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# A DIRECT PROOF OF A THEOREM OF REPRESENTATION OF MULTILINEAR OPERATORS ON $XC_0(T_I)$

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# MOTIVATION OF THE MULTILINEAR REPRESENTATION THEOREM OF DOBRAKOV

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In 1910, F.Riesz proved his famous representation theorem for the bounded linear forms on $C[0,1]$, which has been generalized later to compact and locally compact Hausdorff spaces by Markoff, Kakutani and others. Let us present this theorem in the general form as follows:

**THEOREM 1. (Riesz representation theorem)** *Let $T$ be a locally compact Hausdorff space and let $\mathcal{B}(T)$ be the $\sigma$-algebra of the Borel sets of $T$. Let $C_0(T)$ be the Banach space of all complex valued continuous functions on $T$ vanishing at infinity, with the supremum norm $||f||_T = \sup\{|f(t)| : t \in T\}$. Then the dual $C_0(T)^*$ of all continuous linear forms on $C_0(T)$ can be identified with the Banach space $M(T)$ of all regular (Complex) Borel measures on $T$, in the sense that there exists an isometric isomorphism $F : C_0(T)^* \to M(T)$ such that*

$$\phi(f) = \int_T f dF(\phi),\ f \in C_0(T)$$

*for $\phi \in C_0(T)^*$, and $||\phi|| = |F(\phi)|(T)$, where $|\mu|$ is the variation of the (complex) measure $\mu$.*

Later, in 1955, Bartle, Dunford and Schwartz extended the above theorem to bounded linear operators $T : C(S) \to Y$, where $S$ is a compact Hausdorff space and $Y$ is a Banach space. For this, they developed the theory of integration of scalar functions with respect to a vector measure. Now we shall state their representation theorem.

**THEOREM 2. (Bartle-Dunford-Schwartz representation theorem)** *Let $S$ be a compact Hausdorff space and let $U : C(S) \to Y$ be a bounded*

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*linear operator, where $Y$ is a Banach space. Then there exists a weak* $\sigma$-additive measure $G(.)$ on $\mathcal{B}(S)$ with values in $Y^{**}(y^*G(\bigcup_1^\infty E_i) = \sum_1^\infty y^*G(E_i)$ for each $y^* \in Y^*$, whenever $(E_i)_1^\infty$ is a disjoint sequence in $\mathcal{B}(S))$ such that*

*(i) $y^*G(.)$ is a regular $\sigma$-additive Borel measure (a complex Borel measure) for each $y^* \in Y^*$;*

*(ii) the mapping $y^* \to y^*G(.)$ of $Y^*$ into $C(S)^*$ is weak* to weak*-continuous;*

*(iii) $y^*U(f) = \int_T f d(y^*G)$, for each $y^* \in Y^*$; and*

*(iv) $||U|| = ||G||(S)$, where*

$$||G||(S) = \sup\{||\sum_{i=1}^{r} \alpha_i G(E_i)|| \, : \, E_i \cap E_j = \phi, \, i \neq j, \, E_i \in \mathcal{B}(S), \, |\alpha_i| \leq 1\}$$

*is called the semivariation of $G(.)$ on $S$.*

Conversely, if $G(.)$ is any $Y^{**}$-valued vector measure (=additive set function) defined on $\mathcal{B}(S)$ for which (i) and (ii) hold, then (iii) defines a bounded linear operator $U : C(S) \to Y$ which satisfies (iv). $G(.)$ is called the representing measure of $U$ and is unique by (iii).

They also proved the following results.

Recall that a Banach space $Y$ is said to be weakly complete if, each sequence of vectors in $Y$ which is Cauchy in the weak topology, is weakly convergent to a vector in $Y$.

**THEOREM 3. (Bartle-Dunford-Schwartz representation theorem)** *If $Y$ is weakly complete, then the represeting measure $G(.)$ assumes values in $Y$ itself and moreover, $G(.)$ is $\sigma$-additive in the norm topology of $Y$.*

A bounded linear operator $U : X \to Y$, $X, Y$ Banach spaces, is said to be weakly compact if $\{||Ux|| : ||x|| \leq 1\}$ is relatively weakly compact in $Y$.

**THEOREM 4. (Bartle-Dunford-Schwartz)** *If $U : C(S) \to Y$ is weakly compact, then the representing measure $G(.)$ of $U$ assumes values in $Y$ and $G(.)$ is $\sigma$-additive in the norm topology of $Y$.*

In 1967, Kluvánek extended the Bartle-Dunford-Schwartz representation theorem (theorem 2) to locally compact Hausdorff spaces $T$, with $\mathcal{B}(S)$ being replaced by $\sigma\mathcal{B}(T)$, the $\sigma$-ring generated by the compact subsets of $T$.

In 1958, Bessaga and Pelczyński studied the Banach spaces which behave well like weakly complete spaces. Let $c_0 = \{(a_n)_1^\infty : a_n \in \mathbf{C}, \lim_n \mathrm{A_n} = 0\}$ with $||(a_n)|| = \sup_n |a_n|$. A Banach space $X$ is said to contain a copy of $c_0$ if there exists a closed subspace $Z$ of $X$ such that $Z$ is topologically isomophic to $c_0$; i.e. there is a linear bicontinuous isomophism from $Z$ onto $c_0$, $Z$ being endowed with the relative topology.

**THEOREM 5. (Bessaga-Pelczyński)** *A Banach space $Y$ does not contain a copy of $c_0$ (in symbols, $c_0 \not\subset Y$) if and only if, for each sequence $(y_n)_1^\infty$ of vectors in $Y$ with $\sum_1^\infty |y^*(y_n)| < \infty$ for each $y^* \in Y^*$, the formal series $\sum y_n$ is unconditionally convergent in norm of $Y$.*

Pelczyński extended Theorem 3 of Bartle-Dunford-Schwartz to Banach spaces $Y \not\supset c_0$. Moreover, all these theorems can be extended to locally compact Hausdorff spaces suitably.

Thus we have the following representation theorem:

**THEOREM 6.** *Suppose $U$ is a bounded linear operator from $C_0(T)$ to $Y$, where $(A)$ $U$ is weakly compact, or $(B)$ $c_0 \not\subset Y$. Then the representing* *measure $G(.)$ of $U$ takes values in $Y$, $G(.)$ is $\sigma$-additive in norm of $Y$ and*

$$Uf = \int_T f dG,\ f \in C_0(T)$$

*where $G(.) : \sigma\mathcal{B}(T) \to Y$ is $\sigma\mathcal{B}(T)$-regular. Moreover,*

$$||U|| = ||G||(T) = \sup\{||G||(A) : A \in \sigma\mathcal{B}(T)\}.$$

*Besides, if $c_0 \not\subset Y$, then every bounded linear operator $U : C_0(T) \to Y$ is necessarily weakly compact.*

In this context, it will be interesting to know whether such a representation theorem can be given to bounded multilinear operators $U : \overset{d}{\underset{1}{\mathrm{X}}} C_0(T_i) \to Y$ when $c_0 \not\subset Y$, or when $U$ is weakly compact. This type of study naturally leads to the notion of vector valued multimeasures (=polymeasures).

Historically speaking, the study of bimeasures goes back to Morse, M., and Transue, W. They studied complex valued bimeasures in their memoirs

*"C-bimeasures $\Delta$ and their superior integral $\Delta^*$", Rend. Circ. Mat. Palermo (2) 4 (1955) 270-300*

and

*"C-measures $\Delta$ and their integral extension", Ann. of Math (2) 64 (1956), 480-504*

using an aproach similar to Bourbaki. Unfortunately, their theory was not sufficiently developed to give such representation theorems.

K. Ylinen introduced the notion of vector bimeasures in 1978 and these bimeasures were found useful in the study of stochastic processes and also in Harmonic Analysis.

Later, from 1987 onwards, Dobrakov published a series of papers on multilinear integration with respect to operator valued polymeasures: "*On integration in Banach spaces, VIII,...,XIII*" in Czech. Math. J. His theory was influenced by his earlier theory on vector integration given in "*On integration in Banach spaces, I,II,...,VII*" in the same journal.

