UNIVERSIDAD DE LOS ANDES
FACULTAD DE CIENCIAS

NOTAS DE MATEMATICA

A GENERALIZED PETTIS MEASURABILITY CRITERION AND INTEGRATION OF VECTOR FUNTIONS

POR

DOBRAKOV I. AND PANCHAPAGESAN T.V.

DEPARTAMENTO DE MATEMATICA
MERIDA - VENEZUÉLA

Universidad de los Andes

Facultad de Ciencias

Departamento de Matemática

A generalized Pettis measurability criterion and integration of vector functions

DOBRAKOV I. AND PANCHAPAGESAN T.V.

Notas de Matemática

Serie: Pre-Print

No. 207

Mérida - Venezuela

2000

Notas de Matemática, No. 207
Mérida, 2000.

A generalized Pettis measurability criterion and integration of vector functions

DOBRAKOV I.* AND PANCHAPAGESAN T.V.†

Abstract

For Banach space valued functions, the concepts of $\mathcal{P}$-measurability, λ-measurabiqlity and $\mathbf{m}$-measurability are defined, where $\mathcal{P}$ is a δ-ring of subsets of a non void set T, λ is a σ-subadditive submeasure on $\sigma(\mathcal{P})$ and $\mathbf{m}$ is an operator valued measure on $\mathcal{P}$. Various characterizations are given for $\mathcal{P}$-measurable (resp. λ-measurable, $\mathbf{m}$-measurable) vector functions on T. Using them and other auxiliary results proved here, the basic theorems of [6] are rigorously established.

1. Introduction. The first author developed a theory of integration for Banach space valued functions with respect to an operator valued measure (σ-additive in the strong operator topology) in a series of papers as cited in [17], and among them the papers [6] and [7] are fundamental. This theory has many interesting features which are not shared by other Lebesgue-type integrals. For example, there are four distinct L_1 spaces here; in contrast to the abstract Lebesgue integral and Bochner integral, all the integrable functions cannot be defined through convergence in measure (see Remark 12); this integral is a complete generalization of the abstract Lebesgue integral in the sense of Remark 11; it can be used to represent certain types of operators which arise naturally in analysis (see [8]); etc. Though this work is very interesting, it is not widely known due to its inaccessibility to readers. In fact, his papers have been written briefly lacking details in the proofs of many theorems and there are many results simply stated without proof, which are indispensable either for the development of the theory or for distinguishing it from other Lebesgue-type integration theories. Moreover, there is a lacuna in the proofs of some of the basic theorems of [6].

* The research was done before the demise of I. Dobrakov.

† Supported by the C.D.C.H.T. project C-845-B-97 of the Universidad de los Andes, Mérida, Venezuela.

The aim of the present paper and the succeeding one [9] is to provide proofs of the unproved results mentioned in [6,7] (one such important result is the the stronger version of Pettis measurability criterion), to clarify the statements made in the proofs of certain theorems of [6,7], to give rigorous proofs of the theorems whose original proofs have a lacuna and to strengthen the statements of some of these theorems, and finally, to discuss in detail some of the the distinguishing features of the theory through examples which are much simpler than those given in [6,7]. We hope that these two papers will be very helpful to the interested readers to understand the theory of integration developed in [6,7] and in other papers cited in [17].

The set up of δ-rings is used as the integral representation theorems given in [8] are for σ-rings and δ-rings. Moreover, using some of the ideas of [6,7] and of Thomas [20], the second author has studied a generalization of the Bartle-Dunford-Schwartz integral of scalar functions (see [1]) when the σ-additive measure is defined on a δ-ring with values in a quasicomplete locally convex Hausdorff space. This integral defined on δ-rings plays a key role in another work of the second author which generalizes the results of [15,16] to Radon vector measures treated in [20]. In this context we would like to remark that Thomas' work [20] is based on the locally compact version of Theorem 6 of Grothendieck [11]. But, contrary to Remark 2 on p.161 of [11], the techniques of Grothendieck [11] are not powerful enough to obtain the said version. In fact, his techniques can be used to prove the said version if and only if the locally compact Hausdorff space is further σ-compact (see [19]). However, the said version for arbitrary locally compact Hausdorff spaces with many more equivalent statements has recently been proved in [18] and hence the work of Thomas [20] remains valid.

In Section 2 we fix notation and terminology and state some definitions and results from the literature, sometimes with their proof. In Section 3, following the techniques of [14] and in the set up of σ-rings, we obtain the Kelley-Srinivasan measurability criterion (see Lemma 3) without using the Bochner integral unlike the original proof in [14]. We give a detailed proof of Theorem 1 which is essentially Corollary 1.5 of [14] (not proved in [14]) and which gives several characterizations of $\mathcal{P}$-measurable vector functions in the set up of σ-rings, including a stronger version of Pettis measurability criterion (which is stated without proof on p.518 of [6]).

In Section 4 we introduce the concept of λ-measurability (resp. $\mathbf{m}$-measurability) for Banach space valued vector functions and using Theorem 1, we obtain in Theorem 2 a generalization of Theorems III.6.10 and III.6.11

of [10] for these functions. One of these characterizations is a generalized Pettis measurability criterion. We give a direct proof of Theorem 3 which is the same as the last part of Theorem 2, to the effect that the set of all λ-measurable (resp. $\mathbf{m}$-measurable) vector functions is closed under the formation of a.e. sequential limits. Using Theorem 1 we prove the two unproved results on convergence in measure and semivariation mentioned on p.519 of [6] (see Proposition 8).

In Section 5 we prove the Egoroff-Lusin theorem for a continuous submeasure and obtain an analogue of Pettis theorem on absolute continuity for σ-subadditive submeasures. In Section 6 we establish rigorously Theorems 1, 2, 10, 14 and 15 of [6] rectifying the lacuna in the original proofs (thanks to Theorems 1 and 3 we can define integrability not only for $\mathbf{m}$-measurable functions which are not necessarily $\mathcal{P}$-measurable, but also strengthen the statements of some of these theorems). Using Theorem 2 we deduce that the Bartle-Dunford-Schwartz integral in [1] is a particular case of the integral treated here (see Remarks 5 and 8). We also give a strengthened version of Theorem 14 of [6] and using Proposition 8 we provide a detailed proof of Theorem 13 of [6].

2. Preliminaries. In this section we fix notation and terminology and give some definitions and results from the literature.

T denotes a non void set. $\mathcal{P}$ (resp. $\mathcal{S}$) is a δ-ring (resp. a σ-ring) of subsets of T. $\sigma(\mathcal{P})$ denotes the σ-ring generated by $\mathcal{P}$. X, Y are Banach spaces over $\mathbb{K}$, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), with norm denoted by $|\cdot|$. $L(X,Y)$ denotes the Banach space of all continuous linear maps $U : X \to Y$, with $|U| = \sup_{|x|\le 1} |Ux|$. The dual X^* of X is the Banach space $L(X, \mathbb{K})$.

DEFINITION 1. An additive set function $\gamma : \mathcal{P} \to X$ is called a vector measure. It is said to be σ-additive if $|\gamma(\bigcup_1^\infty E_i) - \sum_1^n \gamma(E_i)| \to 0$ as $n \to \infty$, whenever $(E_i)_1^\infty$ is a disjoint sequence in $\mathcal{P}$ with $\bigcup_1^\infty E_i \in \mathcal{P}$. Then $\gamma(\bigcup_1^\infty E_i) = \sum_1^\infty \gamma(E_i)$.

DEFINITION 2. A family $(\gamma_i)_{i\in\Omega}$ of X-valued σ-additive vector measures defined on the σ-ring $\mathcal{S}$ is said to be uniformly σ-additive if, given $\epsilon > 0$ and a sequence $E_n \searrow \emptyset$ of members of $\mathcal{S}$, there exists n_0 such that $\sup_{i\in\Omega} |\gamma_i(E_n)| < \epsilon$ for $n \ge n_0$.

The following result, known as the Vitali-Hahn-Saks-Nikodym theorem, plays a crucial role in the definition of the integral of vector functions in Section 6. We shall refer to it as VHSN.

PROPOSITION 1 (VHSN). *Let $\gamma_n : \mathcal{S} \to X$, $n \in \mathbb{N}$, be σ-additive and let $\lim_n \gamma_n(E) = \gamma(E)$ exist in X for each $E \in \mathcal{S}$. Then $(\gamma_n)_1^\infty$ are uniformly σ-additive and consequently, γ is a σ-additive vector measure on $\mathcal{S}$.*

The first part of the above theorem is given for σ-algebras in Theorem I.4.8 of [4]. However, the result is easily extended to σ-rings by an argument of negation. The last part is obvious.

DEFINITION 3. A set function $\lambda : \mathcal{S} \to [0, \infty]$ is called a submeasure if $\lambda(\emptyset) = 0$ and is monotone and subadditive. A submeasure λ on $\mathcal{S}$ is said to be continuous (resp. σ-subadditive) if $\lambda(E_n) \searrow 0$ whenever the sequence $E_n \searrow \emptyset$ in $\mathcal{S}$ (resp. if $\lambda(\bigcup_1^\infty E_n) \le \sum_1^\infty \lambda(E_n)$ for any sequence $(E_n)_1^\infty$ in $\mathcal{S}$).

DEFINITION 4. Let $\gamma : \mathcal{P} \to X$ be a vector measure. Then the semivariation $||\gamma|| : \sigma(\mathcal{P}) \to [0, \infty]$ of γ is defined by

$$||\gamma||(E) = \sup\left\{\left|\sum_1^r a_i\gamma(E \cap E_i)\right| : (E_i)_1^r \subset \mathcal{P}\,\text{disjoint},\, a_i \in \mathbb{K}, |a_i| \le 1, r \in \mathbb{N}\right\}$$

for $E \in \sigma(\mathcal{P})$. We define $||\gamma||(T) = \sup\{||\gamma||(E) : E \in \sigma(\mathcal{P})\}$.

The suprenation $\bar{\gamma}$ of γ is defined by

$$\bar{\gamma}(E) = \sup\{|\gamma(F)| : F \subset E, F \in \mathcal{P}\}$$

for $E \in \sigma(\mathcal{P})$ and we define $\bar{\gamma}(T) = \sup\{\bar{\gamma}(E) : E \in \sigma(\mathcal{P})\}$.

By Proposition I.1.11 of [4] which holds also for rings of sets and by Theorem I.2.4 of [4] which is vaild for σ-rings too, we have the following

PROPOSITION 2. *Let $\gamma : \sigma(\mathcal{P}) \to X$ be a σ-additive vector measure. Then:*

(i) $\bar{\gamma}(E) \le ||\gamma||(E) \le 4\bar{\gamma}(E)$ for $E \in \sigma(\mathcal{P})$, and moreover, $||\gamma||(T) < \infty$.

(ii) $||\gamma||$, $\bar{\gamma} : \sigma(\mathcal{P}) \to [0, \infty)$ are continuous submeasures.

By Proposition I.3.1 of Bombal [3] which holds for σ-rings too, we have the following

PROPOSITION 3. *Let $\gamma_n : \mathcal{S} \to X$, $n \in I\!N$, be uniformly σ-additive. Then, given a sequence $(E_k)_1^\infty \subset \mathcal{S}$ with $E_k \searrow \emptyset$ and $\epsilon > 0$, there exists k_0 such that $||\gamma_n||(E_k) < \epsilon$ for all n and for $k \geq k_0$.*

DEFINITION 5. A set function $\mathbf{m} : \mathcal{P} \to L(X, Y)$ is called an operator valued measure if $\mathbf{m}(\cdot)x : \mathcal{P} \to Y$ is a σ-additive vector measure for each $x \in X$; in other words, if $\mathbf{m}$ is σ-additive in the strong operator topology of $L(X, Y)$.

DEFINITION 6. Let $\mathbf{m} : \mathcal{P} \to L(X, Y)$ be an operator valued measure. Then we define the semivariation $\hat{\mathbf{m}}(E)$ and the scalar semivariation $||\mathbf{m}||(E)$ for $E \in \sigma(\mathcal{P})$ by

$$\hat{\mathbf{m}}(E) = \sup \left\{ \left| \sum_1^r \mathbf{m}(E \cap E_i)x_i \right| : (E_i)_1^r \subset \mathcal{P} \text{ disjoint}, x_i \in X, |x_i| \leq 1, r \in I\!N \right\}$$

and

$$||\mathbf{m}||(E) = \sup \left\{ \left| \sum_1^r a_i \mathbf{m}(E \cap E_i) \right| : (E_i)_1^r \subset \mathcal{P} \text{disjoint}, a_i \in I\!K, |a_i| \leq 1, r \in I\!N \right\}.$$

We define $\hat{\mathbf{m}}(T) = \sup\{\hat{\mathbf{m}}(E) : E \in \sigma(\mathcal{P})\}$ and $||\mathbf{m}||(T) = \sup\{||\mathbf{m}||(E) : E \in \sigma(\mathcal{P})\}$.

