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Some Relations Between Contiguous Bibasic q -Appell Functions.

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1 Introduction

The basic analogous of Appell hypergeometric functions were first studied and defined by Jackson [2,3]. In a recent paper [5] I have defined bibasic q-Appell functions. In that paper we had shown that an Appell series with two basis can be reduced to an expression with only one base. We had also given an expansion formula for the Appell function.

In this paper we define contiguous bibasic q-Appell functions and find relations between such functions. In the sequel certain q-partial derivative relations between these functions have also been derived.

2 Notations

Following Jackson [2,3] we define the four bibasic Appell functions on two independent bases q and q_1 as follows:

$$\begin{aligned} \overline{\Phi}^{(1)}(a; b, b'; c; x, y; q, q_1) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q)_m (b'; q)_n x^m y^n}{(c; q_1)_{m+n} (1; q)_m (1; q)_n} \\ \overline{\Phi}^{(2)}(a; b, b'; c, c'; x, y; q, q_1) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q_1)_m (b'; q)_n x^m y^n}{(c; q_1)_m (c'; q)_n (1; q_1)_m (1; q)_n} \\ \overline{\Phi}^{(3)}(a, a'; b, b'; c; x, y; q, q_1) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_m (a'; q_1)_n (b; q_1)_m (b'; q_1)_n x^m y^n}{(c; q_1)_{m+n} (1; q_1)_m (1; q)_n} \\ \overline{\Phi}^{(4)}(a; b; c, c'; x, y; q, q_1) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q_1)_{m+n} x^m y^n}{(c; q_1)_m (c'; q)_n (1; q_1)_m (1; q)_n} \end{aligned}$$

where $(a; p)_n = (1 - p^a)(1 - p^{a+1}) \dots (1 - p^{a+n-1})$, $|p| < 1$.

3 Some three term relations between contiguous functions of $\overline{\Phi}^{(1)}$.

There can be eight functions contiguous to be the function $\overline{\Phi}^{(1)}$, namely

$$\begin{array}{ll} \overline{\Phi}^{(1)}[a + 1; b, b'; c; x, y; q, q_1], & \overline{\Phi}^{(1)}[a - 1; b, b'; c; x, y; q, q_1] \\ \overline{\Phi}^{(1)}[a; b + 1, b'; c; x, y; q, q_1], & \overline{\Phi}^{(1)}[a; b - 1, b'; c; x, y; q, q_1] \\ \overline{\Phi}^{(1)}[a; b, b' + 1; c; x, y; q, q_1], & \overline{\Phi}^{(1)}[a; b, b' - 1; c; x, y; q, q_1] \\ \overline{\Phi}^{(1)}[a; b, b'; c + 1; x, y; q, q_1], & \overline{\Phi}^{(1)}[a; b, b'; c - 1; x, y; q, q_1] \end{array}$$

These function will be denoted by $\overline{\Phi}^{(1)}(a+)$, $\overline{\Phi}^{(1)}(a-)$, $\overline{\Phi}^{(1)}(b+)$, $\overline{\Phi}^{(1)}(b-)$, $\overline{\Phi}^{(1)}(b'+)$, $\overline{\Phi}^{(1)}(b'-)$, $\overline{\Phi}^{(1)}(c+)$ and $\overline{\Phi}^{(1)}(c-)$. Similar abbreviated notations will be used for other functions contiguous to $\overline{\Phi}^{(2)}$, $\overline{\Phi}^{(3)}$ and $\overline{\Phi}^{(4)}$.

We shall prove the following relations:

$$(1 - q_1^a)\overline{\Phi}^{(1)}(a+) = (1 - q_1^{1+a-c})\overline{\Phi}^{(1)} + q_1^{1+a-c}(1 - q_1^{c-1})\overline{\Phi}^{(1)}(c-) \quad (1)$$

$$\overline{\Phi}^{(1)}(c+) = q_1^c\overline{\Phi}^{(1)}(a; b, b'; c+1; q_1x, q_1y; q, q_1) + (1 - q_1^c)\overline{\Phi}^{(1)} \quad (2)$$

$$\overline{\Phi}^{(1)}(a-) = q_1^{a-1}\overline{\Phi}^{(1)}(a-1; b, b'; c; q_1x, q_1y; q, q_1) + (1 - q_1^{a-1})\overline{\Phi}^{(1)} \quad (3)$$

$$(1 - q_1^a)\overline{\Phi}^{(1)}(a+; c+) = (1 - q_1^{a-c})\overline{\Phi}^{(1)}(c+) + q_1^{a-c}(1 - q_1^c)\overline{\Phi}^{(1)} \quad (4)$$

