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Notas de Matemática
Serie: Pre-Print
No. 165

Mérida - Venezuela
1997

On Weakly Compact Operators on $C_o(T)$

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June 16, 1997

Abstract

Let T ba a locally compact Hausdorff space and let $C_o(T) = \{f : T \to \mathbb{C} | f$ is continuous and vanishes at infinity$\}$ be provided with the supremum norm. Let X be a quasicomplete locally convex Hausdorff space. Suppose $u : C_o(T) \to X$ is a continuous linear operator. By refining the method adopted by Grothendick in [6] and by combining the integration technique of Bartle-Dunford-Schwartz [1], are obtained 32 characterizations for the operator u to be weakly compact, several of which are new. The present method is so powerful as to deduce the isolated result of Dinculeanu and Kluvánek on the regular Borel extension of σ-additive locally convex space valued Baire measures as corollary of the main characterization theorem. Also is included an elegant proof of the range theorem of Tweddle on σ-additive vector measures.

1. Introduction.

For a locally compact Hausdorff space T, let $C_o(T)$ be the Banach space of all continuous complex functions vanishing at infinity in T, endowed with the supremum norm. Then its dual $M(T)$ is the Banach space of all bounded complex Radon measures on T, with $||\mu|| = var(\mu, T)$ for $\mu \in M(T)$. Let X be a quasicomplete locally convex Hausdorff space (briefly,a quasicomplete lcHs) and let $u : C_o(T) \to X$ be a continuous linear mapping. Grothendick gave in [6] some necessary and sufficient conditions for u to be weakly compact.

Grothendieck studied in [6] some topological and range properties of the adjoint u^* and the biadjoint u^{**} of the continuous linear operator $u : C_o(T) \to X$, proved that $C_o(T)$ has the strict Dunford-Pettis property (briefly, strict D.P.P.), characterized weakly compact subsets of $M(T)$ and proved another deep result (Theorem 3 of [6]).He used all these results to characterize weakly compact operators u on $C_o(T)$.

Most of the results obtained in Sections 1.1,1.2,1.3, 2.1 and 3.1 of [6] play a key role in the proof of the said characerization theorem. Moreover, the characterization theorem is proved in [6] only for the space $C(K)$ with K compact Hausdorff and is remarked that the results hold also for $C_o(T)$ with T locally compact and Hausdorff. Later, the results of Grothendieck [6] were proved in detail in Sections 4.21,4.22, 9.1-9.4 in Edwards [5],where the said characterization theorem is proved for $C_o(T)$, T locally compact and Hausdorff. But we would like to bring to the attention of the reader that the proof of (3) $\Rightarrow$ (2 bis) of Theorem 9.4.10 of [5],which is the charcterization theorem of Grothendieck, needs justification and hence the proof of the locally compact case remains unestablished in [5].

*Supported by the C.D.C.H.T. project C-586 of the Universidad de los Andes, Mérida, Venezuela, and by the international cooperation project between CONICIT-Venezuela and CNR-Italy.

Grothendieck also proved in [6] that there exists a bijective correspondence between the set of all σ-additive X-valued Baire measures on K and the set of all weakly compact operators $u : C(K) \to X$, where K is a compact Hausdorff space and X is a complete lcHs. However, he did not develop the theory of vector measures to explore further in this direction.

Later, in 1955, Bartle-Dunford-Schwartz [1] characterized first the relatively weakly compact subsets of $ca(\Sigma)$, the Banach space of all σ-additive complex measures μ defined on the σ-algebra Σ of subsets of a set $\Omega(\neq \emptyset)$, endowed with the norm $||\mu|| = var(\mu, \Omega)$. Then they developed a theory of integration for scalar functions with respect to a σ-additive Banach space valued vector measure on the σ-algebra Σ. Using this theory and the characterization of relatively weakly compact subsets of $ca(\Sigma)$, they characterized weakly compact operators $u : C(K) \to X$, where K is a compact Hausdorff space and X is a Banach space. Further, they also deduced that the space $C(K)$ has strict D.P.P..Moreover, other results of Grothendieck [6] dependent on the strict D.P.P. of $C(K)$ are now immediate due to the validity of the Lebesgue bounded convergence theorem for integrals with respect to a σ-additive vector measure.