Based on this theory, Dobrakov proved in 1989 the multilinear integral representation of bounded multilinear operators $U : C_0(T_i)\,X...X\,C_0(T_d) \to Y$, when $U$ is weakly compact, or when $c_0 \not\subset Y$. But this theorem of representation was derived from the earlier multilinear extension theorem of Pełczyński given in 1963. Since the representation theorem is one of the peaks of the multilinear integration theory, it is desirable to have a proof of this theorem directly, without any reference to Pełczyński's result. In our present note, with Dobrakov as coauthor, we have achieved in giving such a direct proof.

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# A Direct Proof of a Theorem of Representation of Multilinear Operators on $XC_0(T_i)$

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*Dedicated to Professor Mischa Cotlar on the occasion of his eightieth birth day*

**ABSTRACT.** Let $T_i, i = 1,2, ..., d$, be locally compact Hausdorff spaces and let $C_0(T_i)$ be the Banach space of all scalar valued continuous functions on $T_i$ vanishing at infinity ( with the supremum norm ). Suppose $U : \mathop{\mathbf{X}}\limits_1^d C_0(T_i) \to Y$ is a bounded d-linear operator, where either $(A)$ $Y$ is a Banach space such that $c_0 \not\subset Y$, or $(B)$ $U$ is weakly compact. Using the multilinear extension theorem of Pelczyński, Dobrakov obtained in an ealier work a multilinear integral representation of $U$ with respect to a unique Y-valued Baire d-multimeasure on $\mathop{\mathbf{X}}\limits_1^d \sigma\mathcal{B}_0(T_i)$. The aim of the present note is to provide a direct proof of this representation theorem , without any reference to the said result of Pelczyński. Then the multilinear extension theorem of the latter follows as a corollary.

## 1. Introduction

In [14] Pelczyński proved the following extension theorem of multilinear operators on $\mathbf{X}C(S_i)$, where $C(S_i)$ is the Banach space of all scalar valued continuous functions on the compact Hausdorff space $S_i$.

THEOREM (Pelczyński). *Suppose $S_1, ..., S_d$ are compact Hausdorff spaces and $U : \mathop{\mathbf{X}}\limits_1^d C(S_i) \to Y$ is a bounded d-linear mapping, where either $Y$ is a Banach space such that $c_0 \not\subset Y$, or $U$ is weakly compact. Then there exists a unique bounded d-linear mapping $U^{**} : \mathop{\mathbf{X}}\limits_1^d B^{\Omega}(S_i) \to Y$ such that*

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*(i)* $U^{**}|\overset{d}{\underset{1}{\mathbf{X}}} C(S_i) = U$, *and*

*(ii) if* $(f_{i,n})_{n=1}^{\infty} \subset \mathcal{B}^{\Omega}(S_i)$ *such that*

$$\lim_{n\to\infty} f_{i,n}(s_i) = f_i(s_i), \text{ for each } s_i \in S_i,$$

*and*

$$\sup_{s_i\in S_i, n=1,2,\ldots} |\mathrm{f}_{\mathrm{i,n}}(\mathrm{s_i})| \leq \mathrm{C}(<\infty)$$

*for* $i = 1,2,\ldots,d$, *then*

$$\lim_{n\to\infty} U^{**}(f_{1,n}, \ldots, f_{d,n}) = U^{**}(f_1, \ldots, f_d).$$

*Moreover, in the case (B) the oprator* $U^{**}$ *is also weakly compact.*

Using the above theorem, Dobrakov proved in [8] that there is a d-multimeasure $\Upsilon : \overset{d}{\underset{1}{\mathbf{X}}} \sigma\mathcal{B}_0(T_i) \to Y$ such that

$$U^{**}(g_i) = \int_{(T_i)} (g_i)\, d\Upsilon$$

for each $(g_i) \in \mathbf{X}\mathcal{B}^{\Omega}(T_i)$, where $T_i$ is a locally compact Hausdorff space, $\sigma\mathcal{B}_0(T_i)$ is the $\sigma$-ring of all Baire sets of $T_i$ and $\mathcal{B}^{\Omega}(T_i)$ is the class of all bounded Baire functions on $T_i$,for $i = 1,2,\ldots$,d. Moreover, $U^{**}$ extends $U$ and is a bounded d-linear operator with $||U^{**}|| = ||U|| = ||\Upsilon||(T_i)$, where $||\Upsilon||(T_i)$ is the scalar semivariation of $\Upsilon$ in $(T_i)$.Finally, the range of $\Upsilon$ is relatively weakly compact if and only if $U$ is weakly compact.

The object of the present note is to present a direct proof of the multilinear integral representation theorem of Dobrakov and then to deduce the cited theorem of Pelczyński as a corollary. Then all the results of Dobrakov in [8] remain independent of Pelczyński's multilinear extension theorem.

## 2. Notation and Terminology

In the sequel, $T, T_i$, $i = 1,2,...,d$, are locally compact Hausdorff spaces. $C_0(T)$ is the Banach space of all scalar valued continuous functions on $T$ vanishing at infinity, with the supremum norm $||.||_T$, where $||f||_T = \sup_{t \in T} |f(t)|$. Similarly, we define $C_0(T_i)$, for $i = 1,2,...,d$.

The family of all compact $G_\delta s$ of $T$ is denoted by $\mathcal{K}_0(T)$ and of $T_i$ by $\mathcal{K}_0(T_i)$, for $i = 1,2,...,d$. The $\sigma$-ring generated by $\mathcal{K}_0(T)$ (resp. $\mathcal{K}_0(T_i)$) is denoted by $\sigma\mathcal{B}_0(T)$(resp. $\sigma\mathcal{B}_0(T_i)$), whose members are called Baire sets of $T$ (resp. $T_i$).

The scalar field is denoted by $\mathbf{K}$ ( $= \mathbf{R}$ or $\mathbf{C}$ ). Let $Y$ be a Banach space over $\mathbf{K}$ , the scalar field of $C_0(T)$ and $C_0(T_i)$.

DEFINITION 2.1. *A mapping* $U : \mathop{\mathbf{X}}_1^d C_0(T_i) \to Y$ *is said to be d-linear if it is separately linear on each coordinate.*

*Such a mapping U is said to be bounded if*

$$\sup\{|U(f_i, ..., f_d)| : ||f_i||_{T_i} \leq 1, i = 1, 2, ..., d\} < \infty$$

*where* $|.|$ *denotes the norm of* $Y$. *When U is bounded, the above supremum is denoted by* $||U||$. *If U is a bounded d-linear mapping and if* $\{U(f_i, ..., f_d) : ||f_i||_{T_i} \leq 1,\ f_i \in C_0(T_i),\ i = 1,2,...,d\ \}$ *is relatively weakly compact in* $Y$, *then U is said to be weakly compact.*

We now proceed to state some definitions and results from the theory of multilinear integration of scalar functions. The reader may refer to Dobrakov [5,6,7,8,9].

If $\mathcal{S}_i$, $i = 1,2,...,d$, are $\sigma$-rings of sets in $T_i$, then let

$$\mathop{\mathbf{X}}_1^d \mathcal{S}_i = \{(A_1, ..., A_d) : A_i \in \mathcal{S}_i,\ \ i = 1, 2, ..., d\}.$$

The rectangle $(A_1, ..., A_d)$ is denoted by $(A_i)$.