Proof of (1)

$$\begin{aligned} q_1^{1+a-c}(1 - q_1^{c-1})\overline{\Phi}^{(1)}(c-) &= q_1^{1+a-c} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q)_m (b'; q)_n x^m y^n}{(c; q_1)_{m+n-1} (1; q)_m (1; q)_n} \\ &= q_1^{1+a-c} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q)_m (b'; q)_n x^m y^n (1 - q_1^{c+m+n-1})}{(c; q_1)_{m+n} (1; q)_m (1; q)_n} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q)_m (b'; q)_n x^m y^n}{(c; q_1)_{m+n} (1; q)_m (1; q)_n} [(1 - q_1^{a+m+n}) - (1 - q_1^{1+a-c})] \\ &= (1 - q_1^a)\overline{\Phi}^{(1)}(a+) - (1 - q_1^{1+a-c})\overline{\Phi}^{(1)} \end{aligned}$$

which proves (1).

Proof of (2)

$$\begin{aligned} \frac{q_1^c}{(1 - q_1^c)}\overline{\Phi}^{(1)}(a; b, b'; c+1; q_1x, q_1y; q, q_1) &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+1} (b; q)_m (b'; q)_n x^m y^n q_1^{c+m+n}}{(c; q_1)_{m+n} (1; q)_m (1; q)_n (1 - q_1^{c+m+n})} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+1} (b; q)_m (b'; q)_n x^m y^n (q_1^{c+m+n} - 1 + 1)}{(c; q_1)_{m+n} (1; q)_m (1; q)_n (1 - q_1^{c+m+n})} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c; q_1)_{m+n} (b; q)_m (b'; q)_n x^m y^n}{(c; q_1)_{m+n} (1; q)_m (1; q)_n (1 - q_1^{c+m+n})} \\ &\quad - \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(c; q_1)_{m+n} (b; q)_m (b'; q)_n x^m y^n}{(c; q_1)_{m+n} (1; q)_m (1; q)_n} \\ &= \frac{1}{(1 - q_1^c)}\overline{\Phi}^{(1)}(a; b, b'; c+1; x, y; q, q_1) - \overline{\Phi}^{(1)} \end{aligned}$$

which proves (2).

Similarly, we can prove (3). Writing $c+1$ por c (3) we have (4). Similar result can be obtained for functions contiguous to b and b' .

4 Some three term relations between contiguous functions of $\overline{\Phi}^{(2)}$

There are ten functions contiguous to the function $\overline{\Phi}^{(2)}$. We shall prove the following two relations:

$$(1 - q_1^b)\overline{\Phi}^{(2)}(b+) - q_1^{b-c+1}(1 - q_1^{c-1})\overline{\Phi}^{(2)}(c-) = (1 - q_1^{b-c+1})\overline{\Phi}^{(2)} \quad (5)$$

$$(1 - q_1^{b'})\overline{\Phi}^{(2)}(b'+) - q_1^{b'-c'+1}(1 - q_1^{c'-1})\overline{\Phi}^{(2)}(c'-) = (1 - q_1^{b'-c'+1})\overline{\Phi}^{(2)} \quad (6)$$

$$(1 - q_1^b)\overline{\Phi}^{(2)}(b+; c+) - (1 - q_1^{b-c})(c+) = q_1^{b-c}(1 - q_1^c)\overline{\Phi}^{(2)} \quad (7)$$

$$(1 - q_1^{b'})\overline{\Phi}^{(2)}(b'+; c'+) - (1 - q_1^{b'-c'})\overline{\Phi}^{(2)}(c'+) = q_1^{b'-c'}(1 - q_1^{c'})\overline{\Phi}^{(2)} \quad (8)$$

To prove (5):

$$\begin{aligned} q_1^{b-c+1}(1 - q_1^{c-1})\overline{\Phi}^{(2)}(c-) &= q_1^{b-c+1} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n}(b; q_1)_n(b'; q)_n x^m y^n}{(c; q_1)_{m-1}(c'; q)_n(1; q_1)_m(1; q)_n} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n}(b; q)_m(b'; q)_n x^m y^n q_1^{b-c+1}(1 - q_1^{c+m-1})}{(c; q_1)_m(c'; q)_n(1; q_1)_m(1; q)_n} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n}(b; q)_m(b'; q)_n x^m y^n (q_1^{b-c+1} - 1 + 1 - q_1^{b+m})}{(c; q_1)_m(c'; q)_n(1; q_1)_m(1; q)_n} \\ &= (1 - q_1^b)\overline{\Phi}^{(2)}(b+) - (1 - q_1^{b-c+1})\overline{\Phi}^{(2)} \end{aligned}$$

which proves (5).

The relation (6) follows from (5) by symmetry between the parameters b and b' and c and c' with base q_1 replaced by q .

The relation (7) and (8) follow by writing $c+1$ for c and $c'+1$ for c' in (5) and (6) respectively.