In the sixties, Dinculeanu and Kluvánek developed extensively the theory of σ-additive vector measures. In particular, they studied the regularity property of σ-additive vector valued Baire measures in a locally compact Hausdorff space T with values in a lcHs and showed that they admit a unique regular σ-additive Borel extension (see [4,7]). Later, first Tweddle [16] and then Kluvanek [8], by different methods, proved that the range of a σ-additive vector measure defined on a σ-algebra of sets with values in a lcHs is relatively weakly compact. The reader may note that the latter result was first proved by Bartle-Dunford-Schwartz in [1] for Banach space valued σ-additive vector measures. by a method more direct and natural than that adopted by Tweddle [16] or by Kluvánek [8] for the lcHs case.

In the monograph of Diestel and Uhl [2] is given a synthesis of the methods employed by Grothendieck [6] and Bartle-Dunford-Schwartz [1] to present the important results in the study of weakly compact operators $u : C(\Omega) \to X$, including their characterizations, where Ω is a compact Hausdorff space and X is a Banach space.

In the present work, refining the method adopted by Grothendieck [6] and combining it with the integration technique of [1], we obtain various characterizations of weakly compact operators $u : C_o(T) \to X$, where X is a quasicomplete lcHs. In fact, first we characterize weakly compact subsets of $M(T)$ in terms of the Baire and σ-Borel restrictions of their members. Then combining this result along with Lemmas 1 and 2 and Theorem 2 of Grothendieck [6] (we emphasise the fact that no other result of [6] is used) and the integration of bounded measurable scalar functions with respect to a σ-additive vector measure with values in a lcHs, the characterizations of Grothendieck [6] as well as several new characterizations of quasicomplete lcHs-valued weakly compact operators on $C_o(T)$ are obtained. Also is given a generalization of Proposition 8 of Dieudonné [3] to locally compact spaces, which is more natural than that of Grothendieck [6].

We would like to bring to the attention of the reader that our proofs are directly given for the locally compact case without any reduction to the compact case or to the compact metrizable case as is done in Grothendieck [6]. In other words, our approach is quite natural. Moreover, the advantage of the present study is that it permits us to deduce the above said isolated results of Dinculeanu and Kluvánek on σ-additive vector valued Baire measures as corollaries.Also, as in Bartle-Dunford-Schwartz [1], we deduce that $C_o(T)$ has strict D.P.P. so that all the results of Grothendieck [6] based on this property of $C_o(T)$ are now immediate.Finally, combining Lemmas 1 and 2 of [6] with a result of [1] we also provide an elegant proof of the range theorem of Tweddle [16].

2. Preliminaries.

In this section we fix notation and terminology and also give some definitions and results from [5,6].

Let T be a locally compact Hausdorff space and let $C_o(T)$ be the Banach space of all complex continuous functions vanishing at infinity in T, endowed with the supremum norm $||\cdot||_T$. The dual of $C_o(T)$ is the Banach space $M(T)$ of all bounded complex Radon measures on T, with $||\mu|| = var(\mu, T)$ for $\mu \in M(T)$. $\mathcal{K}(T)$ (resp. $\mathcal{K}_o(T)$) is the class of all compact subsets (resp. compact G_δ subsets) of T. $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$,$\mathcal{B}_o(T)$) is the σ-ring generated by the class of all closed subsets of T (resp. by $\mathcal{K}(T)$, by $\mathcal{K}_o(T)$) and thus $\mathcal{B}(T)$ (resp. $\mathcal{B}_c(T)$, $\mathcal{B}_o(T)$) is the σ-algebra (resp. σ-ring) of the Borel (resp. the σ-Borel, the Baire) sets in T.

A vector measure is an additive set function defined on a ring of sets with values in a lcHs. In the sequel X denotes a lcHs with topology τ.

