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On the class $\mathbb{A}_{\aleph_0}^N(\mathcal{H})^*$

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Abstract

In this note we consider polynomially bounded N-tuples of commuting operators that generate w^*-continuous, invertible representations of $H^\infty(\mathbb{D}^N)$. We introduce the classes $\mathbb{A}_{\aleph_0}^N(\mathcal{H})$ $(1 \leq p, q \leq \aleph_0)$ similarly to the one variable case, and state a necessary condition for membership in the class $\mathbb{A}_{\aleph_0}^N(\mathcal{H})$ under certain technical assumptions.

1 Introduction

Let $\mathbb{D}$ denote the unit disc in the complex plane $\mathbb{C}$, and let $\mathbb{D}^N$the unit polydisc in $\mathbb{C}^N$. We denote by $\mathbb{T}$ the boundary of $\mathbb{D}$, so that $\mathbb{T}^N$is the distinguished boundary of $\mathbb{D}^N$. Let $H^\infty(\mathbb{D}^N)$ be the algebra of the bounded analytic functions on $\mathbb{D}^N$. Recall that $H^\infty(\mathbb{D}^N)$ can be identified with a subspace of $L^\infty(\mathbb{T}^N)$, the space of essentially bounded functions (with respect to m, normalized Lebesgue measure) and, thus $H^\infty(\mathbb{D}^N)$ can be seen as the dual of the Banach space $L^1(\mathbb{T}^N)/^\perp H^\infty(\mathbb{D}^N)$where

$$^\perp H^\infty(\mathbb{D}^N) = \left\{ g \in L^1(\mathbb{T}^N) : \int_{\mathbb{T}^N} gf\, dm = 0 \text{ for all } f \in H^\infty(\mathbb{D}^N) \right\}.$$

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operator on $\mathcal{H}$. We recall that $\mathcal{L}(\mathcal{H})$ can be identified with the dual of the trace class $\mathcal{C}_1(\mathcal{H})$, and the duality is given by the bilinear form

$$\langle T, S \rangle = \mathrm{tr}\,(TS), \qquad T \in \mathcal{C}_1(\mathcal{H}),\ S \in \mathcal{L}(\mathcal{H}).$$

These considerations show that $H^\infty(\mathbb{D}^N)$ and $\mathcal{L}(\mathcal{H})$ are *dual algebras* in the sense of [Con91, I.2.8]. A *representation* of $H^\infty(\mathbb{D}^N)$ is an homomorphism of algebras $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ with $\Phi(1) = I$.

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Let $T = (T_1, \ldots, T_N)$ be an N-tuple of commuting operators on $\mathcal{H}$. We say that $T = (T_1, \ldots, T_N)$ is *polynomially bounded* if there exists a constant $M \geq 1$ such that

$$\|p(T_1, \ldots, T_N)\| \leq M \sup_{z \in \mathbb{D}^N} |p(z)|,$$

for every polynomial $p \in \mathbb{C}[z_1, \ldots, z_N]$. We write $PB^N(\mathcal{H})$ for the class of such N-tuples.

If $T \in PB^N(\mathcal{H})$, the map $p \to p(T)$, defined in the natural way for polynomials, extends to a norm continuous algebra homomorphism $\Psi : A(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$. We are interested in the case that Ψ extends to a w^*- continuous representation of $H^\infty(\mathbb{D}^N)$. We will say that this representation is *generated* by T. In [Cha] necessary and sufficient conditions are given for the existence of such representations.

The class of polynomially bounded N-tuples which generate w^*-continuous representations of $H^\infty(\mathbb{D}^N)$ will be denoted by $ACPB^N(\mathcal{H})$ (cf. [Cha]). In particular if $T \in ACPB^N(\mathcal{H})$ generates an invertible representation Φ, it is easy to see (cf. [BCP79]) that Φ is a w^*-homeomorphism of $H^\infty(\mathbb{D}^N)$ onto $\mathcal{A}_T$, the dual algebra generated by T.