Remark 1. For an operator valued measure $\mathbf{m}$ on $\mathcal{P}$, $||\mathbf{m}||(E) \leq \hat{\mathbf{m}}(E)$ and $||\mathbf{m}||(E) = 0$ if and only if $\hat{\mathbf{m}}(E) = 0$ for $E \in \sigma(\mathcal{P})$.

It is easy to deduce the following result from Definition 6.

PROPOSITION 4. *Let $\mathbf{m} : \mathcal{P} \to L(X, Y)$ be an operator valued measure. Then $\hat{\mathbf{m}}$ and $||\mathbf{m}||$ are σ-subadditive submeasures on $\sigma(\mathcal{P})$.*

DEFINITION 7. A function $s : T \to X$ is said to be a $\mathcal{P}$- simple function if the range of s is a finite set of vectors $(x_i)_1^n$ such that $s^{-1}(\{x_i\}) \in \mathcal{P}$ for each $x_i \neq 0$, $i = 1, 2, ..., n$. Then an X-valued $\mathcal{P}$-simple function s is of the form $s = \sum_1^r x_i \chi_{E_i}, (E_i)_1^r \subset \mathcal{P}$ being disjoint and $x_i \neq 0$, $i = 1, 2, ..., r$.

Notation 1. $S(\mathcal{P}, X) = \{s : T \to X |\ s\ \mathcal{P}-simple\}$ is a normed space under the operations of pointwise addition and scalar multiplication with norm $|| \cdot ||_T$ given by $||s||_T = \max_{t \in T} |s(t)|$. For a bounded function $f : T \to X$

and $A \subset T$, $||f||_A = \sup_{t\in A} |f(t)|$.

Notation 2. Following Halmos [12], for a function $f : T \to X$, $N(f)$ denotes the set $\{t \in T : f(t) \neq 0\}$.

DEFINITION 8. Let $\mathbf{m} : \mathcal{P} \to L(X,Y)$ be an operator valued measure. For an X-valued $\mathcal{P}$-simple function $s = \sum_1^r x_i \chi_{E_i}$, with $x_i \neq 0$ for all i and with $(E_i)_1^r$ disjoint in $\mathcal{P}$, we define $\int_E s d\mathbf{m} = \sum_1^r \mathbf{m}(E \cap E_i)x_i \in Y$ for $E \in \sigma(\mathcal{P})$ and we define $\int_T s d\mathbf{m} = \int_{N(s)} s d\mathbf{m}$.

Note that the above integrals are well defined.

The following result is immediate from Definitions 6 and 8.

PROPOSITION 5. *Let* $\mathbf{m} : \mathcal{P} \to L(X,Y)$ *be an operator valued measure and let* $s \in S(\mathcal{P}, X)$. *Then:*

(i) $|\int_E s d\mathbf{m}| \leq ||s||_E \cdot \hat{\mathbf{m}}(E)$ *for* $E \in \sigma(\mathcal{P})$.

(ii) If $\gamma(\cdot) = \int_{(\cdot)} s d\mathbf{m}$, *then* $\gamma : \sigma(\mathcal{P}) \to Y$ *is* σ*-additive.*

3. Stronger version of Pettis measurability criterion. Using a theorem of representation for Bochner integrable functions Kelley and Srinivasan [14] characterized X-valued $\mathcal{P}$-measurable functions as σ-simple functions with respect to $\mathcal{P}$. Employing the techniques of [14] we give a direct proof of this characterization, avoiding the use of Bochner integrals and in the set up of σ-rings. Then we pass on to obtain several characterizations of these functions, including the stronger version of the Pettis measurability criterion (which is stated without proof on p.518 of [6]). These characterizations are given in Corollary 1.5 of [14] but the corollary is not proved.

DEFINITION 9. Let $\mathcal{M}(\mathcal{P}, X) = \{f : T \to X|$ there exists a sequence$(s_n)_1^\infty \subset S(\mathcal{P}, X)$ such that $s_n(t) \to f(t)$ for all $t \in T\}$. The members of $\mathcal{M}(\mathcal{P}, X)$ are called X-valued $\mathcal{P}$-measurable functions. When $X = I\!K$, we denote $\mathcal{M}(\mathcal{P}, I\!K)$ by $\mathcal{M}(\mathcal{P})$. (In [6] X-valued $\mathcal{P}$-measurable functions are called measurable functions.)
Let us recall from §20 of Halmos [12] that a function $f : T \to I\!K$ is $\sigma(\mathcal{P})$-measurable if $N(f) \cap f^{-1}(B) \in \sigma(\mathcal{P})$ for each Borel set B in $I\!K$. Then by Theorem 20.B of Halmos [12], such a function f is the pointwise limit of a sequence $(s_n)_1^\infty$ of $\sigma(\mathcal{P})$-simple functions. As $N(f) \in \sigma(\mathcal{P})$ and

$\mathcal{P}$ is a δ-ring, there exists an increasing sequence $(E_n)_1^\infty$ in $\mathcal{P}$ such that $N(f) = \bigcup_1^\infty E_n$. Then $s_n\chi_{E_n}$ are $\mathcal{P}$-simple and converge pointwise to f in T. Thus $f \in \mathcal{M}(\mathcal{P})$. Conversely, if $f \in \mathcal{M}(\mathcal{P})$, then by Theorem 20.A and ex.9 in §18 of Halmos [12], it follows that f is $\sigma(\mathcal{P})$-measurable in the sense of Halmos[12]. Thus we have the following

PROPOSITION 6. *A scalar function on T belongs to $\mathcal{M}(\mathcal{P})$ if and only if it is $\sigma(\mathcal{P})$-measurable in the sense of Halmos [12].*

Following Kelley and Srinivasan [14], we give the following definition.

DEFINITION 10. If $f : T \to X$ is of the form $f = \sum_1^\infty x_i\chi_{E_i}$ where $(E_i)_1^\infty \subset \mathcal{P}$ and $\sum_1^\infty |x_i|\chi_{E_i}(t) < \infty$ for each $t \in T$, then f is called an X-valued σ-simple function with respect to $\mathcal{P}$.

If the sets $(E_i)_1^\infty$ and vectors $(x_i)_1^\infty$ can further be chosen so that $(E_i)_1^\infty$ is a disjoint sequence in $\mathcal{P}$ and $x_i \neq 0$ for all i, then f is called an X-valued $\mathcal{P}$-elementary function.

LEMMA 1. *If $f : T \to X$ has separable range and if $f^{-1}(\bar{B}(x,r)) \cap N(f) \in \sigma(\mathcal{P})$ for all $x \in X$ and for all real $r > 0$, then f is the uniform limit of a sequence of X-valued $\mathcal{P}$-elementary functions belonging to $\mathcal{M}(\mathcal{P}, X)$. (Here $\bar{B}(x,r) = \{y \in X : |x - y| \leq r\}$).*

Proof. Let $D = \{w_n : n \in \mathbb{N}, w_n \neq 0\}$ be dense in $f(T)$. Let $A_{n,p} = \{t \in T : |f(t) - w_n| \leq \frac{1}{p}\} \cap N(f)$ for $n, p \in \mathbb{N}$ and let $B_{n,p} = A_{n,p} \setminus \bigcup_{i<n} A_{i,p}$. Then $(B_{n,p})_{n=1}^\infty$ are disjoint in $\sigma(\mathcal{P})$. Since D is dense in $f(T)$ we have $N(f) = \bigcup_{n=1}^\infty B_{n,p}$ for each p. Particularly, $N(f) \in \sigma(\mathcal{P})$ and hence there exists a disjoint sequence (F_n) in $\mathcal{P}$ such that $\bigcup_1^\infty F_n = N(f)$. Let $g_{n,p} = \sum_{i,j<n} w_i\chi_{F_i \cap B_{j,p}}$ and let $f_p = \sum_{n,m=1}^\infty w_n\chi_{B_{n,p}\cap F_m}$. Then $(g_{n,p})_{n=1}^\infty$ are $\mathcal{P}$-simple and converge pointwise to f_p on T. Thus the $\mathcal{P}$-elementary functions f_p belong to $\mathcal{M}(\mathcal{P}, X)$ and clearly, $f_p \to f$ uniformly on T.

LEMMA 2. *Suppose $f : T \to X$ has separable range and $x^*f \in \mathcal{M}(\mathcal{P})$ for each $x^* \in X^*$. Then f is the uniform limit of a sequence of X-valued $\mathcal{P}$-elementary functions.*

Proof. Let D be a countable set of nonzero vectors in X such that $\bar{D} \supset f(T)$. Let $x_0 \in X$ and let X_0 be the closed linear subspace spanned by $D \cup \{x_0\}$. Then by Theorem 2.5 of [13] there exists a sequence $(x_n^*)_1^\infty$ in the closed unit ball of X_0^* such that $|x| = \sup_n |x_n^*(x)|$ for each $x \in X_0$.

Consequently, $(x_n^*)_1^\infty$ is total in X_0 and hence $N(f) = \bigcup_1^\infty N(x_n^* f)$. Then by hypothesis and Proposition 6 it follows that $N(f) \in \sigma(\mathcal{P})$. Moreover, for a real $r > 0$, we have $f^{-1}(\bar{B}(x_0, r)) \cap N(f) = \bigcap_1^\infty [\{(x_n^* f)^{-1}(\bar{B}(x_n^*(x_0), r))\} \cap N(f)] \in \sigma(\mathcal{P})$. Since x_0 is arbitrary, the result follows from Lemma 1.

The following lemma is a consequence of Theorem 1.4 of [14]. We modify the proof of the said theorem avoiding the use of the Bochner integral.

LEMMA 3 (Kelley-Srinivasan [14]). *Suppose $f : T \to X$ is the uniform limit of a sequence of $\mathcal{P}$-elementary functions on T. Then f is σ-simple with respect to $\mathcal{P}$. Consequently, f is $\mathcal{P}$-measurable.*

Proof. Clearly f satisfies the hypothesis of Lemma 2 and hence, as shown in the proof of the said lemma, $N(f) \in \sigma(\mathcal{P})$ and $N(f) = \bigcup_1^\infty B_n$ where $B_n = N(f) \cap f^{-1}(\bar{B}(0, n)) = \{t \in T : 0 < |f(t)| \leq n\} \in \sigma(\mathcal{P})$. If $E_n = \{t \in T : n - 1 < |f(t)| \leq n\}$, then $(E_n)_1^\infty$ is a disjoint sequence in $\sigma(\mathcal{P})$ and $N(f) = \bigcup_1^\infty E_n$. Clearly, $f = \sum_{n=1}^\infty f\chi_{E_n}$, and for each n, by hypothesis $f\chi_{E_n}$ is the uniform limit of a sequence $(f_k^{(n)})_{k=1}^\infty$ of X-valued $\mathcal{P}$-elementary functions on T vanishing outside E_n. Then we can choose a subsequence $(f_{k_r}^{(n)})_{r=1}^\infty$ of $(f_k^{(n)})_{k=1}^\infty$ such that $||f_{k_{r+1}}^{(n)} - f_{k_r}^{(n)}||_T < \frac{1}{2^n} \cdot \frac{1}{2^r}$ for $r = 1, 2, \ldots$. Then

$$f(t)\chi_{E_n}(t) = f_{k_1}^{(n)}(t) + \sum_{r=1}^\infty (f_{k_{r+1}}^{(n)}(t) - f_{k_r}^{(n)}(t)), \quad t \in T$$

and the series converges uniformly to $f\chi_{E_n}$ and is absolutely convergent for each $t \in T$. Moreover, from the above representation we have

$$||f_{k_1}^{(n)}||_T \leq n + \frac{1}{2^n} \sum_1^\infty \frac{1}{2^r} < n + 1.$$