DEFINITION 2.2. *Suppose $\Upsilon : \mathbf{X}\mathcal{S}_i \to Y$ is a set function such that it is separately $\sigma$-additive in norm of $Y$. Then $\Upsilon$ is called a $Y$ – valued d-multimeasure (or d-polymeasure).*

DEFINITION 2.3. *Let $f_i = \sum_{j=1}^{r_i} a_{ij}\chi_{A_{ij}}, a_{ij} \in \mathbf{K},\ A_{ij} \cap A_{ij'} = \emptyset$ for $j \neq j'$, $A_{ij} \in \mathcal{S}_i$, for $j = 1, 2, ..., r_i$ and $i$ = 1,2,...,d. Such functions $f_i$ are called $\mathcal{S}_i$-simple functions and $(f_i)$ is said to be $\mathbf{X}\mathcal{S}_i$-simple. The set of all $\mathbf{X}\mathcal{S}_i$-simple functions is denoted by $\mathbf{X}S(\mathcal{S}_i)$. If $\Upsilon : \mathbf{X}\mathcal{S}_i \to Y$ is a d-multimeasure, then we define*

$$\int_{(A_i)} (f_i) d\Upsilon = \sum_{j_1=1}^{r_1} \cdots \sum_{j_d=1}^{r_d} a_{1j_1} a_{2j_2} \ldots a_{dj_d} \Upsilon(A \cap A_{ij_i})$$

*where $(f_i)$ is given as above.*

DEFINITION 2.4. *For $(A_i) \in \mathbf{X}\mathcal{S}_i$ and for a d-multimeasure $\Upsilon : \mathbf{X}\mathcal{S}_i \to Y$, we define the scalar semivariation $||\Upsilon||(A_i)$ by*

$$||\Upsilon||(A_i) = \sup\{|\int_{(A_i)} (f_i) d\Upsilon| : (f_i) \in \mathbf{X}S(\mathcal{S}_i), ||f_i||_{T_i} \leq 1, i = 1, ..., d\}$$

*and $||\Upsilon||(T_i) = \sup\{||\Upsilon||(A_i) : (A_i) \in \mathbf{X}\mathcal{S}_i\}$.*

THEOREM 2.5. *For a $Y$-valued d-multimeasure $\Upsilon : \mathbf{X}\mathcal{S}_i \to Y, ||\Upsilon||(T_i)$ is finite.*

THEOREM 2.6. *Let $\overline{S(\mathcal{S}_i)}$ be the closure of $S(\mathcal{S}_i)$ with respect to the topology of uniform convergence in the space of the bounded scalar functions on $T_i$. Let $f_i \in \overline{S(\mathcal{S}_i)}$ and let $(f_{i,n_i})_{n_i=1}^{\infty} \subset S(\mathcal{S}_i)$ be such that*

$$\lim_{n_i \to \infty} ||f_i - f_{i,n_i}||_{T_i} = 0$$

*for $i$ = 1,2,...,d. If $\Upsilon : \mathbf{X}\mathcal{S}_i \to Y$ is a d-multimeasure, then*

$$\lim_{n_1, n_2, \ldots, n_d \to \infty} \int_{(A_i)} (f_{i,n_i}) d\Upsilon$$

*exists in $Y$, uniformly with respect to $(A_i) \in \mathbf{X}\mathcal{S}_i$. Moreover, this limit is independent of the converging sequences $(f_{i,n_i})$.*

The above theorem motivates the following

DEFINITION 2.7. *Let $f_i \in \overline{S(\mathcal{S}_i)}$ and let $(f_{i,n_i})_{n_i=1}^{\infty}$ and $\Upsilon$ be as in Theorem 2.6. Then we say that $(f_i)$ is $\Upsilon$-integrable and the $\Upsilon$-integral of $(f_i)$ over $(A_i) \in \mathbf{X}\mathcal{S}_i$ is defined as*

$$\int_{(A_i)} (f_i) d\Upsilon = \lim_{n_1,\ldots,n_d} \int_{(A_i)} (f_{i,n_i}) d\Upsilon.$$

*Moreover, the $\Upsilon$-integral of $(f_i)$ over $(T_i)$ is defined as that on $\mathbf{X}N(f_i)$, where $N(f_i) = \{t_i \in T_i : f_i(t_i) \neq 0\}$, $i = 1,2,\ldots,d$.*

We have the following generalized Lebesgue bounded convergence theorem (shortly, LBCT) for the $\Upsilon$-integrable functions in $\mathbf{X}\overline{S(\mathcal{S}_i)}$.See Theorem 3 of [8].

THEOREM 2.8. *Suppose $f_{i,n_i} \in \overline{S(\mathcal{S}_i)}$ for $n_i = 1,2,\ldots$ and $f_{i,n_i}(t_i) \to f_i(t_i)$ as $n_i \to \infty$, for each $t_i \in T_i$ and for $i = 1,2,\ldots,d$. Also suppose*

$$\sup_{n_i=1,2,\ldots,\, i=1,2,\ldots,d} ||f_{i,n_i}||_{T_i} \leq C \, (< \infty).$$

*Then*

$$\lim_{n_1,\ldots,n_d \to \infty} \int_{(A_i)} (f_{i,n_i}) d\Upsilon = \int_{(A_i)} (f_i) d\Upsilon$$

*for all $(A_i) \in \mathbf{X}\mathcal{S}_i$.*

Let $\mathcal{B}^{\Omega}(T_i)$ denote the smallest class of bounded scalar functions on $T_i$ containing $C_0(T_i)$, which is closed under the operation of pointwise limits of uniformly bounded sequences of functions. In other words, if $C_0(T_i) \subset \mathcal{C}$ and if, for $(f_n)_1^{\infty} \subset \mathcal{C}$ with $\sup_n ||f_n||_{T_i} < \infty$ and with $f_n(t_i) \to f(t_i)$ as $n \to \infty$, for each $t_i \in T_i$, it follows that $f \in \mathcal{C}$, then $\mathcal{B}^{\Omega}(T_i) \subset \mathcal{C}$. Thus $\mathcal{B}^{\Omega}(T_i)$ is the class of all bounded Baire functions on $T_i$, which also coincides with the family of all bounded Baire measurable scalar functions on $T_i$.

THEOREM 2.9. *Let* $\Upsilon : \overset{d}{\underset{1}{\mathbf{X}}} \sigma\mathcal{B}_0(T_i) \to Y$ *be a Baire d-multimeasure. Then:*

$$||\Upsilon||(T_i) = \sup\{|\int_{(A_i)} (f_i) d\Upsilon| : f_i \in C_0(T_i), ||f_i||_{A_i} \le 1, A_i \in \sigma\mathcal{B}_0(T_i)\}.$$

## 3. A Theorem of Uniqueness

The following theorem states that a $Y$-valued d-multimeasure $\Upsilon$ on $\mathbf{X}\sigma\mathcal{B}_0(T_i)$ is determined by the integrals $\int_{(T_i)}(f_i)d\Upsilon, (f_i) \in \mathbf{X}C_0(T_i)$.

THEOREM 3.1. *Suppose* $\Upsilon_1, \Upsilon_2 : \overset{d}{\underset{1}{\mathbf{X}}} \sigma\mathcal{B}_0(T_i) \to Y$ *are d-multimeasures such that*

$$\text{(1)} \qquad \int_{(T_i)} (f_i) d\Upsilon_1 = \int_{(T_i)} (f_i) d\Upsilon_2$$

*for all* $(f_i) \in \overset{d}{\underset{1}{\mathbf{X}}} C_0(T_i)$. *Then* $\Upsilon_1 = \Upsilon_2$.

***Proof.*** Let $C_i \in \mathcal{K}_0(T_i)$. By Theorem 55.B of Halmos [12] there exists a sequence $\{h_{i,n_i}\}_{n_i=1}^{\infty}$ in $C_0(T_i)$ such that $h_{i,n_i}(t_i) \searrow \chi_{C_i}(t_i)$, for each $t_i \in T_i$ and for $i$ = 1,2,...,d. Then by (1) and by LBCT (Theorem 2.8) we have $\Upsilon_1(C_1) = \Upsilon_2(C_2)$. Thus

$$\text{(2)} \qquad \Upsilon_1(C_1, ..., C_d) = \Upsilon_2(C_1, ..., C_d)$$

for $C_i \in \mathcal{K}_0(T_i)$, $i$ = 1,2,...,d.