The strong topology $\beta(X^*, X)$ of X^* is the locally convex topology induced by the seminorms $\{p_B : B \text{ bounded in} X\}$, where $p_B(x^*) = \sup_{x\in B} |x^*(x)|$. X^{**} denotes the dual of $(X^*, \beta(X^*, X))$ and is endowed with the locally convex topology τ_e of uniform convergence in equicontinuous subsets of X^*.

It is well known that the canonical injection $J : X \to X^{**}$ given by $< Jx, x^* >=< x, x^* >$ for all $x \in X$ and $x^* \in X^*$, is linear. On identifying X with $JX \subset X^{**}$, one has $\tau_e|JX = \tau$.

A linear mapping $u : C_o(T) \to X$ is called a weakly compact operator on $C_o(T)$ if $\{uf : ||f||_T \leq 1\}$ is relatively weakly compact in X. If $u : C_o(T) \to X$ is linear and continuous, then $u^* : X^* \to C_o^*(T) = M(T)$ given by

$$< u^*x^*, f >=< x^*, uf >= x^*(uf)$$

for all $x^* \in X^*$ and $f \in C_o(T)$, is well defined, linear and continuous, when X^* is given the topology $\beta(X^*, X)$ and $M(T)$ the usual norm topology. Similarly, the mapping $u^{**} : C_o^{**}(T) \to X^{**}$ is defined by

$$< u^{**}(w), x^* >=< w, u^*x^* >$$

for all $w \in C_o^{**}(T)$ and $x^* \in X^*$. It is well known that u^{**} is well defined and linear and moreover,

$$u^{**} : (C_o^{**}(T), ||\cdot||) \to (X^{**}, \tau_e)$$

is continuous. See Edwards [5] for details.

In order to compare our method of proof with that of Grothendieck (see Section 5) we state the following results from [6].

THEOREM (A). *Let E and F be lcHs with F quasicomplete. If $u : E \to F$ is linear and continuous, then u maps bounded subsets of E into relatively weakly compact subsets of F if and only if $u^*(A)$ is relatively $\sigma(E^*, E^{**})$-compact in E^* for each equicontinuous subset A of F^*.*

THEOREM (B). *Let V be the vector space spanned by the characteristic functions χ_F, where F runs through all closed G_δ subsets of T. Then a bounded set A in $M(T)$ is relatively weakly compact if and only if A is relatively compact in $\sigma(M(T), V)$-topology.*

THEOREM (C). *$C_o(T)$ has strict D.P.P.. That is, if $u : C_o(T) \to X$, X a quasicomplete lcHs, is a weakly compact operator, then for every weakly Cauchy sequence (x_n) in $C_o(T)$, the sequence (ux_n) is convergent in the topology of X.*

THEOREM (D). *Let $u : C_o(T) \to X$ be a continuous linear mapping, where X is a quasi-complete lcHs. Then the following statements are equivalent:*

(a) u is weakly compact.

(b) $u^{**}(\chi_A) \in X$ *for all closed sets* A *in* X.

(c) $u^{**}(\chi_A) \in X$ *for all closed* G_δ *sets* A *in* X.

(d) u *transforms non decreasing sequences of non negative functions in* $C_o(T)$ *bounded by 1 into weakly convergent sequences in* X.

3. Main results.

Since a detailed exposition of the present work will be quite long, we shall just state the principal results, omitting their proof. Interested reader may refer to [11, 12, 13] for details.