Let $g_1^{(n)} = f_{k_1}^{(n)}$ and $g_r^{(n)} = f_{k_{r+1}}^{(n)} - f_{k_r}^{(n)}$. Then $g_r^{(n)}$ are X-valued $\mathcal{P}$-elementary functions vanishing on $T \backslash E_n$, and hence there exist disjoint sequences $(E_{r,j}^{(n)})_{j=1}^\infty$ of subsets of E_n belonging to $\mathcal{P}$ and of non zero vectors $(x_{r,j}^{(n)})_{j=1}^\infty$ in X for $r = 1, 2, \ldots$ such that $g_r^{(n)} = \sum_{j=1}^\infty x_{r,j}^{(n)} \chi_{E_{r,j}^{(n)}}$. Thus

$$f\chi_{E_n} = \sum_{j=1}^\infty x_{1,j}^{(n)} \chi_{E_{1,j}^{(n)}} + \sum_{r=1}^\infty \sum_{j=1}^\infty x_{r,j}^{(n)} \chi_{E_{r,j}^{(n)}}$$

and

$$|f(t)\chi_{E_n}(t)| \quad \leq \sum_{j=1}^{\infty} |\chi_{ij}^{n}| \chi_{E_{i,j}}^{()n)} + \sum_{r=1}^{\infty} \sum_{j=1}^{\infty} |\chi_{r,j}^{(n)}| \chi_{E_{r,j}}^{(n)}$$
$$\leq n+1+\sum_{r=1}^{\infty} \frac{1}{2^n}\frac{1}{2^r} < n+2$$

for $t \in T$. Hence, the series is absolutely convergent for each $t \in T$, and therefore we can rewrite $f\chi_{E_n} = \sum_{j=1}^{\infty} x_j^{(n)} \chi_{A_j^{(n)}}$ with $(A_j^{(n)})$ a sequence of subsets of E_n belonging to $\mathcal{P}$ with $\sum_{j=1}^{\infty} |x_j^{(n)}| \chi_{A_j^{(n)}}(t) < n+2$ for $t \in T$. Thus $\sum_{j=1}^{\infty} x_j^{(n)} \chi_{A_j^{(n)}}$ is a σ-simple function with respect to $\mathcal{P}$. Since E_n are disjoint, $f(t) = \sum_{n=1}^{\infty} f\chi_{E_n}(t) = \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} x_j^{(n)} \chi_{A_j^{(n)}}(t)$ for $t \in T$ and the series is absolutely convergent for each $t \in T$. Thus f is an X-valued σ-simple function with respect to $\mathcal{P}$. Moreover, $f(t) = \lim_m \sum_{n=1}^{m} \sum_{j=1}^{m} x_j^{(n)} \chi_{A_j^{(n)}}(t)$ for $t \in T$ and hence $f \in \mathcal{M}(\mathcal{P}, X)$.

Using the above lemmas we prove the following theorem which is essentially Corollary 1.5 of [14].

THEOREM 1. *Let $\mathcal{P}$ be a δ-ring of subsets of T and let $f : T \to X$ be a vector function. Then the following conditions are equivalent:*

(i) f is $\mathcal{P}$-measurable.

*(ii) (***The stronger version of Pettis measurability criterion***) f has separable range on T and is weakly $\mathcal{P}$-measurable (i.e. x^*f is $\mathcal{P}$-measurable for each $x^* \in X^*$).*

(iii) f has separable range on T and $f^{-1}(G) \cap N(f) \in \sigma(\mathcal{P})$ for each open set G in X.

(iv) f has separable range on T and $f^{-1}(E) \cap N(f) \in \sigma(\mathcal{P})$ for each Borel subset E of X.

(v) f is the uniform limit of a sequence of X-valued $\mathcal{P}$-elementary functions.

*(vi) (***The Kelley-Srinivasan measurability criterion***) f is an X-valued σ-simple function.*

Consequently, the set $\mathcal{M}(\mathcal{P}, X)$ of all X-valued $\mathcal{P}$-measurable functions is closed under the formation of sequential pointwise limits on T.

Proof. While the implication(i)$\Rightarrow$(ii) is obvious, (ii)$\Rightarrow$(i) by Lemmas 2 and 3.

(i)$\Rightarrow$(iii) Clearly, $f(T)$ is separable. Let $(s_k)_1^\infty \subset S(\mathcal{P}, X)$ such that $s_k(t) \to f(t)$ for each $t \in T$. Let G be a non void open set in X.

Let $G_n = \{x \in G\backslash\{0\} : B(x, \frac{1}{n}) \subset G\backslash\{0\}\}$. Then $G\backslash\{0\} = \bigcup_1^\infty G_n$. Then $f^{-1}(G\backslash\{0\}) = \bigcup_{n=1}^\infty f^{-1}(G_n) = \bigcup_{n=1}^\infty \bigcup_{m=1}^\infty \bigcap_{k=m}^\infty s_k^{-1}(G_{2n})$. In fact, $f(t) \in G_n$ implies that $B(f(t), \frac{1}{n}) \subset G\backslash\{0\}$ and hence there exists k_0 such that $s_k(t) \in B(f(t), \frac{1}{2n})$ for $k \geq k_0$ and hence $B(s_k(t), \frac{1}{2n}) \subset G\backslash\{0\}$ so that $s_k(t) \in G_{2n}$ for $k \geq k_0$. Since s_k are $\mathcal{P}$-simple, it follows that $f^{-1}(G) \cap N(f) = f^{-1}(G\backslash\{0\}) \in \sigma(\mathcal{P})$ and hence (iii) holds.

By a routine argument one can show that (iii)$\Rightarrow$(iv). If (iv) holds, then $f(T)$ is separable and $N(f) \cap f^{-1}(\bar{B}(x,r)) \in \sigma(\mathcal{P})$ for $r > 0$ and $x \in X$. Then by Lemma 1, (v) holds. Clearly (v)$\Rightarrow$(vi)$\Rightarrow$(i) by Lemma 3. Finally, by Proposition 6 the last part is immediate from (ii).

This completes the proof of the theorem.

Remark 2. The results mentioned without proof in paragraphs 4 and 5 of Section 1.2 on p.518 of [6] are the same as the equivalences among (i),(ii) and (iii) of the above theorem.

The following proposition is mentioned without proof in the second paragraph of Section 1.2 on p.518 of [6] and used in the proof of Theorem 14 of [6].

PROPOSITION 7. *Let $f \in \mathcal{M}(\mathcal{P}, X)$. Then $|f(\cdot)| \in \mathcal{M}(\mathcal{P})$. Moreover, there exists a sequence $(s_n)_1^\infty \subset S(\mathcal{P}, X)$ such that $s_n(t) \to f(t)$ and $|s_n(t)| \nearrow |f(t)|$ for $t \in T$.*

Proof. Let $(u_n)_1^\infty \subset S(\mathcal{P}, X)$ such that $u_n(t) \to f(t)$ for $t \in T$. Then $|u_n(\cdot)| \to |f(\cdot)|$ in T and hence by Proposition 6, $|f(\cdot)|$ is $\sigma(\mathcal{P})$-measurable. Therefore, by

Theorem 20.B of Halmos [12] there exists a non decreasing sequence $(h_n)_1^\infty$ of nonnegative $\sigma(\mathcal{P})$-simple functions such that $h_n(t) \nearrow |f(t)|$ for

$t \in T$. Since $N(f) \in \sigma(\mathcal{P})$, there exists $(E_n)_1^\infty \subset \mathcal{P}$ such that $E_n \nearrow N(f)$. Then $\psi_n = h_n \chi_{E_n}$ are $\mathcal{P}$-simple and $\psi_n(t) \nearrow |f(t)|$ for $t \in T$. Define $s_n(t) = \frac{u_n(t)\psi_n(t)}{|u_n(t)|}$ for $t \in N(f) \cap N(u_n)$ and $s_n(t) = 0$ otherwise. Clearly, the sequence $(s_n)_1^\infty$ satisfies the conditions of the proposition.

4. A generalized Pettis measurability criterion. In this section we introduce the concept of X-valued λ-measurable (resp. $\mathbf{m}$-measurable) functions where λ is a σ-subadditive submeasure on $\sigma(\mathcal{P})$ (resp. $\mathbf{m} : \mathcal{P} \to L(X, Y)$ is an operator valued measure), and using Theorem 1 we characterize these functions in Theorem 2, thereby generalizing Theorems III.6.10 and III.6.11 of Dunford and Schwartz [10] (see Remark 4). As a consequence, we deduce that the class of all X-valued λ- measurable (resp. $\mathbf{m}$-measurable) functions is closed under the formation of a.e. sequential limits. Theorem 2 also permits us to show that the Bartle-Dunford-Schwartz integral of [1] is a particular case of the integral defined in Section 6 (see Remarks 5 and 8).

The results about convergence in measure-$\mathbf{m}$ and in semivariation $\hat{\mathbf{m}}$ stated without proof on p.519 of [6] are needed to prove Theorem 13 of [6] and hence they are treated in Proposition 8.

DEFINITION 11. Let $\mathcal{S}$ be a σ-ring of sets and let $\lambda : \mathcal{S} \to [0, \infty]$ be a σ-subadditive submeasure. The generalized Lebesgue completion (briefly, GL-completion) $\tilde{\mathcal{S}}$ of $\mathcal{S}$ with respect to λ is defined by $\tilde{\mathcal{S}} = \{E \cup N : E \in \mathcal{S}, N \subset M \in \mathcal{S} \text{ with } \lambda(M) = 0\}$. The generalized Lebesgue completion (briefly, GL-completion) $\tilde{\lambda}$ of λ with respect to $\mathcal{S}$ is defined by $\tilde{\lambda}(E \cup N) = \lambda(E)$ where $E \cup N \in \tilde{\mathcal{S}}$ with E and N as above.

LEMMA 4. *Let $\mathcal{S}$ be a σ-ring of sets and $\lambda : \mathcal{S} \to [0, \infty]$ be a σ-subadditive submeasure on $\mathcal{S}$. Then the GL-completion $\tilde{\mathcal{S}}$ of $\mathcal{S}$ with respect to λ is a σ-ring containing $\mathcal{S}$ and the GL-completion $\tilde{\lambda}$ of λ is well defined, extends λ and is a σ-subadditive submeasure.*

Proof. Since λ is monotone, σ-subadditive and $\lambda(\emptyset) = 0$, the proofs of Theorem 13.B of [12] and

Theorem III.5.17 of [10] can be combined to prove the present lemma. The details are left to the reader.

Remark 3. If λ is a positive measure on a σ-ring $\mathcal{S}$, then the Lebesgue completion of $\mathcal{S}$ with respect to λ and of λ with respect to $\mathcal{S}$ coincide with their respective GL-completions.

DEFINITION 12. Let $\mathcal{P}$ be a δ-ring of subsets of T, $\lambda : \sigma(\mathcal{P}) \to [0, \infty]$ be a σ-subadditive submeasure and $f : T \to X$. Then:

(i) A sequence $(f_n)_1^\infty$ of X-valued functions on T is said to converge to f λ-a.e. in T if there exists $N \in \sigma(\mathcal{P})$ with $\lambda(N) = 0$ such that $f_n(t) \to f(t)$ for all $t \in T \backslash N$.

(ii) f is said to be λ-measurable if there exists a sequence $(s_n)_1^\infty \subset S(\mathcal{P}, X)$ such that $\lim_n s_n(t) = f(t)$ λ-a.e. in T; in other words, if there exists a set $M \in \sigma(\mathcal{P})$ with $\lambda(M) = 0$ such that $f\chi_{T\backslash M}$ is $\mathcal{P}$-measurable.

(iii) f is said to have λ-essentially separable range on T if there exists a set $N \in \sigma(\mathcal{P})$ with $\lambda(N) = 0$ such that $f(T\backslash N)$ is separable.

(iv) f is said to be weakly λ-measurable if x^*f is λ-measurable for each $x^* \in X^*$.

Suppose $\mathbf{m} : \mathcal{P} \to L(X, Y)$ is an operator valued measure. Note that $\hat{\mathbf{m}}(E) = 0$ if and only if $||\mathbf{m}||(E) = 0$ for $E \in \sigma(\mathcal{P})$ and by Proposition 4 $||\mathbf{m}||$ and $\hat{\mathbf{m}}$ are σ-subadditive submeasures on $\sigma(\mathcal{P})$. Thus with $\lambda = \hat{\mathbf{m}}$ or $\lambda = ||\mathbf{m}||$ in the above definition, we note that an X-valued function f on T is $\hat{\mathbf{m}}$-measurable if and only if it is $||\mathbf{m}||$-measurable and in that case, we say that f is $\mathbf{m}$-measurable; we say that $f_n \to f$ $\mathbf{m}$-a.e. if $f_n \to f$ $\hat{\mathbf{m}}$-a.e. (equivalently, $||\mathbf{m}||$-a.e.) in T.