Let

$$\Sigma_1 = \{E_1 \in \sigma\mathcal{B}_0(T_1) : \Upsilon_1(E_1, C_2, ..., C_d) = \Upsilon_2(E_1, C_2, ..., C_d)$$
$$\text{for}\, C_i \in \mathcal{K}_0(T_i), i = 2, 3, ..., d\}.$$

By (2) it follows that $\mathcal{K}_0(T_1) \subset \Sigma_1$. Consequently,by the separate finite additivity of $\Upsilon_1$ and $\Upsilon_2$ we conclude that $R(\mathcal{K}_0(T_1))$,the ring generated by

$\mathcal{K}_0(T_1)$ is contained in $\Sigma_1$.

Let $(E_n)_{n=1}^{\infty}$ be a monotone sequence in $\Sigma_1$, with $E = \lim\limits_n E_n$. Then by the separate $\sigma$-additivity of $\Upsilon_1$ and $\Upsilon_2$ we have

$$\begin{aligned}\Upsilon_1(E, C_2, ..., C_d) = \lim_n \Upsilon_1(E_n, C_2, ..., C_d) &= \lim_n \Upsilon_2(E_n, C_2, ..., C_d) \\ &= \Upsilon_2(E, C_2, ..., C_d)\end{aligned}$$

for each $C_i \in \mathcal{K}_0(T_i)$, $i$ = 2,3,...,d, since $E_n \in \Sigma_1$ for all $n$. Thus $E \in \Sigma_1$ and consequently, $\Sigma_1$ is a monotone class containing $R(\mathcal{K}_0(T_1))$. Then by Theorem 6.B of Halmos [12], $\Sigma_1$ coincides with $\sigma\mathcal{B}_0(T_1)$ and thus

$$\Upsilon_1(E_1, C_2, ..., C_d) = \Upsilon_2(E_1, C_2, ..., C_d) \tag{3}$$

for all $E_1 \in \sigma\mathcal{B}_0(T_1)$ and for all $C_i \in \mathcal{K}_0(T_i)$, $i$ = 2,3,...,d.

Now let

$$\begin{aligned}\Sigma_2 = \{E_2 \in \sigma\mathcal{B}_0(T_2) : \Upsilon_1(E_1, E_2, C_3, ..., C_d) = \Upsilon_2(E_1, E_2, C_3, ..., C_d) \\ \text{for all } E_1 \in \sigma\mathcal{B}_0(T_1) \text{ and for all } C_i \in \mathcal{K}_0(T_i),\ i = 3,...,\text{d} \}.\end{aligned}$$

By (3), $\mathcal{K}_0(T_2) \subset \Sigma_2$. By an argument similar to that given above for $\Sigma_1$, it is easy to show that $R(\mathcal{K}_0(T_2)) \subset \Sigma_2$ and that $\Sigma_2$ is a monotone class. Then by Theorem 6.B of Halmos [12] we conclude that $\Sigma_2 = \sigma\mathcal{B}_0(T_2)$. Continuing this argument step by step, in the d$^{\text{th}}$ step we have

$$\Upsilon_1(E_1, E_2, ...E_{d-1}, C_d) = \Upsilon_2(E_1, E_2, ..., E_{d-1}, C_d)$$

for all $E_i \in \sigma\mathcal{B}_0(T_i)$, $i$ = 1,2,...,d-1 and for $C_d \in \mathcal{K}_0(T_d)$, which shows that $\mathcal{K}_0(T_d) \subset \Sigma_d$, where

$$\begin{aligned}\Sigma_d = \{E_d \in \sigma\mathcal{B}_0(T_d) : \Upsilon_1(E_1, ..., E_{d-1}, E_d) = \Upsilon_2(E_1, ..., E_{d-1}, E_d) \\ \text{for } E_i \in \sigma\mathcal{B}_0(T_i), i = 1, 2, ..., d-1\}.\end{aligned}$$

Then, as in the above, $\Sigma_d$ is a monotone class containing $R(\mathcal{K}_0(T_d))$ and hence $\Sigma_d = \sigma\mathcal{B}_0(T_d)$. This shows that $\Upsilon_1 = \Upsilon_2$.

## 4. Direct Proof of Theorem 2 of Dobrakov [8]

With the preparation given in the earlier sections, we shall now present a direct proof of the said theorem of Dobrakov (Theorem 2 of [8]) and then deduce the multilinear extension theorem of Pelczyński [14] as a corollary.

THEOREM 4.1. *Let* $U : \mathop{\mathsf{X}}\limits_1^d C_0(T_i) \to Y$ *be a bounded d-linear operator. Suppose either* $(A)$ $c_0 \not\subset Y$, *or* $(B)$ $U$ *is weakly compact. Then there exists a unique d-multimeasure* $\Upsilon$ *on* $\mathop{\mathsf{X}}\limits_1^d \sigma\mathcal{B}_0(T_i)$ *with values in* $Y$ *such that*

$$U(f_i) = \int_{(T_i)} (f_i) d\Upsilon, \ \ (f_i) \in \mathop{\mathsf{X}}\limits_1^d C_0(T_i).$$

*If* $\hat{U}^d : \mathop{\mathsf{X}}\limits_1^d \mathcal{B}^{\Omega}(T_i) \to Y$ *is defined by*

$$\hat{U}^d(g_i) = \int_{(T_i)} (g_i) d\Upsilon, \ (g_i) \in \mathop{\mathsf{X}}\limits_1^d \mathcal{B}^{\Omega}(T_i)$$

*then* $\hat{U}^d$ *is well defined, bounded and d-linear. Moreover,* $\hat{U}^d$ *extends* $U$, $||\hat{U}^d|| = ||U|| = ||\Upsilon||(T_i)$ *and* $\hat{U}^d$ *satisfies the following property* $(P)$*:*

*Let* $(f_{i,n_i})_{n_i=1}^{\infty} \subset \mathcal{B}^{\Omega}(T_i)$ *with*

$$\sup_{n_i=1,2,\ldots} ||f_{i,n_i}||_{T_i} \leq C < \infty$$

*and with*

$$\lim_{n_i \to \infty} f_{i,n_i}(t_i) = f_i(t_i)$$

*for each* $t_i \in T_i$ *and for* $i = 1, 2, \ldots, d$. *Then*

$$\lim_{n_1, \ldots, n_d \to \infty} \hat{U}^d(f_{1,n_1}, \ldots, f_{d,n_d}) = \hat{U}^d(f_1, \ldots, f_d).$$

*If* $U$ *is weakly compact, then* $\hat{U}^d$ *is also weakly compact.*

*Finally, the bounded d-linear extension* $\hat{U}^d$ *is determined uniquely either by property* $(P)$ *or by the multilinear integral representation; the range of* $\Upsilon$ *is relatively weakly compact if and only if* $U$ *is weakly compact.*

*Proof.* Let us prove the theorem by induction on d. Let d=1. Suppose $c_0 \not\subset Y$. Let $\hat{T}_1$ be the Alexandroff compactification of $T_1$ by adjunction of the point $\{\infty\}$ and let $\hat{U} : C(\hat{T}_1) \longrightarrow \mathbf{K}$ be defined by $\hat{U}(f) = U(f - f(\infty))$. Then by Theorem VI.2.15 of [2],$\hat{U}$ is weakly compact and hence U $= \hat{U}|C_0(T_1)$ is weakly compact. Thus, in both the cases (A) and (B), by Lemma 2 of Kluvánek [13] there exists a unique $Y$-valued $\sigma$-additive regular Borel measure G on $\mathcal{B}(T_1)$ such that

$$Uf = \int_{T_1} f dG, \quad f \in C_0(T_1).$$

Let $\Upsilon = G|\sigma\mathcal{B}_0(T_1)$. Then by Theorem 8 of [3], $f$ is $\Upsilon$- integrable,

$$Uf = \int_{T_1} f d\Upsilon, \quad f \in C_0(T_1)$$

and by Theorem 1 of [8],$||U|| = ||\Upsilon||(T_1)$. Clearly, for the second adjoint $U^{**}$ of $U$ we have

$$U^{**}f = \int_{T_1} f d\Upsilon, \quad f \in \mathcal{B}^{\Omega}(T_1).$$

Now let us define $\hat{U}^1 = U^{**}|\mathcal{B}^{\Omega}(T_1)$ .Condition (P) holds in virtue of LBCT. As $\mathcal{B}^{\Omega}(T_1)$ is closed for pointwise limits of bounded sequences, it follows that property $(P)$ implies the uniqueness of the extension $\hat{U}^1$ of $U$ by an argument of transfinite induction. The integral representation of $U$ also determines $\hat{U}^1$ uniquely by Theorem 3.1. Moreover, if $U$ is weakly compact, then $U^{**}$ is weakly compact by the Gantmacher theorem and hence $\hat{U}^1$ is weakly compact.