DEFINITION 1. *Let* $\mathcal{R}$ *be a* σ*-ring of subsets of* T, *containing* $\mathcal{K}_o(T)$ *or* $\mathcal{K}(T)$. *Let* X *be a lcHs with topology* τ *and let* $m : \mathcal{R} \to X$ *be a vector measure. Then* m *is said to be* $\mathcal{R}$*-regular or simply regular (resp.* $\mathcal{R}$*- outer regular or simply outer regular,* $\mathcal{R}$*-inner regular or simply inner regular) in* $E \in \mathcal{R}$ *if, given* $\varepsilon > 0$ *and a* τ*-continuous seminorm* p *in* X, *there exists a compact set* $K \in \mathcal{R}$ *and an open set* $U \in \mathcal{R}$ *with* $K \subset E \subset U$ *(resp.there exists an open set* $U \in \mathcal{R}$ *with* $E \subset U$, *there exists a compact set* $K \in \mathcal{R}$ *with* $K \subset E$*) such that for each* $B \in \mathcal{R}$ *with* $B \subset U \backslash K$ *(resp. with* $B \subset U \backslash E$, *with* $B \subset E \backslash K$*),* $p(m(B)) < \varepsilon$. *Even though* T *does not belong to* $\mathcal{R}$ *one can define* $\mathcal{R}$*-inner regularity of* m *in* T *on similar lines. The vector measure* m *is said to be* $\mathcal{R}$*- regular or simply regular (resp.* $\mathcal{R}$*-outer regular or simply outer regular,* $\mathcal{R}$*-inner regular or simply inner regular) if it is so in each* $E \in \mathcal{R}$. *When* $\mathcal{R} = \mathcal{B}(T)$ *(resp.* $\mathcal{B}_c(T)$, $\mathcal{B}_o(T)$*), we use the terminolgy Borel (resp.* σ*-Borel, Baire) regularity or outer regularity or inner regularity.*

LEMMA 1. *Let* $\mu \in M(T)$ *and let* $|\mu|(\cdot) = var(\mu, (\cdot))$ *in* $\mathcal{B}(T)$. *Then*

$$|\mu||_{\mathcal{B}_o(T)} = var(\mu|_{\mathcal{B}_o(T)}, (\cdot)) \ \ and \ \ |\mu||_{\mathcal{B}_c(T)}(\cdot) = var(\mu|_{\mathcal{B}_c(T)}, (\cdot)).$$

Notation. For $\mu \in M(T)$, let $|\mu|(\cdot) = var(\mu, (\cdot))$ in $\mathcal{B}(T)$.

THEOREM 1. *Let* A *be a bounded set in* $M(T)$. *Then the following statements are equivalent:*

(i) A *is relatively weakly compact.*

(ii) For each disjoint sequence (U_i) *of open Baire sets in* T, $\lim_i \mu(U_i) = 0$ *uniformly in* $\mu \in A$.

(iii) *a) For each open Baire set* U *in* T *and for each* $\varepsilon > 0$, *there exists* $K \in \mathcal{K}_o(T)$ *with* $K \subset U$ *such that*

$$\sup_{\mu \in A} |\mu| (U \backslash K) < \varepsilon.$$

b) For each $\varepsilon > 0$, *there exists a* $K \in \mathcal{K}_0(T)$ *such that*

$$\sup_{\mu \in A} |\mu|(T \backslash K)) < \varepsilon.$$

(iv) $A|_{\mathcal{B}_o(T)}$ *is uniformly Baire regular in the sense that ,given* $E \in \mathcal{B}_o(T)$ *and* $\varepsilon > 0$, *there exists a compact* $K \in \mathcal{K}_o(T)$ *and an open Baire set* U *in* T *with* $K \subset E \subset U$ *such that*

$$\sup_{\mu \in A} |\mu|(U \backslash K) < \varepsilon.$$

COROLLARY 1 (Generalization of Proposition 8 of [3]) *A bounded sequence* (μ_i) *in* $M(T)$ *is weakly convergent if and only if for each open Baire set* U *in* T, $\lim_i \mu_i(U)$ *exists in* $\mathbb{C}$.