LEMMA 5. Let $\mathcal{P}$ be a δ-ring of subsets of T and let $\lambda : \sigma(\mathcal{P}) \to [0, \infty]$ be a σ-subadditive submeasure. Let $\widetilde{\sigma(\mathcal{P})}$ and $\tilde{\lambda}$ be the GL-completions of $\sigma(\mathcal{P})$ and λ, respectively.

Then:

(i) A scalar function f on T is λ-measurable if and only if it is $\widetilde{\sigma(\mathcal{P})}$-measurable (in the sense of Halmos [12]).

(ii) If $f, g : T \to I\!K$ are equal λ-a.e. in T and if g is $\widetilde{\sigma(\mathcal{P})}$-measurable, then the same holds for f also.

(iii) Suppose $f, f_n : T \to I\!K$, $f_n, n = 1, 2, ...,$ are λ-measurable and $f_n \to f$ λ-a.e. in T. Then f is λ-measurable.

Proof. The proof is similar to the classical case and so we leave it to the reader.

The following theorem is a generalization of Theorems III.6.10 and III.6.11 of Dunford and Schwartz [10] and Corollary 1.5 of Kelley and Srinivasan [14] (i.e. Theorem 1 above) to X-valued λ-measurable (resp. $\mathbf{m}$-measurable) functions on T.

THEOREM 2. Let $\mathcal{P}$ be a δ-ring of subsets of T. Let $\lambda : \mathcal{S} = \sigma(\mathcal{P}) \to [0, \infty]$ be a σ-subadditive submeasure, or let $\lambda = \hat{\mathbf{m}}$ or $||\mathbf{m}||$ where $\mathbf{m} : \mathcal{P} \to L(X, Y)$ is an operator valued measure. Let $f : T \to X$. Then the following conditions are equivalent:

(i) f is λ-measurable.

(ii) (**A generalized Pettis measurability criterion**) f has λ-essentially separable range on T and is weakly λ-measurable.

(iii) f has λ-essentially separable range on T and $f^{-1}(G) \cap N(f) \in \tilde{\mathcal{S}}$ for each open set G in X.

(iv) f has λ-essentially separable range on T and $f^{-1}(E) \cap N(f) \in \tilde{\mathcal{S}}$ for each Borel subset E of X.

(v) There exists a set $M \in \mathcal{S}$ with $\lambda(M) = 0$ such that $f\chi_{T\backslash M}$ is the uniform limit of a sequence of $\mathcal{P}$-elementary functions on T.

(vi) (**A generalized Kelley-Srinivasan measurability criterion**) There exists a set $M \in \mathcal{S}$ with $\lambda(M) = 0$ such that $f\chi_{T\backslash M}$ is a σ-simple function with respect to $\mathcal{P}$.

Consequently, the set $\mathcal{M}(\mathcal{P}, X, \lambda)$ of all X-valued λ-measurable functions is closed under the formation of λ-a.e. sequential limits in T. When $\lambda = \hat{\mathbf{m}}$ or $||\mathbf{m}||$, $\mathcal{M}(\mathcal{P}, X, \lambda)$ is denoted by $\mathcal{M}(\mathcal{P}, X, \mathbf{m})$.

Proof.

(i)$\Rightarrow$(ii) By hypothesis there exists $M \in \mathcal{S}$ such that $\lambda(M) = 0$ and such that $f\chi_{T\backslash M}$ is $\mathcal{P}$-measurable. Then by (i)$\Rightarrow$(ii) of Theorem 1 $f(T\backslash M)$ is separable and $x^*f\chi_{T\backslash M}$ is $\mathcal{S}$-measurable for each $x^* \in X^*$. Consequently, x^*f is $\tilde{\mathcal{S}}$-measurable and hence x^*f is λ-measurable by Lemma 5(i). Thus (ii) holds.

(ii)$\Rightarrow$(i) By hypothesis and Lemma 5(i) there exists $M \in \mathcal{S}$ with $\lambda(M) = 0$ such that $f(T\backslash M)$ is separable and x^*f is $\tilde{\mathcal{S}}$-measurable for each $x^* \in X^*$.

Then by Theorem 1 it follows that $f\chi_{T\setminus M}$ is $\tilde{\mathcal{S}}$- measurable and hence there exists a sequence $(s_n)_1^\infty$ of X-valued $\tilde{\mathcal{S}}$-simple functions converging pointwise to $f\chi_{T\setminus M}$ in T. Then $N(f\chi_{T\setminus M}) = N(|f|\chi_{T\setminus M}) \in \tilde{\mathcal{S}}$ and hence $N(f\chi_{T\setminus M}) = E \cup N$, where $E \in \mathcal{S}$ and $N \subset H \in \mathcal{S}$ with $\lambda(H) = 0$. Let $(E_n)_1^\infty \subset \mathcal{P}$ such that $E_n \nearrow E$. Let u_n be $\mathcal{S}$-simple such that $s_n = u_n$ λ-a.e. in T, for $n = 1, 2, \ldots$ Let $w_n = u_n\chi_{E_n}$. Then $(w_n)_1^\infty$ are $\mathcal{P}$-simple and converge to $f\chi_{T\setminus M}$ λ-a.e. in T and hence f is λ-measurable. Thus (i) holds.

(i)⇒(iii) Let $M \in \mathcal{S}$ with $\lambda(M) = 0$ such that $f\chi_{T\setminus M} \in \mathcal{M}(\mathcal{P}, X)$. Then by (i)⇒(iii) of Theorem 1 we have $f(T\setminus M)$ is separable and

$$(f\chi_{T\setminus M})^{-1}(G) \cap N(f\chi_{T\setminus M}) \in \mathcal{S} \text{ and consequently, } f^{-1}(G) \cap N(f) \in \tilde{\mathcal{S}}$$

for all open sets G in X. Hence (iii) holds. By a routine argument, one can show that (iii)⇒(iv).

(iv)⇒(i) By hypothesis there exists $M \in \mathcal{S}$ with $\lambda(M) = 0$ such that $f(T\setminus M)$ is separable and $f^{-1}(E) \cap N(f) \in \tilde{\mathcal{S}}$ for all Borel subsets E of X. Hence $N(f) \cap f^{-1}(E) \cap (T\setminus M) \in \tilde{\mathcal{S}}$.

Then by the equivalence of (i) and (iv) of Theorem 1, $f\chi_{T\setminus M}$ is $\tilde{\mathcal{S}}$-measurable. Therefore, there exists a sequence $(s_n)_1^\infty$ of $\tilde{\mathcal{S}}$-simple functions converging pointwise to $f\chi_{T\setminus M}$ on T. Then following an argument similar to that in the proof of (ii)⇒(i), we conclude that f is λ-measurable and hence (i) holds.

(i)⇒(v) Since there exists $M \in \mathcal{S}$ with $\lambda(M) = 0$ such that $f\chi_{T\setminus M}$ is $\mathcal{P}$-measurable, by (i)⇒ (v) of Theorem 1, (v) holds. (v)⇒(vi)⇒(i) by Lemma 3 applied to $f\chi_{T\setminus M}$. By Lemma 5(iii) the last part is immediate from the equivalence of (i) and (ii).

This completes the proof of the theorem.

Remark 4. Clearly the above theorem subsumes Theorem 3.5.3 of [13], Theorem 2, §6 of [5] and Theorems III.6.10 and III.6.11 of [10].

The proofs of the said theorems of [5] and [10] make use of the Egoroff theorem which is not available for countably subadditive submeasures. However, thanks to the ingenious techniques of Kelley and Srinivasan [14], we are able to generalize the above mentioned classical theorems to X-valued λ-measurable (resp. $\mathbf{m}$-measurable) functions when λ is a σ-subadditive submeasure (resp. when $\mathbf{m}$ is an operator valued measure).

Remark 5. Let $\nu : \Sigma \to X$ be σ-additive, where Σ is a σ-algebra of subsets of T. If we define $\mathbf{m}(E)(\alpha) = \alpha \cdot \nu(E)$ for $\alpha \in I\!K$, then $\mathbf{m} : \Sigma \to L(I\!K, X)$ is an operator valued measure and it is well known that $\hat{\mathbf{m}} = ||\nu||$. If μ is the control measure of ν, then $\mu(E) = 0$ if and only if $||\nu||(E) = 0$ and hence if and only if $\hat{\mathbf{m}}(E) = 0$.

Therefore, the Lebesgue completion Σ^* of Σ with respect to μ as in Section IV.10 of [10] coincides with the GL-completion of Σ with respect to $\hat{\mathbf{m}}$ and then Theorem 2 implies that a scalar function is ν-measurable according to the definition on p.322 of [10] if and only if it is $\mathbf{m}$-measurable in our sense.

Because of the importance of the last part of the above theorem in the theory of integration of vector functions, we state it as a separate theorem and also prove it directly.

THEOREM 3. *Let λ be a σ-subadditive submeasure on $\sigma(\mathcal{P})$ (resp. $\mathbf{m}$ be an operator valued measure on $\mathcal{P}$). Then $\mathcal{M}(\mathcal{P}, X, \lambda)$ (resp. $\mathcal{M}(\mathcal{P}, X, \mathbf{m})$) is closed under the formation of λ-a.e. (resp. $\mathbf{m}$-a.e.) sequential limits.*

Proof. It suffices to prove the proposition for λ. Let $(f_n)_{n=1}^{\infty} \subset \mathcal{M}(\mathcal{P}, X, \lambda)$. If $f_0 : T \to X$ and if $f_n \to f_0$ λ-a.e. in T, then there exist $(N_i)_{i=0}^{\infty} \subset \sigma(\mathcal{P})$ with $\lambda(N_i) = 0$ for $i = 0, 1, 2, \ldots$ such that $f_n(t) \to f_0(t)$ for $t \in T \backslash N_0$ and $f_n \chi_{T\backslash N_n} \in \mathcal{M}(\mathcal{P}, X)$ for $n \in I\!N$. If $N = \bigcup_{n=0}^{\infty} N_n$, then $N \in \sigma(\mathcal{P})$, $\lambda(N) = 0$, $(f_n \chi_{T\backslash N})_1^{\infty} \subset \mathcal{M}(\mathcal{P}, X)$ and $f_n(t)\chi_{T\backslash N}(t) \to f_0(t)\chi_{T\backslash N}(t)$ for $t \in T$.

Therefore, by the last part of Theorem 1 we conclude that $f_0 \chi_{T\backslash N} \in \mathcal{M}(\mathcal{P}, X)$. Since $\lambda(N) = 0$, we conclude that f_0 is λ-measurable in T.

DEFINITION 13. Let $\mathbf{m} : \mathcal{P} \to L(X, Y)$ be an operator valued measure and let $f, f_n : T \to X, n \in I\!N$, be $\mathbf{m}$-measurable . Then $(f_n)_1^{\infty}$ is said to converge to f in measure-$\mathbf{m}$ (resp. in semivariation $\hat{\mathbf{m}}$) if, for each $\eta > 0$, $\lim_{n\to\infty} ||\mathbf{m}||(\{t \in T : |f_n(t) - f(t)| \geq \eta\}) = 0$ (resp. $\lim_{n\to\infty} \hat{\mathbf{m}}(\{t \in T : |f_n(t) - f(t)| \geq \eta\}) = 0$). Similarly, as in Halmos [12], the concepts of fundamental in measure-$\mathbf{m}$ (resp. in semivariation $\hat{\mathbf{m}}$), and almost uniform convergence in measure-$\mathbf{m}$ (resp. in semivariation $\hat{\mathbf{m}}$) are defined.

The proofs of the two results mentioned in the first two paragraphs on p.519 of [6] are based on Theorem 1 and as these results are indispensable for proving Theorem 13 of [6], the following proposition treats these results.

PROPOSITION 8. *Let $\mathbf{m} : \mathcal{P} \to L(X, Y)$ be an operator valued measure* *and let $f_n : T \to X$, $n \in I\!N$, be* $\mathbf{m}$*- measurable. Then:*

(i) If $(f_n)_1^\infty$ is fundamental in measure-$\mathbf{m}$ (resp. in semivariation $\hat{\mathbf{m}}$), then there exist a subsequence $(f_{n_k})_{k=1}^\infty$ and an $\mathbf{m}$-measurable function $f : T \to X$ such that $f_{n_k} \to f$ almost uniformly in measure-$\mathbf{m}$ (resp. in semivariation $\hat{\mathbf{m}}$) in T. Consequently, $f_{n_k} \to f$ $\mathbf{m}$-a.e. in T.