In this paper we will consider the class $ACPB^N(\mathcal{H})$ and similarly to the one variable case, will introduce the class $\mathbb{A}^N_{p,q}(\mathcal{H})$, $(1 \leq p, q \leq \aleph_0)$ in order to study the predual of $\mathcal{A}_T$. In the Section 2, we give the definitions and preliminary results, and in the Section 3 we consider in particular, the class $\mathbb{A}^N_{\aleph_0}(\mathcal{H})$, and give a necessary geometric condition for membership in this class, under certain technical assumption. This give a partial answer to a question in [Oct95] and extends a result in [LP95].

2 The classes $\mathbb{A}^N_{p,q}(\mathcal{H})$

We introduce some definitions that are natural extensions of those for one operator (cf. [BFP85] and [LP95]).

Definition 2.1. *The class $\mathbb{A}^N(\mathcal{H})$ consists of those N-tuples T in $ACPB^N(\mathcal{H})$ such that the w^*-continuous representation $\Phi :\ H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ generated by T is invertible.*

Let $\mathcal{Q}_T = \mathcal{C}_1(\mathcal{H})/{}^\perp\mathcal{A}_T$ be the predual of $\mathcal{A}_T$, it is easy to see that the bounded, linear, one-one operator $\Phi_* : \mathcal{Q}_T \to L^1(\mathbb{T}^N)/{}^\perp\, H^\infty(\mathbb{D}^N)$ such that $(\Phi_*)^* = \Phi$ is an isomorphism of $\mathcal{Q}_T$ onto $L^1(\mathbb{T}^N)/{}^\perp\, H^\infty(\mathbb{D}^N)$.

We denote by $x \otimes y$ the range one operator

$$x \otimes y : \mathcal{H} \to \mathcal{H}, \qquad z \to \langle z, y\rangle x.$$

Let $[x \otimes y]$ be the element in $\mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}_T$ that contains the operator $x \otimes y$. We see that $\Phi_*(x \otimes y)$ is the functional w^*-continuous on $H^\infty(\mathbb{D}^N)$ such that

$$\langle \Phi_*([x \otimes y]), u\rangle = \langle [x \otimes y], \Phi(u)\rangle = tr\left((x \otimes y)\Phi(u)\right) = \langle \Phi(u)x, y\rangle,$$

for every $u \in H^\infty(\mathbb{D}^N)$. Thus, in the sequel, we will write $x \otimes y$ $(x, y \in \mathcal{H})$ for the one range operator and also for the w^*-continuous functional

$$x \otimes y : H^\infty(\mathbb{D}^N) \to \mathbb{C}, \qquad f \to \langle \Phi(f)x, y\rangle.$$

For every $\lambda \in \mathbb{D}^N$, the functional

$$\mathcal{E}_\lambda : \ H^\infty(\mathbb{D}^N) \to \mathbb{C}, \qquad u \to u(\lambda),$$

is w^*-continuous and we will write $[L_\lambda]$ to denote the element in $\mathcal{C}_1(\mathcal{H})/^{\perp}\mathcal{A}_T$ such that $\Phi_*([L_\lambda]) = \mathcal{E}_\lambda$. Observe that

$$u(\lambda) = \langle \mathcal{E}_\lambda, u\rangle = \langle \Phi_*([L_\lambda]), u\rangle = \langle [L_\lambda], \Phi(u)\rangle,$$

for all $u \in H^\infty(\mathbb{D}^N)$.

Definition 2.2. *Let p, q be cardinal numbers with $1 \le p, q \le \aleph_0$. We will say that $T \in \mathbb{A}^N(\mathcal{H})$ is in the class $\mathbb{A}^N_{p,q}(\mathcal{H})$ if for every doubly indexed family $\{[L_{ij}]\}$, $(0 \le i < p,\ 0 \le j < q)$ in $\mathcal{Q}_T$, there exist vectors $\{x_i\}_{0\le i<p}$ and $\{y_i\}_{0\le j<q}$ in $\mathcal{H}$ such that*

$$[L_{ij}] = [x_i \otimes y_j], \qquad (0 \le i < p, 0 \le j < q).$$

If $p = q$, we will write $\mathbb{A}^N_p(\mathcal{H})$ for $\mathbb{A}^N_{p,p}(\mathcal{H})$.