Remark 1. When T is compact, the proof of Proposition 9 in [3] holds to show that the hypothesis that $\lim_i \mu_i(U)$ exists in $\mathbb{C}$ for each open set U in T implies that (μ_i) is bounded. When T is locally compact and Hausdorff,arguing with the one-point compactification of T as on p.177 of Thomas [15], one can show that the above hypothesis ensures the boundedness of (μ_i). Again, when T is compact, using Theorem 50.D of P.R.Halmos,Measure Theory, Van Nostrand,1950,and the Baire regularity of $\mu_i|_{\mathcal{B}_o(T)}$ we can modify the proof of Proposition 9 in [3] to show that (μ_i) is bounded when $\lim_i \mu_i(O)$ exists in $\mathbb{C}$ for each open Baire set O in T. However, when T is locally compact and not compact,we do not know whether the boundedness condition can be dispensed with in the above corollary. When T is metrizable and compact, $\mathcal{B}(T) = \mathcal{B}_o(T)$ and hence the above corollary reduces to Proposition 8 of Dieudonné [3]. Thus the present generalization is more natural than that of Grothendieck on p.150 of [6].

THEOREM 2. *Let* A *be a bounded set in* $M(T)$. *Then the following statements are equivalent:*

(i) A *is relatively weakly compact.*

(ii) For each disjoint sequence (U_i) *of* σ*-Borel open sets in* T, $\lim_i \mu(U_i) = 0$ *uniformly in* $\mu \in A$.

(iii) a) *For each* σ*-Borel open set* U *in* T *and for each* $\varepsilon > 0$, *there exists* $K \in \mathcal{K}(T)$ *with* $K \subset U$ *such that*

$$\sup_{\mu \in A} |\mu|(U \backslash K) < \varepsilon.$$

b) *For each* $\varepsilon > 0$, *there exists a compact* K *in* T *such that*

$$\sup_{\mu \in A} |\mu|(T \backslash K)) < \varepsilon.$$

(iv) $A|_{\mathcal{B}_c(T)}$ *is uniformly* σ*-Borel regular, in the sense that, given* $\varepsilon > 0$ *and a set* $E \in \mathcal{B}_c(T)$,*there exists a* σ*-Borel open set* U *and a compact set* K *in* T *with* $K \subset E \subset U$ *such that*

$$\sup_{\mu \in A} |\mu|(U \backslash K) < \varepsilon.$$

THEOREM 3. *Let* $u : C_0(T) \to X$ *be a continuous linear operator. Then there exists a vector measure* m *in* $\mathcal{B}(T)$ *with values in* X^{**} *such that the following hold:*

(i) $x^*m \in M(T)$ *for all* $x^* \in X^*$. *Thus* m *is* σ*-additive in* $\sigma(X^{**}, X^*)$*-topology.*

(ii) The mapping $x^* \to x^*m$ *of* X^* *in* $M(T)$ *is weak** *- weak** *continuous.*

(iii) $x^*uf = \int_T f dx^*m$ *for all* $f \in C_o(T)$ *and* $x^* \in X^*$.

(iv) $\{m(E) : E \in \mathcal{B}(T)\}$ *is* τ_e*-bounded in* X^{**}.

Conversely, if $m : \mathcal{B}(T) \to X^{**}$ *is a vector measure which satisfies (i) and (ii), then there exists a unique continuous linear transformation* $u : C_o(T) \to X$ *such that (iii) holds.Moreover,* $m(E) = u^{**}(\chi_E)$ *for* $E \in \mathcal{B}(T)$ *and* m *verifies (iv).*

Finally, the vector measure m *satisfying (i)-(iii) is uniquely determined by the continuous linear transformation* u *and has* τ_e*-bounded range in* X^{**}.

DEFINITION 2. *Given a continuous linear transformation $u : C_o(T) \to X$, the unique X^{**}-valued vector measure m satisfying (i)-(iii) of Theorem 3 is called the representing measure of u.*

THEOREM 4. *Let $u : C_o(T) \to X$ be a continuous linear operator, where X is a quasicomplete lcHs and let m be the representing measure of u. Then the following statements are equivalent:*

(i) u *is weakly compact.*

(ii) $m(\mathcal{B}(T)) \subset X$.