(ii) If $(f_n)_1^\infty$ converges to an $\mathbf{m}$-measurable function $f : T \to X$ in measure-$\mathbf{m}$ or in semivariation $\hat{\mathbf{m}}$ in each set $E \in \mathcal{P}$, then there exists a subsequence $(f_{n_k})_1^\infty$ converging to f $\mathbf{m}$-a.e. in T.

Proof. Let $\nu = ||\mathbf{m}||$ or $\hat{\mathbf{m}}$. Then by Proposition 4 ν is a σ-subadditive submeasure on $\sigma(\mathcal{P})$.

(i) By hypothesis and the σ-subadditivity of ν there exists $M \in \sigma(\mathcal{P})$ with $\nu(M) = 0$ such that $(f_n\chi_{T\setminus M})_1^\infty$ are $\mathcal{P}$-measurable.

Let $\epsilon > 0$ and let $E_{n,p}(\epsilon) = \{t \in T\setminus M : |f_n(t) - f_p(t)| \geq \epsilon\}$. Proceeding as in the proof of

Theorem 22.D of Halmos [12], we can construct a subsequence $(n_k)_{k=1}^\infty$ of $I\!N$ such that $\nu(E_{n,p}(\frac{1}{2^k})) < \frac{1}{2^k}$ for $n, p \geq n_k$. Defining $E_k = E_{n_k,n_{k+1}}(\frac{1}{2^k})$, let $F_k = \bigcup_{i\geq k} E_i$.

Then $E_i, F_k \in \sigma(\mathcal{P})$ for all i, k. Then as in the proof of the said theorem of Halmos [12] it can be shown that (f_{n_i}) is Cauchy for uniform convergence on $T\setminus M\setminus F_k$ for each k and consequently, as X is complete, $\lim_i f_{n_i}(t) = f(t)$ (say) exists in X for each $t \in T\setminus M\setminus F_k$. Moreover, as ν is σ-subadditive, $\nu(F_k) \leq \frac{1}{2^{k-1}}$ for each k and hence $(f_{n_i})_{i=1}^\infty$ is almost uniformly Cauchy (in ν) in $T\setminus M$. Let $N = \bigcap_{k=1}^\infty F_k$. Then $\nu(N) = 0$ and hence $\hat{\mathbf{m}}(N) = 0$. If we define $f(t) = 0$ for $t \in M \cup N$, then $f : T \to X$ and as seen above, $f_{n_i}(t) \to f(t)$ for $t \in \bigcup_{k=1}^\infty (T\setminus M\setminus F_k) = T\setminus M\setminus N$. In other words, $f_{n_i}\chi_{T\setminus M\setminus N}$ converges pointwise to $f\chi_{T\setminus M\setminus N}$ in T and hence by Theorem 1, $f\chi_{T\setminus M\setminus N}$ is $\mathcal{P}$-measurable.

Since $\nu(M \cup N) = 0$, we conclude that f is ν-measurable and $(f_{n_i})_1^\infty$ converges to f ν-a.e. in T.

(ii) Let $f_0 = f$. Take M as in the proof of (i) so that $f_n\chi_{T\setminus M}, n \in I\!N \cup \{0\}$, are $\mathcal{P}$-measurable. Let $F = \bigcup_{n=0}^\infty (T\setminus M) \cap N(f_n)$. Then $F \in \sigma(\mathcal{P})$. Choose an increasing sequence $(F_k)_1^\infty \subset \mathcal{P}$ such that $F = \bigcup_1^\infty F_k$. By hypothesis and (i), there exist a subsequence $(f_{1,i})_{i=1}^\infty$ of $(f_n)_1^\infty$, a set $N_1 \in \sigma(\mathcal{P}) \cap F_1$ with $\nu(N_1) = 0$ and a $\mathcal{P} \cap (F_1\setminus N_1)$-measurable function g such that $f_{1,i} \to g$ almost uniformly in ν in F_1.

Then by adapting the proofs of Theorems 22.B and 22.C of Halmos [12], we conclude that $f = g$ ν-a.e. in F_1 and consequently, there exists $\tilde{N}_1 \subset F_1$, $N_1 \in \sigma(\mathcal{P})$ with $\nu(\tilde{N}_1) = 0$ such that $f_{1,i}(t) \to f(t)$ for $t \in F_1 \backslash \tilde{N}_1$.

Repeating the argument with the subsequence $(f_{1,i})_{i=1}^{\infty}$ we get a subsequence $(f_{2,i})_{i=1}^{\infty}$ and a set $\tilde{N}_2 \subset F_2$, $N_2 \in \sigma(\mathcal{P})$ with $\nu(\tilde{N}_2) = 0$ such that $f_{2,i}(t) \to f(t)$ for $t \in F_2 \backslash \tilde{N}_2$. Repeating this process successively, in the n^{th} stage we obtain a subsequence $(f_{n,i})_{i=1}^{\infty}$ of $(f_{n-1,i})_{i=1}^{\infty}$ and a set $\tilde{N}_n \subset F_n$, $\tilde{N}_n \in \sigma(\mathcal{P})$ with $\nu(\tilde{N}_n) = 0$ such that $f_{n,i}(t) \to f(t)$ for $t \in F_n \backslash \tilde{N}_n$. Let $N = \bigcup_1^{\infty} \tilde{N}_n$. Then $N \in \sigma(\mathcal{P})$, $\nu(N) = 0$ and the diagonal sequence $(f_{n,n})_1^{\infty}$, which is a subsequence of $(f_n)_1^{\infty}$, converges to f pointwise in $F \backslash N$. Since $T = F \cup M$ and $\nu(M \cup N) = 0$, (ii) holds.

5. Submeasures which are continuous or σ-subadditive. Important results such as the Egoroff theorem and the Egoroff-Lusin theorem (resp. Pettis theorem on absolute continuity of measures) are generalized to continuous (resp. σ-subadditive) submeasures. The Egoroff-Lusin theorem and Pettis theorem are used in Section 6.

PROPOSITION 9. *A continuous submeasure λ defined on a σ-ring $\mathcal{S}$ is σ-subadditive.*

Proof. Since λ is monotone, it suffices to show that $\lambda(\bigcup_1^{\infty} E_n) \leq \sum_1^{\infty} \lambda(E_n)$ for any disjoint sequence $(E_n)_1^{\infty} \subset \mathcal{S}$. For such a sequence, let $E = \bigcup_1^{\infty} E_n$ and let $F_n = \bigcup_{k=n}^{\infty} E_k$. Then $F_n \searrow \emptyset$. As λ is finitely subadditive, we have $\lambda(E) \leq \sum_{k=1}^{n-1} \lambda(E_k) + \lambda(F_n)$. Taking the limit as $n \to \infty$, we have $\lambda(E) \leq \sum_1^{\infty} \lambda(E_n)$ since λ is continuous and $F_n \searrow \emptyset$.

In the proof of the classical Egoroff theorem with respect to a finite positive measure μ, only the continuity from above and the σ-subadditivity of μ are used. Thus, in the light of Proposition 9, we can adapt the proof of the classical Egoroff theorem to generalize it to the case of continuous submeasures. Thus we have:

THEOREM 4 (Egoroff). *Let $\lambda : \mathcal{S} \to [0, \infty]$ be a continuous submeasure on the σ-ring $\mathcal{S}$ and let $f, f_n : T \to X$, $n \in \mathbb{N}$, be $\mathcal{S}$-measurable.*

If $f_n \to f$ λ-a.e. in T, then, given $\epsilon > 0$, there exists a set $E_\epsilon \in \mathcal{S}$ such that $\lambda(E_\epsilon) < \epsilon$ and $f_n \to f$ uniformly on $T \backslash E_\epsilon$.

From the above theorem we deduce the following result, known as the Egoroff- Lusin theorem.

THEOREM 5 (Egoroff-Lusin). *Let $\mathcal{P}$ be a δ-ring of subsets of T and let $\lambda : \sigma(\mathcal{P}) \to [0,\infty]$ be a continuous submeasure. Let $f, f_n : T \to X$, $n = 1,2,..$ be $\mathcal{P}$-measurable and suppose $f_n(t) \to f(t)$ for $t \in T$. If $F = \bigcup_{n=1}^\infty N(f_n)$, then there exist $N \in \sigma(\mathcal{P})$ with $\lambda(N) = 0$ and a sequence $(F_k)_1^\infty \subset \mathcal{P}$ with $F_k \nearrow F \backslash N$ such that $f_n \to f$ uniformly on every F_k.*

Proof. By applying the Egoroff theorem successively with $\epsilon = \frac{1}{n}$ in the n^{th} step, we can construct a decreasing sequence $(G_n)_1^\infty \subset \sigma(\mathcal{P})$ such that $\lambda(G_n) < \frac{1}{n}$ and $f_n \to f$ uniformly on $G_{n-1} \backslash G_n$ where $G_0 = F$. Let $N = \bigcap_1^\infty G_n$. Then $N \in \sigma(\mathcal{P})$ and $\lambda(N) = 0$. Moreover, $F \backslash N = \bigcup_1^\infty (F \backslash G_n)$ and $F \backslash G_n \nearrow$. Clearly, $f_n \to f$ uniformly on $F \backslash G_n = \bigcup_{k=1}^n (G_{k-1} \backslash G_k)$ for each n. As $F \backslash G_n \in \sigma(\mathcal{P})$ there exists an increasing sequence $(H_{n,m})_{m=1}^\infty \subset \mathcal{P}$ such that $\bigcup_{m=1}^\infty H_{n,m} = F \backslash G_n$. Let $F_n = \bigcup_{p,m=1}^n H_{p,m}$. Then $F_n \in \mathcal{P}$ for all n, $F_n \nearrow F \backslash N$ and $f_k \to f$ uniformly on each F_n.

The easy proof of the following corollary is left to the reader.

COROLLARY 1. *If $\mu : \mathcal{S} = \sigma(\mathcal{P}) \to [0,\infty]$ is a σ-finite measure, then the Egoroff-Lusin theorem holds for μ.*

DEFINITION 14. Let λ be a submeasure on a σ-ring $\mathcal{S}$ and let $\gamma : \mathcal{S} \to X$ be σ-additive. We say that γ is absolutely continuous with respect to λ and write $\gamma \ll \lambda$ (resp. λ-continuous) if $\lambda(E) = 0$ implies $\gamma(E) = 0$ (resp. if $\lim_{\lambda(E)\to 0} \gamma(E) = 0$) for $E \in \mathcal{S}$.

THEOREM 6 (Pettis). *Let $\mathcal{S}$ be a σ-ring of subsets of T. Let $\lambda : \mathcal{S} \to [0,\infty]$ be a σ-subadditive submeasure and let $\gamma : \mathcal{S} \to X$ be σ-additive. Then $\gamma \ll \lambda$ if and only if γ is λ-continuous.*

Proof. Clearly the condition is sufficient. Suppose $\gamma \ll \lambda$ and γ is not λ-continuous. Then there would exist an $\epsilon > 0$ such that, for each $n \in I\!N$, there would exist a set $E_n \in \mathcal{S}$ with $\lambda(E_n) < \frac{1}{2^n}$ for which $|\gamma(E_n)| \geq \epsilon$. If $E = \limsup E_n$ and $A_n = \bigcup_{k=n}^\infty E_k$, then we have $\lambda(E) = \lambda(\bigcap_{n=1}^\infty A_n) \leq \lambda(A_n) \leq \sum_{k=n}^\infty \lambda(E_k) < \frac{1}{2^{n-1}}$ for each n and hence $\lambda(E) = 0$. Then by hypothesis $\gamma(E) = 0$. Clearly, $A_n \searrow E$ and hence by Proposition 2(ii) $\lim_n ||\gamma||(A_n \backslash E) = 0$. Thus, there exists n_o such that $||\gamma||(A_n \backslash E) < \epsilon$ for $n \geq n_0$. Since $\lambda(E) = 0$ implies $\lambda(F) = 0$ for all $F \subset E$, $F \in \mathcal{S}$, by hypothesis we have $\gamma(F) = 0$ for $F \subset E$, $F \in \mathcal{S}$, and hence $||\gamma||(E) = 0$. Therefore we have $||\gamma||(A_n) = ||\gamma||(A_n) - ||\gamma||(E) \leq ||\gamma||(A_n \backslash E) < \epsilon$ for $n \geq n_0$. This is impossible since $||\gamma||(A_n) \geq ||\gamma||(E_n) \geq |\gamma(E_n)| \geq \epsilon$ for all n. Thus the theorem holds.