5 Some three relations between contiguous functions of π and $\overline{\Phi}^{(4)}$

$$(1 - q_1^a)\overline{\Phi}^{(3)}(a+) - q_1^{a-b}(1 - q_1^b)\pi(b+) = (1 - q_1^{a-b})\overline{\Phi}^{(3)} \quad (9)$$

$$(1 - q_1^a)\overline{\Phi}^{(4)}(a+) - q_1^{a-b}(1 - q_1^b)\overline{\Phi}^{(4)}(b+) = (1 - q_1^{a-b})\overline{\Phi}^{(4)} \quad (10)$$

The proofs of these are as in (5) and (6).

6 Some relations between basic partial derivatives of the ϕ -functions and their contiguous functions.

Following Jackson [4] we define the basic operator $D_p f(x)$ by

$$D_p f(x) = \frac{f(x) - f(px)}{x}, \quad q = 1 - \epsilon$$

which is the basic analogue of differential operator $\frac{d}{dx}$.

We easily have

$$D_p x^p = (1 - q^p)x^{p-1}.$$

For a function of two variables in x, y we shall write the basic partial derivatives with respect to x and y as $\left(\frac{\partial}{\partial x}\right)_p, \left(\frac{\partial}{\partial y}\right)_p$ respectively.

It is easily seen that

$$\begin{aligned} \frac{\partial^{s+t}\overline{\Phi}^{(1)}}{(\partial x)_q^s(\partial y)_q^t} &= \frac{(a; q_1)_{s+t}(b; q)_s(b'; q)_t}{(c; q_1)_{s+t}} \overline{\Phi}^{(1)}[a + s + 1; b + s, b' + t; c + s + t; x, y, q, q_1] \\ \frac{\partial^{s+t}\overline{\Phi}^{(2)}}{(\partial x)_{q_1}^s(\partial y)_q^t} &= \frac{(a; q_1)_{s+t}(b; q)_s(b'; q)_t}{(c; q_1)_s(c'; q)_t} \overline{\Phi}^{(2)}[a + s + t; b + s, b' + t, c + s; x, y, q, q_1] \\ \frac{\partial^{s+t}\overline{\Phi}^{(3)}}{(\partial x)_{q_1}^s(\partial y)_q^t} &= \frac{(a; q_1)_s(a'; q_1)_t(b; q_1)_s(b; q)_t}{(c; q_1)_{s+t}} \overline{\Phi}^{(3)}[a + s; a' + t; b + s, b' + t; c + s + t; x, y, q, q_1] \\ \frac{\partial^{s+t}\overline{\Phi}^{(4)}}{(\partial x)_{q_1}^s(\partial y)_q^t} &= \frac{(a; q_1)_{s+t}(b; q_1)_{s+t}}{(c; q_1)_s(c'; q)_t} \overline{\Phi}^{(4)}[a + s + t; b + s + t; c + s, c' + t; x, y, q, q_1] \end{aligned}$$

We shall get some relations between these partial derivatives and the contiguous functions, analogous to these known for the Appell's function [11].

we have

$$\begin{aligned} x \frac{\partial \overline{\Phi}^{(1)}}{(\partial x)_q} &= x \frac{(1 - q_1^a)(1 - q^b)}{(1 - q_1^c)} \overline{\Phi}^{(1)}[a + 1; b + 1, b'; c + 1; x, y, q, q_1] \\ &= \frac{1 - q^b}{q^b} (\overline{\Phi}^{(1)}(b+) - \overline{\Phi}^{(1)}) \\ y \frac{\partial \overline{\Phi}^{(1)}}{(\partial y)_q} &= y \frac{(1 - q_1^a)(1 - q^{b'})}{(1 - q_1^c)} \overline{\Phi}^{(1)}[a + 1; b, b' + 1; c + 1; x, y, q, q_1] \\ &= \frac{1 - q^{b'}}{q^{b'}} (\overline{\Phi}^{(1)}(b'+) - \overline{\Phi}^{(1)}) \end{aligned}$$

which can be written as

$$(1 - q^b)\overline{\Phi}^{(1)}(b+) = (1 - q^b)\overline{\Phi}^{(1)} + xq^p \frac{\partial \overline{\Phi}^{(1)}}{(\partial x)_q} \quad (11)$$

and

$$(1 - q^{b'})\overline{\Phi}^{(1)}(b'+) = (1 - q^{b'})\overline{\Phi}^{(1)} + yq^p \frac{\partial \overline{\Phi}^{(1)}}{(\partial y)_q} \quad (12)$$