Upto some stage we closely follow the proof of Pelczyński [14]. Suppose the result holds for d-1. For $(f_i) \in \mathop{\mathbf{X}}\limits_{1}^{d-1} C_0(T_i)$, let

$$U_{f_1,\ldots,f_{d-1}} : C_0(T_d) \longrightarrow Y$$

be given by

$$U_{f_1,\ldots,f_{d-1}}(f_d) = U(f_1, \ldots, f_d), \quad f_d \in C_0(T_d).$$

Then $U_{f_1,\ldots,f_{d-1}}$ is a bounded linear operator on $C_0(T_d)$ with values in $Y$. Then, by the case d=1 established above, for both the cases $(A)$ and $(B)$

there is a unique $\sigma$-additive vector measure

$$\Upsilon_{f_1,\dots,f_{d-1}} : \sigma\mathcal{B}_0(T) \to Y$$

such that

$$U_{f_1,\dots,f_{d-1}}(f_d) = \int_{T_d} f_d \, d\Upsilon_{f_1,\dots,f_{d-1}}, \quad f_d \in C_0(T_d).$$

Fixing $f_d$ in $C_0(T_d)$, let us define

$$U_{f_d} : \mathop{\mathbf{X}}_{1}^{d-1} C_0(T_i) \to Y$$

by

$$U_{f_d}(f_1, \dots, f_{d-1}) = U(f_1, \dots, f_d) \in Y.$$

Clearly, $U_{f_d}$ is a bounded (d-1)-linear mapping. When $U$ is weakly compact, clearly $U_{f_d}$ is weakly compact. Thus, when $c_0 \not\subset Y$, or when $U$ is weakly compact, the induction hypothesis implies that there is a (d-1)-multimeasure $\Upsilon_{f_d} : \mathop{\mathbf{X}}_{1}^{d-1} \sigma\mathcal{B}_0(T_i) \to Y$ such that

$$(1) \qquad U_{f_d}(f_1, \dots, f_{d-1}) = \int_{(T_i)_1^{d-1}} (f_1, \dots, f_{d-1}) d\Upsilon_{f_d}$$

for $(f_i)_1^{d-1} \in \mathop{\mathbf{X}}_{1}^{d-1} C_0(T_i)$. On the other hand,

$$(2) \qquad U_{f_d}(f_1, \dots, f_{d-1}) = U(f_1, \dots, f_d) = U_{f_1,\dots,f_{d-1}}(f_d) = \int_{T_d} f_d \, d\Upsilon_{f_1,\dots,f_{d-1}}.$$

Thus by (1) and (2)

$$(3) \qquad \int_{(T_i)_1^{d-1}} (f_1, \dots, f_{d-1}) \, d\Upsilon_{f_d} = \int_{T_d} f_d \, d\Upsilon_{f_1,\dots,f_{d-1}}$$

for all $f_i \in C_0(T_i)$, $i$ = 1,2,...,d.

For $g_d \in \mathcal{B}^{\Omega}(T_d)$, let us define

$$U_{g_d} : \mathop{\mathbf{X}}_{1}^{d-1} C_0(T_i) \to Y$$

by

$$(4) \qquad U_{g_d}(f_1, \ldots, f_{d-1}) = \int_{T_d} g_d \, d\Upsilon_{f_1,\ldots,f_{d-1}}.$$

Since $g_d$ is a bounded Baire measurable function, $U_{g_d}$ is well defined.Moreover,

$$\begin{aligned}
\textstyle\int_{T_d} f_d \, d\Upsilon_{f_1,\ldots,f_{i-1},\alpha f_i+\beta f_i',f_{i+1},\ldots,f_{d-1}} &= U(f_1, \ldots, f_{i-1}, \alpha f_i + \beta f_i', f_{i+1}, \ldots, f_d) \\
&= \alpha U(f_1, \ldots, f_d) + \beta U(f_1, \ldots, f_i', \ldots, f_d) \\
&= \alpha \int_{T_d} f_d \, d\Upsilon_{f_1,\ldots,f_{d-1}} + \\
&+ \beta \int_{T_d} f_d \, d\Upsilon_{f_1,\ldots,f_{i-1},f_i',f_{i+1},\ldots,f_{d-1}}
\end{aligned}$$

for $\alpha, \beta \in \mathbf{K}$ ,$f_i \in C_0(T_i)$, $i$ = 1,2,...,d. Then by Theorem 3.1 it follows that

$$\Upsilon_{f_1,\ldots,f_{i-1},\alpha f_i+\beta f_i',f_{i+1},\ldots,f_{d-1}} = \alpha \Upsilon_{f_1,\ldots,f_{d-1}} + \beta \Upsilon_{f_1,\ldots,f_{i-1},f_i',f_{i+1},\ldots,f_{d-1}}.$$

Using the above equality in (4), we conclude that $U_{g_d}$ is a (d-1)-linear operator.

We claim that $U_{g_d}$ is bounded. In fact,

$$\begin{aligned}
||U_{g_d}|| &= \sup\{|U_{g_d}(f_1, \ldots, f_{d-1})| \, : \, ||f_i||_{T_i} \leq 1, \, f_i \in C_0(T_i), \, 1 \leq i \leq d-1\} \\
&\leq ||g_d||_{T_d} \sup\{||\Upsilon_{f_1,\ldots,f_{d-1}}||(T_d) \, : \, ||f_i||_{T_i} \leq 1, \, f_i \in C_0(T_i), \, 1 \leq i \leq d-1\}.
\end{aligned}$$

Since $U_{f_1,\ldots,f_{d-1}}(f_d) = \int_{T_d} f_d \, d\Upsilon_{f_1,\ldots,f_{d-1}}$ by (2),

$$||U_{f_1,\ldots,f_{d-1}}|| = ||\Upsilon_{f_1,\ldots,f_{d-1}}||(T_d).$$

Therefore,

$$(5) \qquad \begin{aligned}
||U_{g_d}|| &\leq ||g_d||_{T_d} \sup\{|U_{f_1,\ldots,f_{d-1}}(f_d)| \, : \, ||f_i||_{T_i} \leq 1, \, f_i \in C_0(T_i)\} \\
&= ||g_d||_{T_d} ||U|| < \infty
\end{aligned}$$

If $c_0 \not\subset Y$, then by induction hypothesis there exists a (d-1)-multimeasure

$$\Upsilon_{g_d} : \mathop{\mathbf{X}}_1^{d-1} \sigma\mathcal{B}_0(T_i) \to Y$$

such that

$$U_{g_d}(f_1, ..., f_{d-1}) = \int_{(T_i)_i^{d-1}} (f_1, ..., f_{d-1}) d\Upsilon_{g_d} \tag{6A}$$

for each $g_d \in \mathcal{B}^\Omega(T_d)$, $f_i \in C_0(T_i)$, i$= 1, 2, ..., d-1$. Moreover, its unique bounded (d-1)-linear extension $\tilde{U}_{g_d}$ satisfying property (P) is given by

$$\tilde{U}_{g_d}(g_1, ..., g_{d-1}) = \int_{(T_i)_i^{d-1}} (g_1, ..., g_{d-1}) d\Upsilon_{g_d} \tag{7A}$$

for $g_i \in \mathcal{B}^\Omega(T_i)$,i$= 1, 2, ..., d-1$. Further, by induction hypothesis, $||U_{g_d}|| = ||\tilde{U}_{g_d}|| = ||\Upsilon_{g_d}||(T_i)_1^{d-1}$.