(iii) m *is σ-additive in the topology τ_e of X^{**}.*

(iv) $m(U) \in X$ *for all open sets U in T.*

(v) $m(F) \in X$ *for all closed sets F in T.*

(vi) $m(U) \in X$ *for all open sets U in T which are σ-Borel.*

(vii) $m(U) \in X$ *for all open Baire sets U in T.*

(viii) $m(U) \in X$ *for all open sets U in T which are σ-compact in T.*

(ix) $m(F) \in X$ *for all closed sets F in T which are G_δ.*

(x) $m(U) \in X$ *for all open sets U in T which are a countable union of closed sets in T.*

(xi) For each non decreasing sequence $(f_n)_1^\infty \subset C_0(T)$, with $0 \leq f_n \leq 1$, (uf_n) converges weakly in X.

(xii) $m_c = m|\mathcal{B}_c(T)$ *is σ-additive in the topology τ_e of X^{**}.*

(xiii) $m_o = m|\mathcal{B}_o(T)$ *is σ-additive in the topology τ_e of X^{**}.*

(xiv) m_c *has range in X.*

(xv) m_o *has range in X.*

(xvi) m *is Borel regular for the topology τ_e of X^{**}.*

(xvii) For the topology τ_e of X^{**}, m is Borel inner regular in each $E \subset \mathcal{B}(T)$.

(xviii) For the topology τ_e of X^{**},m is Borel inner regular in each open set in T.

(xix) For the topology τ_e of X^{**}, m is Borel outer regular in each $K \in \mathcal{K}(T)$, and m is Borel inner regular in the set T.

(xx) For the topology τ_e of X^{**},m_c is σ-Borel regular.

(xxi) For the topology τ_e of X^{**},m_c es σ-Borel inner regular.

(xxii) For the topology τ_e of X^{**}, m_c is σ-Borel inner regular in each open set $U \in \mathcal{B}_c(T)$ and in the set T.

(xxiii) For th: topology τ_e of X^{**},m_c is σ-Borel outer regular in each compact K in T and is σ-Borel inner regular in the set T.

(xxiv) m_o *is Baire regular for the topology τ_e of X^{**}.*

(xxv) For the topology τ_e of X^{**},m_o is Baire inner regular in each $A \in \mathcal{B}_o(T)$.

(xxvi) For the topology τ_e of X^{**}, m_o is Baire inner regular in each open set $U \in \mathcal{B}_o(T)$ and in the set T.

(xxvii) *For the topology τ_e of X^{**}, m_o is Baire outer regular in each $K \in \mathcal{K}_o(T)$ and is Baire inner regular in the set T.*

(xxviii) *All bounded $\mathcal{B}(T)$-measurable scalar functions in T are m-integrable and $\int_T f dm \in X$.*

(xxix) *All bounded $\mathcal{B}_c(T)$-measurable scalar functions in T are m_c-integrable and $\int_T f dm_c \in X$.*

(xxx) *All bounded $\mathcal{B}_o(T)$-measurable scalar functions in T are m_o-integrable and $\int_T f dm_o \in X$.*

(xxxi) *All bounded scalar functions belonging to the first Baire class in T are m_o-integrable and $\int_T f dm_o \in X$.*

(xxxii) *$u^{**}f \in X$ for all bounded scalar functions f in the first Baire class in T.*

4. Applications.

As an immediate consequence of Theorem 4 we deduce the regular Borel extension theorem of Dinculeanu and Kluvánek [4 ,7] , the strict D.P.P. of $C_o(T)$ and a theorem of Pelczyński [14] and Thomas [15] on weakly compact operators on $C_o(T)$. Combining Theorem (A) with the Bartle-Dunford-Schwartz characterization of weakly compact sets in $ca(\Sigma)$ we also deduce the range theorem of Tweddle [16].

THEOREM 5 (Dinculeanu and Kluvánek). *Each σ-additive Baire measure m_0 in T with values in a lcHs X is regular. Moreover, if X is further quasicomplete, then there exists a unique X-valued σ-additive regular Borel (resp. regular σ-Borel) vector measure m on $\mathcal{B}(T)$ (resp. m_c on $\mathcal{B}_c(T)$) such that $m|_{\mathcal{B}_o(T)} = m_o$ (resp. $m_c|_{\mathcal{B}_o(T)} = m_o$). Besides, $m_c = m|_{\mathcal{B}_c(T)}$.*

Remark 2. Especially the last part of the above theorem generalizes Theorem 3.7 of [9] and Theorem 2.4 of [10] to vector valued σ-additive Baire measures.