we can easily prove that

$$\begin{aligned} x \frac{\partial \bar{\Phi}^{(1)}}{(\partial x)_{q_1}} + y \frac{\partial \bar{\Phi}^{(1)}}{(\partial y)_{q_1}} &= (1 - q_1^{(1)}) \bar{\Phi}^{(1)}(c-) + q_1^{c-1} \bar{\Phi}^{(1)} + q_1^{c-1} \bar{\Phi}^{(1)}(a; b, b'; c; q_1^x x, q_1 y; q, q_1) + \\ &\quad + \bar{\Phi}^{(1)}(a; b, b'; c; q_1 x, y; q_2 q_1) + \bar{\Phi}^{(1)}(a; b, b'; c; x, q_1 y; q, q_1) + \bar{\Phi}^{(1)} \\ &= 2\bar{\Phi}^{(1)} - \bar{\Phi}^{(1)}(a; b, b'; c; q, x, y; q, q_1) - \bar{\Phi}^{(1)}(a; b, b'; c; x, q, y; q, q_1) \end{aligned} \quad (13)$$

and

$$\begin{aligned} x \frac{\partial \bar{\Phi}^{(1)}}{(\partial x)_{q_1}} + y \frac{\partial \bar{\Phi}^{(1)}}{(\partial y)_{q_1}} &= (1 - q_1^a) \bar{\Phi}^{(1)}(a+) + q_1^a \bar{\Phi}^{(1)}(a; b, b'; c; q_1 x, q_1 y; q, q_1) - \bar{\Phi}^{(1)}(a; b, b'; c; q_1 x, y; q, q_1) \\ &\quad - \bar{\Phi}^{(1)}(a; b, b'; c; x, q_1 y, q, q_1) + \bar{\Phi}^{(1)}(a; b, b'; c; x, y; v, v_1) \\ &= 2\bar{\Phi}^{(1)} - \bar{\Phi}^{(1)}(a; b, b'; c; q, x, y; q, q_1) - \bar{\Phi}^{(1)}(a; b, b'; c; x, q, y; q, q_1) \end{aligned} \quad (14)$$

We now prove the following relations

$$\begin{aligned} (1 - q_1^{c-1}) \bar{\Phi}^{(1)}[a; b, b'; c - 1; x, xy; q, q_1] &= (1 - q_1^{(1)}) \bar{\Phi}^{(1)}[a; b, b'; c; x, xy; q, q_1] + \\ &\quad + q_1^{(1)} x \frac{\bar{\Phi}^{(1)}[a; b, b'; c; x, xy; q, q_1]}{(\partial x)_{q_1}} \end{aligned} \quad (15)$$

and

$$\begin{aligned} (1 - q_1^a) \bar{\Phi}^{(1)}[a + 1; b, b'; c; x, xy; q, q_1] &= (1 - q_1^a) \bar{\Phi}^{(1)}[a; b, b'; c; x, xy; q, q_1] + \\ &\quad + q_1^a x \frac{\bar{\Phi}^{(1)}[a; b, b'; c; x, xy; q, q_1]}{(\partial x)_{q_1}} \end{aligned} \quad (16)$$

To prove (15):

$$\begin{aligned} (1 - q_1^{(1)}) \bar{\Phi}^{(1)}[a; b, b'; c - 1; x, xy; q, q_1] &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q)_m (b'; q)_n x^{m+n} y^n}{(c; q_1)_{m+n-1} (1; q)_m (1; q)_n} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{(a; q_1)_{m+n} (b; q)_m (b'; q)_n x^{m+n} y^n (1 - q_1^{c+m+n-1})}{(c; q_1)_{m+n-1} (1; q)_m (1; q)_n} \\ &= (1 - q_1^{c+1}) \bar{\Phi}^{(1)}[a; b, b'; c; x, xy; q, q_1] + \\ &\quad + q_1^{c-1} x \frac{\bar{\Phi}^{(1)}[a; b, b'; c; x, xy; q, q_1]}{(\partial x)_{q_1}} \end{aligned}$$

which proves the result.

Similarly we can prove (16).

Also we have the relations:

$$q_1^a x \frac{\partial \bar{\Phi}^{(3)}}{(\partial x)_{q_1}} = (1 - q_1^a) (\bar{\Phi}^{(3)}(a+) - \bar{\Phi}^{(3)}) \quad (17)$$

and

$$q_1^{c-1} x \frac{\partial \bar{\Phi}^{(4)}}{(\partial x)_{q_1}} = (1 - q_1^{c-1})(\bar{\Phi}^{(4)}(c-) - \bar{\Phi}^{(4)}) \quad (18)$$

and

$$q_1^{c'-1} x \frac{\partial \bar{\Phi}^{(4)}}{(\partial x)_q} = (1 - q_1^{c'-1})(\bar{\Phi}^{(4)}(c-) - \bar{\Phi}^{(4)}) \quad (19)$$

and similar relations for $y \frac{\partial \bar{\Phi}^{(3)}}{(\partial x)_{q_1}}$, $y \frac{\partial \bar{\Phi}^{(4)}}{(\partial x)_{q_1}}$ and $y \frac{\partial \bar{\Phi}^{(4)}}{(\partial x)_q}$.

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