Suppose now $U$ is weakly compact. Let $B_i = \{f_i \in C_0(T_i) : ||f_i||_{T_i} \leq 1\}$. Then the range $U(B_1$ X...X $B_d)$ is convex and is relatively weakly compact in $Y$. Therefore, by Corollary V.3.14 of Dunford and Schwartz [11], the norm closure of $U(B_1$ X...X $B_d)$ is weakly compact. Let $K$ = closure of $U(B_1$ X...X $B_d)$.

We claim that $U_{g_d}$ is weakly compact, whenever $U$ is so. In fact, by (4) it suffices to prove the result for $||g_d||_{T_d} \leq 1$.

Let $||g_d||_{T_d} \leq 1$. Let $(f_{d,n})_{n=1}^{\infty} \subset B_d$ and let $\lim_n f_{d,n}(t) = g_d(t)$, for each $t \in T_d$. Let $f_i \in B_i$, $i$ = 1,2,...,d-1. Then by LBCT and by (3) and (4)

$$\begin{aligned} U_{g_d}(f_1, ..., f_{d-1}) &= \int_{T_d} g_d \, d\Upsilon_{f_1,...,f_{d-1}} \\ &= \lim_n \int_{T_d} f_{d,n} \, d\Upsilon_{f_1,...,f_{d-1}} \\ &= \lim_n \int_{(T_i)_1^{d-1}} (f_1, ..., f_{d-1}) d\Upsilon_{f_{d,n}} \\ &= \lim_n U(f_1, ..., f_{d-1}, f_{d,n}) \in K. \end{aligned}$$

Let $\Sigma = \{g_d \in \mathcal{B}^\Omega(T_d) : ||g_d||_{T_d} \leq 1$ and $U_{g_d}(B_1 X ... X B_{d-1}) \subset K\}$. By the above argument, the closed unit ball of the first Baire class $\mathcal{B}^1(T_d)$ of bounded functions is contained in $\Sigma$. By a usual argument of transfinite induction, and by applying LBCT as in the above, it can be shown that $\Sigma$ coincides with the closed unit ball of $\mathcal{B}^\Omega(T_d)$. Thus $U_{g_d}$ is a weakly compact operator for $g_d \in \mathcal{B}^\Omega(T_d)$ with $||g_d||_{T_d} \leq 1$ and consequently, for arbitrary $g_d \in \mathcal{B}^\Omega(T_d)$.

Thus in case $(B)$, by induction hypothesis, there is a (d-1)-multimeasure $\Upsilon_{g_d} : \overset{d-1}{\underset{1}{\mathbf{X}}} \sigma\mathcal{B}_0(T_i) \to Y$ such that

(6B) $$U_{g_d}(f_1, ..., f_{d-1}) = \int_{(T_i)_1^{d-1}} (f_1, ..., f_{d-1}) d\Upsilon_{g_d}$$

for each $g_d \in \mathcal{B}^\Omega(T_d)$, $f_i \in C_0(T_i)$, $i = 1,2,...,$d-1. Moreover, its unique bounded (d-1)-linear extension $\tilde{U}_{g_d}$ satisfying property $(P)$ is given by

(7B) $$\tilde{U}_{g_d}(g_1, ..., g_{d-1}) = \int_{(T_i)_1^{d-1}} (g_1, ..., g_{d-1}) d\Upsilon_{g_d}$$

for $g_i \in \mathcal{B}^\Omega(T_i)$, $i = 1,2,...,$d-1, in virtue of LBCT. Further, by induction hypothesis, $||U_{g_d}|| = ||\tilde{U}_{g_d}|| = ||\Upsilon_{g_d}||(T_i)_1^{d-1}$.

In the light of (6A) and (7A), or (6B) and (7B), the (d-1)-multimeasure $\Upsilon_{g_d}$ is well defined for $g_d \in \mathcal{B}^\Omega(T_d)$ and hereafter let us treat cases (A) and (B) simultaneously.

We define

$$U^d : C_0(T_1)\,X...X\,C_0(T_{d-1})X\mathcal{B}^{\Omega}(T_d) \to Y$$

by putting

$$U^d(f_1,...,f_{d-1},g_d) = U_{g_d}(f_1,...,f_{d-1}), \text{ for } g_d \in \mathcal{B}^{\Omega}(T_d).$$

Since $U_{g_d}$ is (d-1)-linear, $U^d$ is (d-1)-linear in the first (d-1) coordinates. Moreover, by (4) it is clear that $U^d$ is separately linear on the $d^{\text{th}}$ coordinate also. Therefore, $U^d$ is d-linear.

$$\begin{aligned}||U^d|| &= \sup\{|U_{g_d}(f_1,...,f_{d-1})| : ||f_i||_{T_i} \leq 1,\ ||g_d||_{T_d} \leq 1,\ f_i \in C_0(T_i),\ g_d \in \mathcal{B}^{\Omega}(T_d)\} \\ &= \sup\{||U_{g_d}|| : g_d \in \mathcal{B}^{\Omega}(T_d), ||g_d||_{T_d} \leq 1\} \\ &\leq ||U|| < \infty\end{aligned}$$

by (5). Thus $U^d$ is bounded. Moreover, since $U^d|\overset{d}{\underset{1}{\mathbf{X}}}\,C_0(T_i) = U$, it follows that $||U^d|| = ||U||$.

We define

$$\hat{U}^d : \mathcal{B}^{\Omega}(T_1)\,X...X\,\mathcal{B}^{\Omega}(T_d) \to Y$$

by putting

$$\hat{U}^d(g_1,...,g_d) = \int_{(T_i)_1^{d-1}} (g_1,...,g_{d-1})d\Upsilon_{g_d} = \tilde{U}_{g_d}(g_1,...,g_{d-1}).$$

By Theorem 2.6 and Difinition 2.7, the integral, and hence the operator $\hat{U}^d$, is well defined. By (6A) (resp. (6B)) $\hat{U}^d$ extends $U^d$. Obviously, $\hat{U}^d$ is separately linear in the first (d-1) coordinates.

Now

(8) $$\hat{U}^d(g_1,...,g_{d-1},\alpha g_d + \beta g'_d) = \int_{(T_i)_1^{d-1}} (g_1,...,g_{d-1})d\Upsilon_{\alpha g_d+\beta g'_d}$$

and

$$\alpha U^d(g_1, ..., g_{d-1}, g_d) + \beta \hat{U}^d(g_1, ..., g_{d-1}, g'_d) = \alpha \int_{(T_i)_1^{d-1}} (g_1, ..., g_{d-1}) d\Upsilon_{g_d} \tag{9}$$
$$+ \beta \int_{(T_i)_1^{d-1}} (g_1, ..., g_{d-1}) d\Upsilon_{g'_d}$$

for $g_i \in \mathcal{B}^{\Omega}(T_i)$, $i = 1,2,...,d$, and $g'_d \in \mathcal{B}^{\Omega}(T_d)$. Let $g_i = f_i \in C_0(T_i)$, for $i =$ 1,2,...,d-1. Then by (6A) (resp.(6B))

$$\begin{aligned} \hat{U}^d(f_1, ..., f_{d-1}, \alpha g_d + \beta g'_d) &= \int_{T_d} (\alpha g_d + \beta g'_d) d\Upsilon_{f_1,...,f_{d-1}} \\ &= \alpha \int_{(T_i)_1^{d-1}} (f_1, ..., f_{d-1}) d\Upsilon_{g_d} + \\ &+ \beta \int_{(T_i)_1^{d-1}} (f_1, ..., f_{d-1}) d\Upsilon_{g'_d} \\ &= \int_{(T_i)_1^{d-1}} (f_1, ..., f_{d-1}) d(\alpha \Upsilon_{g_d} + \beta \Upsilon_{g'_d}) \end{aligned} \tag{10}$$