THEOREM 6 (Tweddle [16] and Kluvánek [3]). *Let $\mathcal{S}$ be a σ-ring of subsets of a set Ω ($\neq \emptyset$) and let $m : \mathcal{S} \to X$ be a σ-additive vector measure, where X is a quasicomplete lcHs. Let $\bar{S}(\mathcal{S}) = \{f : \mathcal{S} \to \mathbb{C}$, $\mathcal{S}$-measurable and bounded $\}$ and let*

$$u(f) = \int_T f dm, \ f \in \bar{S}(\mathcal{S}).$$

Then u is a weakly compact operator and consequently, the absolutely convex hull of the range of m is relatively weakly compact.

THEOREM 7 (Grothendieck [6]). *$C_o(T)$ has strict D.P.P. Consequently, Grothendieck's results on weakly compact operators on $C_0(T)$ are valid.*

THEOREM 8 (Pelczyński [14] and Thomas [15]). *If the lcHs X contains no copy of c_0, then each continuous linear operator $u : C_0(T) \to X$ is weakly compact.*

5. Concluding remarks.

Theorem D of Section 2 is the same as the characetization theorem of Grothendieck (Theorem 6 and Remark 2 of [6]) and is also the same as Theorem 9.4.10 of Edwards [5], with the hypothesis of completeness of X being replaced by that of quasicompleteness. As remarked in the Introduction, Grothendieck proved his theorem only for the compact case and mentioned in Remark 2 of [6] that the result holds also for locally compact Hausdorff spaces also. Even in the proof of the compact case Grothendieck implicitly makes use of the strict D.P.P. of $C(K)$, K compact. Later, Edwards provided the details for the locally compact case in [5], but unfortunately the proof of (3) $\Rightarrow$ (2 bis) in Theorem 9.4.10 of [5] is incorrect and needs justification. (Moreover, his proof of (1) $\Rightarrow$ (3) is also incorrect,but can be rectified by appealing to the strict D.P.P. of $C_o(T)$.) In the said theorem, the equivalence of conditions (1),(2) and (2 bis) holds, but their euivalence to condition (3) (for locally compact case) seems to be unestablished in the literature. Since the equivalence of the statements in Theorem D is the same as that of the satements (i),(v),(ix) and (xi) of Theorem 4, the locally compact case of the Grothendieck theorem also holds. Morevoer, several new characterizations of weakly compact operators on $C_o(T)$ are also given in Theorem 4. While Grothendieck's proof presupposes the strict D.P.P. of $C_o(T)$, ours is independent of this property. Moreover, in our approach, as in [1], the srtict D.P.P. of $C_o(T)$ is deduced as a consequence of Theorem 4. Further, while Grothendieck uses Theorem 3 of [6] (see Theorem (B) in Section 2), whose proof is very deep and based on the technique of reduction to compact metrizable space case, our proof of Theorem 4 is direct and simple, and does not anywhere use such a reduction technique. Finally, the effectiveness of the present method is also quite evident from the fact that the earlier said isolated results of Dinculeanu and Kluvánek on σ-additive vector valued Baire measures are obtained as corollaries of Theorem 4.

Theorem 6 was first proved by Tweddle [16] for the convex hull of the range using James' powerful characterizations of weakly compact sets in a lcHs. Later, Kluvánek proved in [8] the absolutely convex hull case, using the Frechét-Nikodým topology and other properties of vector measures. Our proof is natural and elegant. Theorem 7 is due to Grothendieck [6]. For Banach space valued operators, Theorem 8 was proved by Pelczyński in [14] and then it was later extended by Thomas [15] to lcHs-valued operators. Our proof of Theorem 8 is immediate from Theorem 4 and nowhere is used the technique of reduction to metrizable compact case as is done in [14,15].

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