6. Integration of X-valued m-measurable functions. Theorem 1 of [6] is used in the proofs of Theorems 2,10,14 and 15 of [6]. If μ is the Y-valued σ-additive measure constructed in the proof of Theorem 1 of [6], $\mu(N) = 0$ does not imply $\int_E f_n \chi_E d\mathbf{m} = 0$, contrary to what is claimed there. Because of this lacuna, the said theorems remain unestablished in [6]. However, using the results of Sections 4 and 5, we modify the original proofs of [6] in this section and establish the said results rigorously. Besides, using Theorems 1 and 3 of Section 4, not only we dispense with the hypothesis of measurability of the limit functions in these theorems but also strengthen the statements of these theorems by using $\mathbf{m}$-measurable functions in place of $\mathcal{P}$-measurable functions. It is also noted in Remark 8 that the Bartle-Dunford-Schwartz integral treated in Section IV.10 of [10] is a particular case of the integral defined here. Employing Proposition 7 we provide a strengthened version of Theorem 14 of [6] and using Proposition 8 we give a detailed proof of Theorem 13 of [6]. Also we clarify certain statements in the proofs of Theorems 10 and 14 of [6].

BASIC ASSUMPTION. In the sequel $\mathbf{m} : \mathcal{P} \to \mathbf{L(X,Y)}$ is σ-additive in the strong operator topology of $L(X,Y)$ with $\hat{\mathbf{m}}(E) < \infty$ for each $E \in \mathcal{P}$.

Remark 6. The finiteness of $\hat{\mathbf{m}}$ on $\mathcal{P}$ has to be imposed and is not a consequence even if $\mathbf{m}$ is σ-additive in the uniform operator topology, contrary to the claim made by Bartle on p.339 of [2]. This has been established in Example 5 on p.517 of [6].

Under the additional hypothesis that $\hat{\mathbf{m}}(E)) < \infty$ for all $E \in \mathcal{P}$, the X-valued $\mathcal{P}$-simple functions are called simple integrable functions.

LEMMA 6. *Let $\gamma_n, \eta_n : \sigma(\mathcal{P}) \to Y, n \in I\!N$, be σ-additive. Let*

$$\lambda(E) = \sum_1^\infty \frac{1}{2^n}\left(\frac{\bar{\gamma}_n(E)}{1+\bar{\gamma}_n(T)} + \frac{\bar{\eta}_n(E)}{1+\bar{\eta}_n(T)}\right), \quad E \in \sigma(\mathcal{P}).$$

Then λ is a continuous submeasure on $\sigma(\mathcal{P})$.

Proof. By Proposition 2 $\bar{\gamma}_n$ and $\bar{\eta}_n$, $n \in I\!N$, are bounded continuous submeasures on $\sigma(\mathcal{P})$ and hence λ is also a bounded submeasure. To show that λ is also continuous, let $\epsilon > 0$ be given. Choose n_0 such that $\frac{1}{2^{n_0}} < \frac{\epsilon}{2}$. Let $(E_n)_1^\infty \subset \sigma(\mathcal{P})$ such that $E_n \searrow \emptyset$. As $\bar{\gamma}_n, \bar{\eta}_n$, $n = 1, 2, ..., n_0$ are continuous, there exists k_0 such that $(\bar{\gamma}_n(E_k) + \bar{\eta}_n(E_k)) < \frac{\epsilon}{2}$ for $k \geq k_0$ and for $n = 1, 2, ..., n_0$. Then it follows that $\lambda(E_k) < \epsilon$ for $k \geq k_0$. Hence λ is continuous.

The following theorem combines Theorems 2 and 7 of [6] for simple integrable functions.

THEOREM 7. *Let $f : T \to X$ be a vector function. If there exists a sequence $(s_n)_1^\infty \subset \mathcal{S}(\mathcal{P}, X)$ such that $\lim_n s_n(t) = f(t)$ $\mathbf{m}$-a.e. in T, then f is $\mathbf{m}$-measurable. Let $\gamma_n(\cdot) = \int_{(\cdot)} s_n d\mathbf{m} : \sigma(\mathcal{P}) \to Y$, $n \in I\!N$. Then the following statements are equivalent:*

(i) $\lim_n \gamma_n(E) = \gamma(E)$ exists in Y for each $E \in \sigma(\mathcal{P})$.

(ii) $\gamma_n(\cdot) : \sigma(\mathcal{P}) \to Y$, $n \in I\!N$, are uniformly σ-additive on $\sigma(\mathcal{P})$.

(iii) $\lim_n \gamma_n(E)$ exists in Y uniformly with respect to $E \in \sigma(\mathcal{P})$.

Moreover, if $(s'_n)_1^\infty$ is another sequence in $\mathcal{S}(\mathcal{P}, X)$ with $\lim_n s'_n(t) = f(t)$ $\mathbf{m}$-a.e. in T, satisfying anyone of the above conditions, then $\lim_n \int_E s_n d\mathbf{m} = \lim_n \int_E s'_n d\mathbf{m}$ for all $E \in \sigma(\mathcal{P})$. Finally, $\gamma : \sigma(\mathcal{P}) \to Y$ is σ-additive and $\hat{\mathbf{m}}$-continuous (resp. $||\mathbf{m}||$-continuous).

Proof. By Theorem 3 f is $\mathbf{m}$-measurable. Since the γ_n are σ-additive on $\sigma(\mathcal{P})$ by Proposition 5(ii), by VHSN (i)$\Rightarrow$(ii) and obviously, (iii)$\Rightarrow$(i). Let (ii) hold. In the definition of the continuous submeasure λ of Lemma 6 let us take γ_n as above and $\eta_n = 0$ for all n. Let $M \in \sigma(\mathcal{P})$ with $\hat{\mathbf{m}}(M) = 0$ such that $s_n(t) \to f(t)$ for $t \in T \backslash M$.

Let $F = \bigcup_{n=1}^\infty N(s_n) \cap (T \backslash M)$. Now by the Egoroff-Lusin theorem applied to λ, there exist $N \in \sigma(\mathcal{P}) \cap F$ with $\lambda(N) = 0$ and an increasing sequence $(F_k)_1^\infty \subset \mathcal{P}$ with $F_k \nearrow F \backslash N$ such that $s_n \to f$ uniformly on each F_k. Let $G = F \backslash N$. Given $\epsilon > 0$, by hypothesis (ii) and Proposition 3 there exists k_0 such that $||\gamma_n||(G \backslash F_{k_0}) < \frac{\epsilon}{3}$ for all n. Since $\hat{\mathbf{m}}(F_{k_0}) < \infty$ and since $s_n \to f$ uniformly on F_{k_0}, there exists n_0 such that $||s_n - s_p||_{F_{k_0}} \cdot \mathbf{m}(F_{k_0}) < \frac{\epsilon}{3}$ for all $n, p \geq n_0$. As $\hat{\mathbf{m}}(M) = 0$, by Proposition 5(i) $\bar{\gamma}_n(M) = 0$ for all n and hence $\lambda(M) = 0$.

Moreover, $\lambda(M) = \lambda(N) = 0$ imply that $\gamma_n(E \cap N) = \gamma_n(E \cap M) = 0$ for all n and for all $E \in \sigma(\mathcal{P})$. Thus we have

$$\begin{aligned}\left|\int_E s_n d\mathbf{m} - \int_E s_p d\mathbf{m}\right| &\leq \left|\int_{E\cap(G\setminus F_{k_0})} s_n d\mathbf{m}\right| + \left|\int_{E\cap(G\setminus F_{k_0})} s_p d\mathbf{m}\right| \\ &+ \left|\int_{E\cap F_{k_0}} (s_n - s_p) d\mathbf{m}\right| \\ &\leq ||\gamma_n||(G\setminus F_{k_0}) + ||\gamma_p||(G\setminus F_{k_0}) \\ &+ ||s_n - s_p||_{F_{k_0}} \cdot \hat{\mathbf{m}}(F_{k_0}) < \epsilon\end{aligned}$$

for all $n, p \geq n_0$ and for all $E \in \sigma(\mathcal{P})$. Thus $\{\gamma_n(E)\}_1^\infty$ is uniformly Cauchy for $E \in \sigma(\mathcal{P})$ and as Y is Banach, (iii) holds. The uniqueness of the limit is established as in the third paragraph on p.522 of [6] by considering the sequence $(g_n)_1^\infty$ with $g_{2n} = s_n$ and $g_{2n-1} = s'_n$ for all n.

By VHSN γ is σ-additive on $\sigma(\mathcal{P})$ and is $\hat{\mathbf{m}}$-continuous (resp. $||\mathbf{m}||$-continuous) by Theorem 6 as $\hat{\mathbf{m}}$ (resp. $||\mathbf{m}||$) is a σ-subadditive submeasure by Proposition 4 and as $\hat{\mathbf{m}}(E) = 0$ implies by Proposition 5(i) that $\gamma_n(E) = 0$ for all n and hence implies that $\gamma(E) = 0$.

This completes the proof of the theorem.

Remark 7. In the above proof we could have defined $\lambda(E) = \sum_1^\infty \frac{1}{2^n} \frac{\mu_n(E)}{1+||\mu_n||}$ for $E \in \sigma(\mathcal{P})$, where μ_n is the control measure of γ_n and $||\mu_n|| = \sup\{\mu_n(E) : E \in \sigma(\mathcal{P})\}$. In that case, λ is a finite positive measure and hence the Egoroff-Lusin theorem applies. We preferred to use the supremations of γ_n as they can directly be described by the vector measures unlike their control measures.

Using the above theorem we extend Definition 2 of [6] to a wider class $\mathcal{I}(\mathbf{m})$ which contains $S(\mathcal{P}, X)$ and which is contained in $\mathcal{M}(\mathcal{P}, X, \mathbf{m})$.

DEFINITION 15. An X-valued $\mathbf{m}$-measurable function f is said to be $\mathbf{m}$-integrable if there exists a sequence $(s_n)_1^\infty \subset S(\mathcal{P}, X)$ such that $s_n \to f$ $\mathbf{m}$-a.e. in T and such that anyone of the conditions of Theorem 7 is satisfied by the integrals $\int_{(\cdot)} s_n d\mathbf{m}$, $n \in I\!N$. In that case, we define $\int_E f d\mathbf{m} = \lim_n \int_E s_n$ for $E \in \sigma(\mathcal{P})$ and it is well defined by the last part of Theorem 7. By $\int_T f d\mathbf{m}$ we mean the integral $\int_{N(f)} f d\mathbf{m}$. The set of all X-valued $\mathbf{m}$-integrable functions is denoted by $\mathcal{I}(\mathbf{m})$.

The above integral includes the Bartle-Dunford-Schwartz integral of [1] as a particular case. In fact, we have the following

Remark 8. Let ν, Σ and $\mathbf{m}$ be as in Remark 5. Then by Remark 5 we note that the Bartle-Dunford-Schwartz integral of scalar functions with respect to ν (see [2] or Definition 4.10.7 of [10]) coincides with the $\mathbf{m}$-integral given in Definition 15. Moreover, in this case, $\mathcal{I}(\mathbf{m}) = \mathcal{L}_\infty(\mathbf{m})$ where $\mathcal{L}_\infty(\mathbf{m})$ is as defined in [7] (see [9, 17]).

In the proof of Theorem 14 of [6], Proposition 7 in Section 3 above guarantees the existence of a sequence $(f_n)_1^\infty$ of X-valued $\mathcal{P}$-simple functions such that $f_n(t) \to f(t)$ and $|f_n(t)| \nearrow |f(t)|$ for $t \in T$.

The Egoroff-Lusin theorem referred to in the proof of the said theorem should be with respect to the continuous submeasure λ of Lemma 6 with $\gamma_n(\cdot) = \int_{(\cdot)} f_n d\mathbf{m}$ and $\eta_n = 0$ for $n \in \mathbb{N}$. Also a clarification is needed in regard to the claim (in the said proof) that $|\int_{E\cap F_k}(f - f_n)d\mathbf{m}| < \frac{\epsilon}{2}$. Since $(f_n)_1^\infty$ converges to f uniformly on F_k, $|\int_{E\cap F_k} f_n d\mathbf{m} - \int_{E\cap F_k} f_p d\mathbf{m}| \leq ||f_n - f_p||_{F_k} \cdot \hat{\mathbf{m}}(F_k)$ by Proposition 5(i) and hence $(\int_{E\cap F_k} f_n d\mathbf{m})_1^\infty$ is uniformly Cauchy (in Y) with respect to $E \in \sigma(\mathcal{P})$. Hence f is $\mathbf{m}$-integrable on $E \cap F_k$ and $\int_{E\cap F_k} f d\mathbf{m} = \lim_p \int_{E\cap F_k} f_p d\mathbf{m}$.Thus by Proposition 5(i) we have $|\int_{E\cap F_k}(f - f_n)d\mathbf{m}| = \lim_p |\int_{E\cap F_k}(f_p - f_n)d\mathbf{m}| \leq \lim_p ||f_n - f_p||_{F_k} \cdot \hat{\mathbf{m}}(F_k) = ||f - f_n||_{F_k} \cdot \hat{\mathbf{m}}(F_k) < \frac{\epsilon}{2}$ for sufficiently large n. However, Theorem 14 of [6] can be improved as follows.

