite sequence of natural numbers. Then the expression

$$x = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cdots}}} \tag{5}$$

is called a *continuos fractions* (see [1]) and we use the notation

$$x = < n_1, n_2, n_3, \ldots >$$

One of the most striking properties of the Golden Number φ is given by its expression as a continuos fraction, namely

$$\varphi = < 1, 1, 1, \ldots >$$

It is a well know fact that if x is a quadratic irrational, and therefore a root of a second degree equation, then its continuos fraction is periodic. That is

$$x = < n_1, n_2, \ldots, n_t, n_1, \ldots, n_t, \ldots >$$

We use the shortest notation

$$x = \overline{< n_1, n_2, \ldots, n_t >}$$

For example $\varphi = < \bar{1} >$

For the Golden Numbers we have the following result, concerning continuos fractions.

Theorem: *For every $n \geq 1$, let φ_n be defined as in (2). Then we have*

$$\varphi_n = \overline{< 1, t, 1 >}$$

where $t = 2n - 1$

Proof: Let $n \geq 1$, and for every $t \geq 1$ make

$$x_t = \overline{< 1, t, 1 >}$$

Then we have

$$x_t = 1 + \cfrac{1}{t + \cfrac{1}{1 + \cfrac{1}{x_t}}}$$

After doing some elementary algebraic operations we get

$$x_t = \frac{(t+2)x_t + t + 1}{(t+1)x_t + t},$$

and from this we obtain the quadratic equation

$$2\left(\frac{t+1}{2}x_t^2 - x_t - \frac{t+1}{2}\right) = 0$$

Thus we have

$$x_t = \varphi_{\frac{t+1}{2}}$$

So we get the result.

□

Example:

$$\varphi_2 = \frac{1+\sqrt{17}}{4} =< 1, 3, 1, 1, 3, 1, \ldots >$$

4 Fibonacci Sequence:

The sequence of natural numbers $1, 1, 2, 3, 5, 8, \cdots$ is called the Fibonacci Sequence. It is easy to show that the Fibonacci Sequence is defined by the recurrence relation

$$F(n) = F(n-1) + F(n-2), \qquad (6)$$

for $n = 2, 3, 4, \ldots$ and initial values given by

$$F(0) = F(1) = 1.$$

It is a well know fact (see [2]) that the Fibonacci Numbers satisfy the equation

$$F(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

for $n = 0, 1, 2, \ldots$

So the Golden Number $\varphi = \dfrac{1+\sqrt{5}}{2}$ is related to this sequence. Now we may ask ourselves: What kind of sequence is related to φ_n, for every $n \geq 2$?

Consider the special case of $\varphi_2 = \dfrac{1+\sqrt{17}}{4}$. We know that φ_2 satisfy the equation:

$$2\varphi_2^2 - \varphi_2 - 2 = 0 \tag{7}$$

On the other hand, we may consider a recurrence relation $F(n)$, given by

$$F(n) = \frac{1}{2}F(n-1) + F(n-2), \tag{8}$$

To get a solution we can set $F(n) = q^n$ for some value of $q \neq 0$ to be determined. Then we have

$$q^n = \frac{1}{2}q^{n-1} + q^{n-2}$$

so

$$q^{n-2}(q^2 - \frac{1}{2}q - 1) = 0$$

Therefore q is a solution of the quadratic equation

$$2x^2 - x - 2 = 0$$

and so we conclude

$$q = \frac{1+\sqrt{17}}{4} = \varphi_2 \text{ or } q = \frac{1-\sqrt{17}}{4}$$

Hence a solution of (6) is given by

$$F(n) = C_1(\frac{1+\sqrt{17}}{4})^{n+1} + C_2(\frac{1-\sqrt{17}}{4})^{n+1}$$

Where C_1 and C_2 are constants to be determined. Thus we have found a generalize Fibonacci sequence given by (6), which is related to the 2-Golden Number φ_2. Let's get the first 10 values of the sequence $F(n)$, with initial values $F(0) = 1, F(1) = 2$.

n	0	1	2	3	4	5	6	7	8	9	10
F(n)	1	2	2	3	3.5	4.75	5.87	8.81	10.27	13.94	17.24

We observe that this sequence increases more slowly then the Fibonacci Sequence. Using the same argument as in obtaining formula (7), we give the next generalization

Theorem: *For any $k \geq 1$, the recurrence relation*

$$F(n) = \frac{1}{k}F(n-1) + F(n-2) \tag{9}$$

has the solution

$$F(n) = \left(\frac{1+\sqrt{1+4k^2}}{2k}\right)^n = (\varphi_k)^n. \tag{10}$$

The set of all solutions of relation (6), is given by

$$F(n) = C_1^k\left(\frac{1+\sqrt{1+4k^2}}{2k}\right)^n + C_2^k\left(\frac{1-\sqrt{1+4k^2}}{2k}\right)^n \tag{11}$$

where C_1^k and C_2^k are constants to be determined.

Proof: It is clear that both $(1+\sqrt{1+4k^2})/2k$ and $(1-\sqrt{1+4k^2})/2k$ are independent solutions of (8). Since the recurrence relation is linear; we conclude that (11) is general solution.

□

Golden Numbers are also connected to limits of quotients of Fibonacci Sequences.

Theorem: *For every $k \geq 1$ let*

$$\varphi_k = \frac{1+\sqrt{1+4k^2}}{2k},$$

and $\{x_n^k\}$ *be the sequence defined by*

$$x_n^k = \frac{1}{k}x_{n-1} + x_{n-2}.$$

Then we have:

$$\lim_{n\to\infty} \frac{x_{n+1}^k}{x_n^k} = \varphi_k$$

Proof: We have

$$\frac{x_{n+1}^k}{x_n^k} = \frac{\frac{1}{k}x_n^k + x_{n-1}^k}{x_n^k}$$
$$= \frac{1}{k} + \frac{1}{\frac{x_n^k}{x_{n-1}^k}}$$

Taking limits in both sides as $n \to \infty$ gives

$$L = \lim_{n\to\infty} \frac{x_{n+1}^k}{x_n^k} = \frac{1}{k} + \frac{1}{L}$$

Thus we obtain

$$kL^2 - L - k = 0,$$

and from this we get the solution

$$L = \varphi_k$$

□

References

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[2] R. Brualdi (1977) - Introductory Combinatorics Elseiver Science Publishing Co. Amsterdam

[3] J. Kappraff (1991) - Connections: The Geometric Bridge Between Art and Science. McGraw-Hill, New York

F. Rivero
Department of Mathematics, Universidad de los Andes,
Facultad de Ciencias, Mérida 5101, Venezuela.
e-mail:lico@ciens.ula.ve