Thus by Theorem 3.1, (8) and (10) we have

$$\alpha \Upsilon_{g_d} + \beta \Upsilon_{g'_d} = \Upsilon_{\alpha g_d + \beta g'_d}$$

and consequently, by (9) it follows that $\hat{U}^d$ is separately linear on the $d^{th}$ coordinate also. Thus $\hat{U}^d$ is d-linear.

$$\begin{aligned} ||\hat{U}^d|| &= \sup\{|\hat{U}^d(g_1, ..., g_d)| : ||g_i||_{T_i} \le 1,\ g_i \in \mathcal{B}^{\Omega}(T_i)\} \\ &= \sup\{|\hat{U}_{g_d}(g_1, ..., g_{d-1})| ||g_i||_{T_i} \le 1,\ g_i \in \mathcal{B}^{\Omega}(T_i),\ 1 \le i \le d\} \\ &= \sup\{||\tilde{U}_{g_d}|| : ||g_d||_{T_d} \le 1,\ g_d \in \mathcal{B}^{\Omega}(T_d)\} \\ &= \sup\{||U_{g_d}|| : ||g_d||_{T_d} \le 1,\ g_d \in \mathcal{B}^{\Omega}(T_d)\} \\ &\le ||U|| < \infty \end{aligned}$$

by Theorem 2.9 and by (5). Thus $\hat{U}^d$ is bounded. Since $\hat{U}^d$ extends $U^d$ and $||U^d|| = ||U||$, we conclude that $||\hat{U}^d|| = ||U||$.

We define $\Upsilon(.) : \overset{d}{\underset{1}{\mathbf{X}}}\, \sigma\mathcal{B}_0(T_i) \to Y$ by putting

$$\Upsilon(A_i) = \Upsilon_{\chi_{A_d}}(A_1, ..., A_{d-1}).$$

Since $\chi_{A_d} \in \mathcal{B}^{\Omega}(T_d)$,

$$\Upsilon(A_i) = \int_{(T_i)_1^{d-1}} (\chi_{A_1}, ..., \chi_{A_{d-1}}) d\Upsilon_{\chi_{A_d}} = \hat{U}^d(\chi_{A_1}, ..., \chi_{A_d}) \in Y$$

and is well defined. By (4) and (6A) (resp. (6B))

$$\int_{T_d} \chi_{A_d} d\Upsilon_{f_1,...,f_{d-1}} = \int_{(T_i)_1^{d-1}} (f_1, ..., f_{d-1}) d\Upsilon_{\chi_{A_d}} \tag{11}$$

for $f_i \in C_0(T_i)$, $i$ = 1,2,...,d-1. By induction hypothesis, $\Upsilon_{\chi_{A_d}}$ is a (d-1)-multimeasure on $\overset{d-1}{\underset{1}{\mathbf{X}}}\, \sigma\mathcal{B}_0(T_i)$ and hence $\Upsilon$ is separately $\sigma$-additive in the first d-1 coordinates. To show that $\Upsilon$ is separately $\sigma$-additive on the d$^{\text{th}}$ coordinate also, let us extend (11) to $(g_i)_1^{d-1}$ in $\overset{d-1}{\underset{1}{\mathbf{X}}}\, \mathcal{B}^{\Omega}(T_i)$.

Let $(f_{i,n})_{n=1}^{\infty} \subset C_0(T_i)$ be bounded and let $\lim_n f_{i,n}(t_i) = g_i(t_i)$, for each $t_i \in T_i$, $i$ = 1,2,...,d-1. Then by LBCT and by (11)

$$\begin{aligned} \lim_{n\to\infty} \int_{T_d} \chi_{A_d} d\Upsilon_{f_{1,n},...,f_{d-1,n}} &= \lim_n \int_{(T_1,...,T_{d-1})} (f_{1,n}, ..., f_{d-1,n}) d\Upsilon_{\chi_{A_d}} \\ &= \int_{(T_1,...,T_{d-1})} (g_1, ..., g_{d-1}) d\Upsilon_{\chi_{A_d}} \in Y. \end{aligned}$$

Thus, for the sequence of $\sigma$-additive set functions $\{\Upsilon_{f_{1,n},f_{2,n},...,f_{d-1,n}}\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \Upsilon_{f_{1,n},...,f_{d-1,n}}(A_d) \text{ exists in Y}$$

for each $A_d \in \sigma\mathcal{B}_0(T_d)$. Since Theorem I.4.8 of [2] is valid for $\sigma$ -rings too and since the uniform $\sigma$-additivity is the same as uniform strong additivity on $\sigma$-rings for $\sigma$-additive vector measures, it follows that there exists a $\sigma$-additive Y-valued measure $\Upsilon_{g_1,...,g_{d-1}}$ (say) on $\sigma\mathcal{B}_0(T_d)$ such that

$$\lim_{n\to\infty} \Upsilon_{f_{1,n},...,f_{d-1,n}}(A_d) = \Upsilon_{g_1,...,g_{d-1}}(A_d)$$

and the measure $\Upsilon_{g_1,...,g_{d-1}}$ depends solely on $g_1, ..., g_{d-1}$ and is independent of the converging sequences $\{f_{i,n}\}_{n=1}^{\infty}$ , $i$ = 1,2,...,d-1. Then

$$
\begin{aligned}
\Upsilon_{g_1,\dots,g_{d-1}}(A_d) &= \int_{T_d} \chi_{A_d} d\Upsilon_{g_1,\dots,g_{d-1}} \\
&= \lim_n \int_{T_d} \chi_{A_d} d\Upsilon_{f_{1,n},\dots,f_{d-1,n}} \\
&= \lim_n \int_{(T_1,\dots,T_{d-1})} (f_{1,n},\dots,f_{d-1,n}) d\Upsilon_{A_d} \\
&= \int_{(T_1,\dots,T_{d-1})} (g_1,\dots,g_{d-1}) d\Upsilon_{A_d}
\end{aligned}
$$

by (11) and by LBCT. Thus (11) holds with $g_i$ in place of $f_i$, for $i = 1,2,\dots,d-1$. This shows that (11) is valid for $(g_i)_1^{d-1} \in \mathop{\mathbf{X}}_1^{d-1} \mathcal{B}^1(T_i)$, where $\mathcal{B}^1(T_i)$ is the first Baire class of bounded functions on $T_i$. Assuming the validity of (11) for $(g_i)_1^{d-1} \in \mathop{\mathbf{X}}_1^{d-1} \mathcal{B}^\beta(T_i)$ for all ordinals $\beta$ strictly less than a countable ordinal $\alpha$, one can show by the above argument that the equality (11) holds for all $(g_i)_1^{d-1} \in \mathop{\mathbf{X}}_1^{d-1} \mathcal{B}^\alpha(T_i)$. Now, by transfinite induction we conclude that (11) holds for all $(g_i)_1^{d-1} \in \mathop{\mathbf{X}}_1^{d-1} \mathcal{B}^\Omega(T_i)$.

Thus $\Upsilon_{g_1,\dots,g_{d-1}} : \sigma\mathcal{B}_0(T_d) \to Y$ is well defined and is $\sigma$-additive for each $(g_i)_1^{d-1} \in \mathop{\mathbf{X}}_1^{d-1} \mathcal{B}^\Omega(T_i)$. Consequently, for $(A_i) \in \mathop{\mathbf{X}}_1^{d-1} \sigma\mathcal{B}_0(T_i)$,

$$
\begin{aligned}
\Upsilon(A_i) &= \Upsilon_{\chi_{A_d}}(A_1,\dots,A_{d-1}) \\
&= \int_{(T_i)_1^{d-1}} (\chi_{A_1},\dots,\chi_{A_{d-1}}) d\Upsilon_{\chi_{A_d}} \\
&= \int_{T_d} \chi_{A_d} d\Upsilon_{\chi_{A_1},\dots,\chi_{A_{d-1}}} \\
&= \Upsilon_{\chi_{A_1},\dots,\chi_{A_{d-1}}}(A_d)
\end{aligned} \tag{12}
$$

because of the validity of (11) for $(\chi_{A_i})_1^{d-1}$. Therefore, $\Upsilon$ is separately $\sigma$-additive on the d$^{\text{th}}$ coordinate too and thus $\Upsilon$ is a d-multimeasure on $\mathop{\mathbf{X}}_1^{d} \sigma\mathcal{B}_0(T_i)$.