THEOREM 8. *If $f \in \mathcal{I}(\mathbf{m})$, then there exist a sequence $(s_n)_1^\infty \subset S(\mathcal{P}, X)$ and a set $M \in \sigma(\mathcal{P})$ with $\hat{\mathbf{m}}(M) = 0$ such that $s_n(t) \to f(t)$ and $|s_n(t)| \nearrow |f(t)|$ for $t \in T\backslash M$ and $\lim_n \int_E s_n d\mathbf{m} = \int_E f d\mathbf{m}$ for $E \in \sigma(\mathcal{P})$, the limit being uniform with respect to $E \in \sigma(\mathcal{P})$. Consequently,*

$$\hat{\mathbf{m}}(E) = \sup\{|\int_E f d\mathbf{m}| : f \in \mathcal{I}(\mathbf{m}), ||f||_E \leq 1\}, \quad E \in \sigma(\mathcal{P})$$

and hence

$$|\int_E f d\mathbf{m}| \leq ||f||_E \cdot \hat{\mathbf{m}}(E)$$

for $f \in \mathcal{I}(\mathbf{m})$ and $E \in \sigma(\mathcal{P})$.

Proof. Let $f \in \mathcal{I}(\mathbf{m})$. By Proposition 7 and Definition 15, there exist two sequences of X-valued $\mathcal{P}$-simple functions $(w_n)_1^\infty$ and $(h_n)_1^\infty$ and a set $M \in \sigma(\mathcal{P})$ with $\hat{\mathbf{m}}(M) = 0$ such that $w_n(t) \to f(t)$, $h_n(t) \to f(t)$ and

$|w_n(t)| \nearrow |f(t)|$ for $t \in T\backslash M$ and such that $\gamma_n(\cdot) = \int_{(\cdot)} h_n d\mathbf{m}$, $n \in \mathbb{N}$, are uniformly σ-additive on $\sigma(\mathcal{P})$ with $\lim_n \gamma_n(E) = \int_E f d\mathbf{m}$ for $E \in \sigma(\mathcal{P})$. Let $\eta_n(\cdot) = \int_{(\cdot)} w_n d\mathbf{m}$, $n \in N$. Let $\nu(E) = \int_E f d\mathbf{m}$, $E \in \sigma(\mathcal{P})$. Let $F = \bigcup_1^\infty \{t \in T\backslash M : |h_n(t)| + |w_n(t)| > 0\}$. Let λ be the continuous submeasure defined as in Lemma 6 with respect to these σ-additive vector measures $(\gamma_n)_1^\infty$ and $(\eta_n)_1^\infty$. Let $u_{2n-1} = h_n$ and $u_{2n} = w_n$ for $n \in \mathbb{N}$. Then $(u_n)_1^\infty \subset S(\mathcal{P}, X)$ converges to f pointwise in $T\backslash M$. So by the Egoroff-Lusin theorem (with respect to λ) there exist $N \in F \cap \sigma(\mathcal{P})$ with $\lambda(N) = 0$ and a sequence $(F_k)_1^\infty \subset \mathcal{P}$ with $F_k \nearrow F\backslash N$ such that $u_n \to f$ uniformly on each F_k.

As $u_n \to f$ uniformly on each F_k, we can select a subsequence $(n_k)_1^\infty$ of $\mathbb{N}$ such that $||h_{n_k} - w_{n_k}||_{F_k} \cdot \hat{\mathbf{m}}(F_k) < \frac{1}{k}$ for each k. Let $s_k = w_{n_k}\chi_N + w_{n_k}\chi_{F_k}$. Clearly the $\mathcal{P}$-simple functions s_k converge pointwise to f in $T\backslash M$ with $|s_k(t)| \nearrow |f(t)|$ for $t \in T\backslash M$. Let $G = F\backslash N$. $\hat{\mathbf{m}}(M) = 0$ implies by Proposition 5(i) that $\bar{\gamma}_n(M) = \bar{\eta}_n(M) = 0$ for all n and hence $\lambda(M) = 0$. Moreover, as $\lambda(N) = \lambda(M) = 0$, $\eta_n(E \cap N) = \eta_n(E \cap M) = \gamma_n(E \cap N) = \gamma_n(E \cap M) = 0$ for all n and clearly $s_k(t) = 0$ for $t \in E \cap (G\backslash F_k)$. Hence we have $|\int_E f d\mathbf{m} - \int_E s_k d\mathbf{m}| \leq |\int_{E\cap F_k}(s_k - h_{n_k})d\mathbf{m}| + |\int_{E\cap(G\backslash F_k)} h_{n_k} d\mathbf{m}| + |\int_E f d\mathbf{m} - \int_E h_{n_k} d\mathbf{m}|$. Consequently, by Proposition 5(i) we obtain $|\int_E f d\mathbf{m} - \int_E s_k d\mathbf{m}| \leq ||w_{n_k} - h_{n_k}||_{F_k} \cdot \hat{\mathbf{m}}(F_k) + ||\gamma_{n_k}||(G\backslash F_k) + |\int_E f d\mathbf{m} - \int_E h_{n_k} d\mathbf{m}|$. Given $\epsilon > 0$, let us choose k_0 such that $\frac{1}{k_0} < \frac{\epsilon}{3}$. By Theorem 7 $\nu(E) = \lim_k \int_E h_{n_k} d\mathbf{m}$ uniformly with respect to $E \in \sigma(\mathcal{P})$ and hence we can choose $k_1 \geq k_0$ such that $|\nu(E) - \int_E h_{n_k} d\mathbf{m}| < \frac{\epsilon}{3}$ for all $k \geq k_1$ and for all $E \in \sigma(\mathcal{P})$. Thus choosing $k \geq k_1$ we have

$$||w_{n_k} - h_{n_k}||_{F_k} \cdot \hat{\mathbf{m}}(F_k) \quad < \quad \frac{\epsilon}{3} \tag{1}$$

and

$$|\nu(E) - \int_E h_{n_k} d\mathbf{m}| \quad < \quad \frac{\epsilon}{3} \text{ for all } E \in \sigma(\mathcal{P}). \tag{2}$$

Now by hypothesis $(\gamma_n)_1^\infty$ are uniformly σ-additive on $\sigma(\mathcal{P})$ and as $(G\backslash F_p) \searrow \emptyset$, by Proposition 3 there exists $k_2 \geq k_1$ such that $||\gamma_n||(G\backslash F_k) < \frac{\epsilon}{3}$ for all $k \geq k_2$ and for all $n \in N$. Thus, in particular,

$$||\gamma_{n_k}||(G\backslash F_k) < \frac{\epsilon}{3} \tag{3}$$

for $k \geq k_2$. Consequently, by (1),(2) and (3) we have $|\int_E s_k d\mathbf{m} - \int_E f d\mathbf{m}| < \epsilon$ for $k \geq k_2$ and for $E \in \sigma(\mathcal{P})$. This proves the first part of the theorem. The remaining parts are immediate from the first and the definition of $\hat{\mathbf{m}}$.

This completes the proof of the theorem.

Remark 9. For any sequence of X-valued $\mathcal{P}$-simple functions (s_n) satisfying the hypothesis of the above theorem, generally $\int_{(\cdot)} f d\mathbf{m} \neq \lim_n \int_{(\cdot)} s_n d\mathbf{m}$. However, it holds if and only if $f \in L_1(\mathbf{m})$. See [7,9,17]. For $L_1(\mathbf{m})$ the condition is sufficient by the Lebesgue dominated convergence theorem (see [7]). The necessity is proved via the construction of a counter example when f does not belong to $L_1(\mathbf{m})$ (see [9]).

Remark 10. The inequality in Theorem 14 of [6] replaces that of Proposition 5(i) to extend the proofs given for simple integrable functions in [6] to general integrable functions. For example, see Theorems 2, 3, 9 and 11 of [6].

Theorem 10 of [6] is valid, but its proof should be corrected by applying the Egoroff-Lusin theorem with respect to the continuous submeasure λ of Lemma 6 (and not by Theorem 1 of [6]), with $\gamma_n(\cdot) = \int_{(\cdot)} s_n d\mathbf{m}$ and $\eta_n = 0$ for $n \in \mathbb{N}$, where $(s_n)_1^\infty \subset S(\mathcal{P}, X)$ and $s_n \to f$ in T.

The following theorem is an improved version of Theorems 15 and 16 of [6] and the original proof of [6] is rectified here by defining suitably the continuous submeasure λ.

THEOREM 9 (**Theorem of closure or of interchange of limit and integral**). Let $f : T \to X$ and suppose $(f_n)_1^\infty \subset \mathcal{I}(\mathbf{m})$ converges to f $\mathbf{m}$-a.e. in T. Then f is $\mathbf{m}$-measurable. Let $\gamma_n(\cdot) = \int_{(\cdot)} f_n d\mathbf{m} : \sigma(\mathcal{P}) \to Y$ for $n \in \mathbb{N}$. Then the following statements are equivalent:

(i) $\lim_n \gamma_n(E) = \gamma(E)$ exists in Y for each $E \in \sigma(\mathcal{P})$.

(ii) γ_n, $n \in \mathbb{N}$, are uniformly σ-additive on $\sigma(\mathcal{P})$.

(iii) $\lim_n \gamma_n(E) = \gamma(E)$ exists in Y uniformly with respect to $E \in \sigma(\mathcal{P})$.

If anyone of the above conditions holds, then f is $\mathbf{m}$-integrable and

$$\int_E f d\mathbf{m} = \int_E (\lim_n f_n) d\mathbf{m} = \lim_n \int_E f_n d\mathbf{m}, \quad E \in \sigma(\mathcal{P})$$

the limit being uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. By Theorem 7, γ_n, $n \in \mathbb{N}$, are σ-additive on $\sigma(\mathcal{P})$. Then by VHSN (i)⇒(ii) and the implication (iii)⇒(i) is obvious.

Suppose (ii) holds. By Theorem 3 f is $\mathbf{m}$-measurable and by hypothesis there exists $M \in \sigma(\mathcal{P})$ with $\hat{\mathbf{m}}(M) = 0$ such that $f_n(t) \to f(t)$ for $t \in T \backslash M$ and $(f_n \chi_{T\backslash M})_{n=1}^{\infty}$ are $\mathcal{P}$-measurable. By Theorem 1 $f\chi_{T\backslash M}$ is also $\mathcal{P}$-measurable. Define λ of Lemma 6 with γ_n as above and $\eta_n = 0$ for $n \in \mathbb{N}$. Let $F = \bigcup_{n=1}^{\infty} N(f_n) \cap (T\backslash M)$. Then $F \in \sigma(\mathcal{P})$. As $\hat{\mathbf{m}}(M) = 0$, by Theorem 8 $|\gamma_n(E)| \le ||f_n||_E \cdot \hat{\mathbf{m}}(E) = 0$ for $E \subset M$, $E \in \sigma(\mathcal{P})$,where we define $0 \cdot \infty = 0$. Thus we have $\bar{\gamma}_n(M) = 0$ for all n and hence $\lambda(M) = 0$. Following the proof of Theorem 7 and applying the Egoroff-Lusin theorem (with respect to λ), using the inequality in Theorem 8 instead of Proposition 5(i) and observing that $\lambda(M) = \lambda(N) = 0$ imply that $\gamma_n(E \cap M) = \gamma_n(E \cap N) = 0$ for all n and for $E \in \sigma(\mathcal{P})$, we deduce that $(\int_E f_n d\mathbf{m})_1^{\infty}$ is uniformly Cauchy for $E \in \sigma(\mathcal{P})$. Since Y is complete, (iii) holds.