By (12) and by the definition of $\hat{U}^d$ we have

$$\hat{U}^d(\chi_{A_1},...,\chi_{A_d}) = \int_{(T_i)_1^{d-1}} (\chi_{A_1},...,\chi_{A_{d-1}}) d\Upsilon_{\chi_{A_d}}$$

$$(13) \qquad = \Upsilon(A_1,...,A_d)$$

$$= \int_{(T_i)_1^d} (\chi_{A_1},...,\chi_{A_d}) d\Upsilon$$

for $A_i \in \sigma\mathcal{B}_0(T_i)$, $i = 1,2,...,d$. Fixing $A_1,...,A_{d-1}$, and replacing $\chi_{A_d}$ by a $\sigma\mathcal{B}_0(T_d)$ - simple function $s$, by the separate linearity of $\hat{U}^d$ and the definition of the integral we deduce from (13) that

$$\hat{U}^d(\chi_{A_1},...,\chi_{A_{d-1}}, s) = \int_{(T_i)_1^d} (\chi_{A_1},...,\chi_{A_{d-1}}, s) d\Upsilon.$$

Since each $g_d \in B^\Omega(T_d)$ is the uniform limit of a sequence of $\sigma\mathcal{B}_0(T_d)$ - simple functions, it then follows by LBCT that

$$\hat{U}^d(\chi_{A_1},...,\chi_{A_{d-1}}, g_d) = \int_{(T_i)_1^d} (\chi_{A_1},...,\chi_{A_{d-1}}, g_d) d\Upsilon.$$

Similarly, replacing $\chi_{A_{d-1}}$ by $g_{d-1} \in B^\Omega(T_{d-1})$ and keeping $\chi_{A_1},...,\chi_{A_{d-2}}, g_d$ fixed, it can be shown that

$$\hat{U}^d(\chi_{A_1},...,\chi_{A_{d-2}}, g_{d-1}, g_d) = \int_{(T_i)_1^d} (\chi_{A_1},...,\chi_{A_{d-2}}, g_{d-1}, g_d) d\Upsilon$$

for $g_{d-1} \in B^\Omega(T_{d-1})$. Proceeding step by step, finally it follows that

$$\hat{U}^d(g_1,...,g_{d-1}, g_d) = \int_{(T_i)_1^d} (g_1,...,g_{d-1}, g_d) d\Upsilon$$

for $g_i \in B^\Omega(T_i)$, $i = 1,2,...,d$.

Since $\hat{U}^d | \mathop{\mathsf{X}}_1^d C_0(T_i) = U$,

$$(14) \qquad U(f_i) = \int_{(T_i)} (f_i) d\Upsilon, \quad (f_i) \in \mathop{\mathsf{X}}_1^d C_0(T_i).$$

Property $(P)$ holds for $\hat{U}^d$ by LBCT.

By Theorem 3.1 and by (14), $\Upsilon$ is determined uniquely by $U$. Consequently, the operator $\hat{U}^d$ is also determined uniquely by $U$. If $\hat{U}$ is a bounded d-linear extension of U to $\mathop{\mathbf{X}}\limits_{1}^{d} B^{\Omega}(T_i)$, satisfying property $(P)$, then by an argument of transfinite induction one can show that property $(P)$ implies $\hat{U} = \hat{U}^d$.

Now, let us show that $U$ is weakly compact if and only if the range of $\Upsilon$ is relatively weakly compact. If $U$ is weakly compact, then let $B_1, ..., B_d$ and $K$ be as in the above, where we proved the weak compactness of the operator $U_{g_d}$ in case (B). If $(f_{i,n})_{n=1}^{\infty} \subset C_0(T_i)$ are bounded sequences with $||f_{i,n}||_{T_i} \leq 1$, and if $f_{i,n}(t_i) \to g_i(t_i)$ for each $t_i \in T_i$, then by LBCT

$$\hat{U}^d(g_1, ..., g_d) = \lim_n \int_{(T_i)_1^d} (f_{i,n}) d\Upsilon \in K.$$

By an argument of transfinite induction it then follows that

$$\{\hat{U}^d(g_1, ..., g_d) : g_i \in B^{\Omega}(T_i), ||g_i||_{T_i} \leq 1, i = 1, 2, ..., d\} \subset K$$

and hence $\hat{U}^d$ is weakly compact. Consequently, the range of $\Upsilon$, being contained in $K$, is relatively weakly compact.

If the range of $\Upsilon$ is relatively weakly compact, then by the argument given on p.292 of [8] we conclude that $\hat{U}^d$ and hence $U$, is weakly compact.

This completes the proof.

COROLLARY 4.2. *Pelczyński's theorem [14] on multilinear extension (see Introduction) holds also for multilinear operators on* $\mathop{\mathbf{X}}\limits_{1}^{d} C_0(T_i)$.

***Proof.*** If we define $U^{**} = \hat{U}^d$, where $\hat{U}^d$ is as in Theorem 4.1, then $U^{**}$ is the required bounded d-linear extension of $U$ as in Pelczyński's theorem.

COROLLARY 4.3. *Condition (ii) in Pelczyński's theorem (in Introduction) is the same as condition $(P)$ given in Theorem 4.1.*

*Proof.* Clearly, condition $(P)$ of Theorem 4.1 implies condition (ii) of Pelczyński's theorem, since $U^{**} = \hat{U}^d$ by Corollary 4.2. Conversely, if condition (ii) of Pelczyński's theorem holds, then as shown by Dobrakov [8],

$$\Upsilon(A_1, ..., A_d) = U^{**}(\chi_{A_1}, ..., \chi_{A_d})$$

is a separately $\sigma$-additive d-multimeasure on $\overset{d}{\underset{1}{\mathbf{X}}} \mathcal{B}^{\Omega}(T_i)$ and

$$U^{**}(g_i) = \int_{(T_i)_1^d} (g_1, ..., g_d) d\Upsilon.$$

Then $U^{**}$ satisfies property $(P)$ by LBCT of multimeasures.

## 5. Concluding Remarks

The range of a $\sigma$-additive Banach space valued measure defined on a $\sigma$-ring of sets is relatively weakly compact. This result is essentially due to Bartle, Dunford and Schwartz [1]. This result implies that, for a $\sigma$-additive vector measure $G(.) : \sigma\mathcal{B}_0(T) \to Y$, $Y$ a Banach space, the operator $U : \mathcal{B}^{\Omega}(T) \to Y$ given by

$$Uf = \int_T f dG$$

is weakly compact, where $T$ is a locally compact Hausdorff space.

Since there are examples of non weakly compact multilinear operators (see p. 385 of [14]) from $C(S)$ into a Hilbert space , $S$ a compact Hausdorff space, the integral representation of a bounded multilinear operator $U$ on $\overset{d}{\underset{1}{\mathbf{X}}} C_0(T_i)$, with range in a Banach space $Y$ not containing $c_0$, does not guarantee the weak compactness of the operator $U$. Consequently, by Theorem 4.1, the range of the associated multimeasure of $U$ is not relatively weakly compact.

Thus the integral representation of a multilinear operator $U$ on $\overset{d}{\underset{1}{\mathbf{X}}} C_0(T_i)$ with respect to a d-multimeasure $\Upsilon$ on $\overset{d}{\underset{1}{\mathbf{X}}} \sigma\mathcal{B}_o(T_i)$ with values in $Y$ does not imply that the multilinear operator $U$ is weakly compact. This is contrary to the situation of bounded linear opearators on $C_0(T)$.This observation has motivated our recent note [10].

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