Since $f\chi_{T\backslash M}$ is $\mathcal{P}$-measurable, there exists a sequence $(w_n)_1^{\infty}$ of $\mathcal{P}$-simple functions such that $w_n(t) \to f(t)\chi_{T\backslash M}(t)$ for $t \in T$. Let $F = \bigcup_{n=1}^{\infty}\{t \in T\backslash M : |f_n(t)| + |w_n(t)| > 0\}$. Let $\eta_n(\cdot) = \int_{(\cdot)} w_n d\mathbf{m}$. Then $F \in \sigma(\mathcal{P})$ and η_n are σ-additive on $\sigma(\mathcal{P})$. Let λ be as in Lemma 6 with η_n and γ_n (as in the above). Taking $u_{2n-1} = f_n$ and $u_{2n} = w_n$, we have $u_n(t) \to f(t)$ for $t \in T\backslash M$. As observed in the above, $\hat{\mathbf{m}}(M) = 0$ implies that $\bar{\gamma}_n(M) = 0$ for all n. Similarly, by Proposition 5(i) $\bar{\eta}_n(M) = 0$ for all n. Thus $\lambda(M) = 0$. By the Egoroff-Lusin theorem (with respect to λ) there exist $N \in F \cap \sigma(\mathcal{P})$ with $\lambda(N) = 0$ and an increasing sequence $(F_k)_1^{\infty} \subset \mathcal{P}$ with $F_k \nearrow F\backslash N$ such that $u_n \to f$ uniformly on each F_k. As γ_n are uniformly σ-additive by hypothesis (ii), we can repeat the argument given in the second paragraph of the proof of Theorem 8 by replacing h_n by f_n, by choosing a subsequence (n_k) of $\mathbb{N}$ such that $||w_{n_k} - f_{n_k}||_{F_k} \cdot \hat{\mathbf{m}}(F_k) < \frac{1}{k}$ and by defining $s_k = w_{n_k}\chi_N + w_{n_k}\chi_{F_k}$. Then $(s_k)_1^{\infty}$ are $\mathcal{P}$-simple and $s_k(t) \to f(t)$ for $t \in T\backslash M$. Let $G = F\backslash N$. As $\lambda(N) = \lambda(M) = 0$, we have $\eta_n(E \cap N) = \eta_n(E \cap M) = \gamma_n(E \cap N) = \gamma_n(E \cap M) = 0$ for all n and $s_k(t) = 0$ for $t \in E \cap (G\backslash F_k)$. Given $\epsilon > 0$, using the inequality in Theorem 8 and arguing as in the proof of Theorem 8, we have

$$\begin{aligned}|\gamma(E) - \int_E s_k d\mathbf{m}| &\le |\int_{E\cap F_k}(s_k - f_{n_k})d\mathbf{m}| + ||\gamma_{n_k}||(E \cap (G\backslash F_k)) \\ &+ |\gamma(E) - \int_E f_{n_k} d\mathbf{m}| \\ &\le ||w_{n_k} - f_{n_k}||_{F_k} \cdot \hat{\mathbf{m}}(F_k) + ||\gamma_{n_k}||(G\backslash F_k)\end{aligned}$$

$$+ \quad |\gamma(E) - \gamma_{n_k}(E)| < \epsilon$$

for sufficiently large k and for all $E \in \sigma(\mathcal{P})$.

Thus f is $\mathbf{m}$-integrable and $\int_E f d\mathbf{m} = \gamma(E) = \lim_n \int_E f_n d\mathbf{m}$ for $E \in \sigma(\mathcal{P})$, the limit being uniform with respect to $E \in \sigma(\mathcal{P})$.

This completes the proof of the theorem.

Remark 11. The above theorem is called closure theorem for the following reason. If the process of Theorem 7 is repeated with sequences of functions in $\mathcal{I}(\mathbf{m})$ instead of X-valued $\mathcal{P}$-simple functions, we obtain only $\mathcal{I}(\mathbf{m})$ and no new $\mathbf{m}$-measurable functions are obtained. Clearly, the theorem gives necessary and sufficient conditions for the validity of the interchange of integral and limit, which hold particularly for abstract Lebesgue integral. Moreover, $\mathcal{I}(\mathbf{m})$ is the smallest class in $\mathcal{M}(\mathcal{P}, X, \mathbf{m})$ containing $S(\mathcal{P}, X)$ for which Theorem 9 holds. More precisely, let $\mathcal{J}(\mathbf{m})$ be another class of X-valued $\mathbf{m}$-measurable functions which are integrable in a different sense $(\mathcal{J})$ with the integral being denoted by $(\mathcal{J}) \int_{(\cdot)} f d\mathbf{m}$ for $f \in \mathcal{J}(\mathbf{m})$. If for each X-valued $\mathcal{P}$-simple function s and for each $E \in \sigma(\mathcal{P})$, $(\mathcal{J}) \int_E s d\mathbf{m} = \int_E s d\mathbf{m}$ and if Theorem 9 holds for $f \in \mathcal{J}(\mathbf{m})$, then $\mathcal{I}(\mathbf{m}) \subset \mathcal{J}(\mathbf{m})$. The last observation shows that Theorem 9 does not hold for the Bochner and Dinculeanu integrable vector functions (see p.102 of [17]). **In other words, among various Lebesgue-type integration theories developed in the literature (see [6,17]), it is only the integral developed by Dobrakov (particular case being the Bartle-Dunford-Schwartz integral (see Remark 8)) that preserves the theorem of interchange of limit and integral for the class of all integrable functions and hence it can be considered as the complete generalization of the abstract Lebesgue integral, while others are only its partial generalizations.**

Using Theorems 7 and 9 and Proposition 8 we provide a detailed proof of the following theorem which is the same as Theorem 13 of [6]. The original proof in [6] is only very sketchy.

THEOREM 10. *Let $f : T \to X$ be $\mathbf{m}$-measurable and let $f_n : T \to X$, $n \in I\!N$, be $\mathcal{P}$-simple functions or more generally, $\mathbf{m}$-integrable functions converging to f in measure-$\mathbf{m}$ (resp. in semivariation $\hat{\mathbf{m}}$) on each $E \in \mathcal{P}$. Let $\gamma_n(\cdot) = \int_{(\cdot)} f_n d\mathbf{m} : \sigma(\mathcal{P}) \to Y$, $n \in I\!N$. Then the following conditions are equivalent:*

(i) $\lim_n \gamma_n(E) = \gamma(E)$ *exists in Y for each $E \in \sigma(\mathcal{P})$.*

(ii) γ_n, $n \in N$, *are uniformly* σ*-additive on* $\sigma(\mathcal{P})$.

(iii) $\lim_n \gamma_n(E) = \gamma(E)$ *exists in* Y *uniformly with respect to* $E \in \sigma(\mathcal{P})$.

If anyone of these conditions holds, then f *is* $\mathbf{m}$*-integrable and* $\int_E f d\mathbf{m} = \lim_n \gamma_n(E)$ *for each* $E \in \sigma(\mathcal{P})$*, the limit being uniform with respect to* $E \in \sigma(\mathcal{P})$.

Proof. The set functions γ_n are σ-additive by Proposition 5(ii) if f_n are simple functions and by Theorem 7 if f_n are integrable functions. Then by VHSN (i)$\Rightarrow$(ii) and (iii)$\rightarrow$(i) obviously. Let (ii) hold. If (iii) does not hold, then there would exist an $\epsilon > 0$, a subsequence $(k_p)_{p=1}^{\infty}$ of $\mathbb{N}$ and a sequence $(E_p)_{p=1}^{\infty} \subset \sigma(\mathcal{P})$ such that $|\gamma_{k_p}(E_p) - \gamma(E_p)| \geq \epsilon$ for $p \in \mathbb{N}$. But, on the other hand, by hypothesis and by Proposition 8 there exists a subsequence $(f_{k_{p_q}})_{q=1}^{\infty}$ of $(f_{k_p})_{p=1}^{\infty}$ such that $f_{k_{p_q}} \rightarrow f$ $\mathbf{m}$-a.e. Then by Theorem 7 in the case of simple functions and by Theorem 9 in the case of $\mathbf{m}$-integrable functions, there exists q_0 such that $|\gamma_{k_{p_q}}(E) - \gamma(E)| < \epsilon$ for all $E \in \sigma(\mathcal{P})$ and for all $q \geq q_0$. This contradiction shows that (ii)$\Rightarrow$(iii). Thus these conditions are equivalent.

By Proposition 8 there exists a subsequence $(f_{n_k})_{k=1}^{\infty}$ such that $f_{n_k} \rightarrow f$ $\mathbf{m}$-a.e. in T. Then by Theorem 7 in the case of simple functions and by Theorem 9 in the case of $\mathbf{m}$-integrable functions, f is $\mathbf{m}$-integrable and $\int_E f d\mathbf{m} = \lim_k \gamma_{n_k}(E) = \lim_n \gamma_n(E)$ for $E \in \sigma(\mathcal{P})$ and by (iii) the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

This completes the proof of the theorem.

Remark 12. In the case of the abstract Lebesgue integral as in Halmos [12] and of he Bochner integral as in Dunford and Schwartz [10], the class of all integrable functions is obtained by starting with sequences of simple functions which converge in measure to a measurable function, satisfying certain Cauchy conditions. But in the present theory of integration of vector functions, there exist functions $f \in \mathcal{I}(\mathbf{m})$ for which there does not exist any sequence of simple functions converging to f in measure-$\mathbf{m}$ or in semivariation $\hat{\mathbf{m}}$ on each $E \in \mathcal{P}$ and satisfying any of the conditions of Theorem 10, even though $\mathcal{P}$ is a σ-algebra. See Example 7" of [6]. A much simpler example is given in [9]. Thus, in contrast to the classical cases of the abstract Lebesgue and Bochner integrals, the class $\mathcal{I}(\mathbf{m})$ cannot be obtained by considering convergence in measure-$\mathbf{m}$ or in semivariation $\hat{\mathbf{m}}$ as in Theorem 10.

References

1. R.G. Bartle, N. Dunford, and J.T. Schwartz, *Weak compactness and vector measures*, Canad. J. Math. 7(1955), 289-305.
2. R.G. Bartle, *A general bilinear vector integral*, Studia Math. 15(1956), 337-352.
3. F. Bombal, *Medidas Vectoriales y Espacios de Funciones Continuas*, Lecture Notes, Facultad de Matemáticas, Univ. Complutense, Madrid, 1984.
4. J. Diestel and J.J. Uhl, *Vector Measures*, in "Survey", Vol. 15, Amer. Math. Soc., Providence, 1977.
5. N. Dinculeanu, *Vector Measures*, Pergamon Press, New York, (1967).
6. I. Dobrakov, *On integration in Banach spaces,I,* Czech. Math. J. 20(1970), 511-536.
7. I. Dobrakov, *On integration in Banach spaces, II,* Czech. Math. J. 20(1970), 680-695.
8. I. Dobrakov, *On representation of linear operators on* $C_0(T, X)$, Czech. Math. J. 21(1971), 13-30.
9. I. Dobrakov, and T.V. Panchapagesan, *Complements to integration of vector functions*, submitted.
10. N. Dunford, and J.T. Schwartz, *Linear Operators. Part I. General Theory*, Interscience, New York, 1958.
11. A. Grothendieck, *Sur les applications linéares faiblement compactes d'espaces du type* $C(K)$, Canad. J. Math. 5(1953), 129-173.
12. P.R. Halmos, *Measure Theory*, Van Nostrand, New York, 1950.
13. E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, Amer. Math. Soc. Colloq. Publ. Providence, 1957.
14. J.L. Kelley, and T.P. Srinivasan, *On the Bochner integral*, in Vector and operator valued measures and applications, Academic Press, New York, 1973, 165-174.

15. T.V. Panchapagesan, *On complex Radon Measures I*, Czech. Math. J. 42(1992), 599-612.

16. T.V. Panchapagesan, *On complex Radon measures II*, Czech. Math. J., 43(1993), 65-82.

17. T.V. Panchapagesan, *On the distinguishing features of the Dobrakov integral*, Divulgaciones Mat. 3(1995), 79-114.

18. T.V. Panchapagesan, *Characterizations of weakly compact operators on* $C_0(T)$, Trans. Amer. Math. Soc., 350(1998), 4849-4867.

19. T.V. Panchapagesan, *On the limitations of the Grothendieck techniques*, to appear in Revista Real Acad. Ciencias Exactas, Madrid.

20. E. Thomas, *L'integration par rapport a une mesure de Radon vectoriele*, Ann. Inst. Fourier (Grenoble), 20(1970), 55-191.

Departamento de Matemáticas,
Facultad de ciencias,
Universidad de los Andes,
Mérida, Venezuela.
E-mail address:panchapa@ciens.ula.ve