Note that $T \in \mathbb{A}^N_{p,q}(\mathcal{H})$ if and only if for every family $\{L_{ij}\}$, $(0 \le i < p,\ 0 \le j < q)$ in $L^1(\mathbb{T}^N)/^{\perp}\ H^\infty(\mathbb{D}^N)$, there exists vectors $\{x_i\}_{0\le i<p}$ and $\{y_i\}_{0\le j<q}$ in $\mathcal{H}$ such that

$$L_{ij} = x_i \otimes y_j \qquad (0 \le i < p, 0 \le j < q).$$

Therefore our definitions are the same that the corresponding in [BHGP88].

We recall also the following definitions (cf. [BFP85]).

Definition 2.3. *Let* $0 \leq \theta < \gamma \leq 1$*, let* $\mathcal{X}_\theta(\mathcal{A}_T)$ *be the set of all the* $[L] \in \mathcal{Q}_T$ *such that there exists sequences* $\{x_n\}$ *and* $\{y_n\}$*, in the closed unit ball of* $\mathcal{H}$ *that satisfy the following conditions:*

$$\limsup_{n\to\infty} \|[x_n \otimes y_n] - [L]\| \leq \theta,$$

and

$$\|[x_n \otimes w]\| + \|[w \otimes y_n]\| \to 0, \qquad w \in \mathcal{H}.$$

We will say that $\mathcal{A}_T$ *has the* $X_{\theta,\gamma}$ *property if* $\mathcal{X}_\theta(\mathcal{A}_T)$ *contain* $(\mathcal{Q}_T)_\gamma$*, the closed ball in* $\mathcal{Q}_T$ *with radius* γ *and center in the origin of* $\mathcal{Q}_T$*.*

It is well known that $\mathcal{X}_\theta(\mathcal{A}_T)$ is a closed absolutely convex subset of $\mathcal{Q}_T$. In similar way to [LP95], using [BFP85, Ths. 3.7 y 9.22], we obtain immediately:

Theorem 2.4. *(cf. [LP95, Th.2.1]) If* $T \in ACPB^N(\mathcal{H})$ *and* $\mathcal{A}_T$ *has the* $X_{\theta,\gamma}$ *property for some* $0 \leq \theta < \gamma \leq 1$*, then* $T \in \mathbb{A}^N_{\aleph_0}(\mathcal{H})$ *and* $\mathcal{A}_T$ *is reflexive.*

In the case of one variable, conditions on the spectrum of the operator and applications of standard techniques of dual algebras carry to results on the existence of invariant subspaces for the operator (cf. [LP95, Th.1.9]). In our case, we obtain similar results.

We recall that a representation $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is of class $C_{0\cdot}$ if for every sequence $\{h_n\}$ in $H^\infty(\mathbb{D}^N)$ such that $h_n @ > w^* >> 0$ we have that $h_n(T) \to 0$ in the strong operator topology (SOT). The representation Φ is in the class $C_{\cdot 0}$, if when $h_n @ > w^* >> 0$, we have that $h_n(T)^* @ > SOT >> 0$.

If Φ is generated by $T \in ACPB^N(\mathcal{H})$, it is well known that Φ is in the class $C_{0\cdot}$ (respectively in the class $C_{\cdot 0}$) if and only if T_i is of class $C_{0\cdot}$ (of class $C_{\cdot 0}$) for every $i = 1, \ldots, N$ (see [Apo80]). Additionally we will say that Φ is in the class C_{00} if it is in the class $C_{0\cdot}$ and also in the class $C_{\cdot 0}$.

We recall that a subset $\Delta \subset \mathbb{D}^N$ is said to be *dominating* for $\mathbb{T}^N$if

$$\sup\{|u(z)| : z \in \Delta\} = \|u\|_\infty$$

for every $u \in H^\infty(\mathbb{D}^N)$.

Several options exist to define the joint spectrum of an N-tuple T. We refer to [Cur88] for the definitions and properties of the different concepts of joint spectrum. We consider the Harte spectrum $\sigma_H(T)$, the joint right spectrum $\sigma_r(T)$, the joint left spectrum $\sigma_l(T)$ and the corresponding essential spectra ($\sigma_{He}(T)$, $\sigma_{re}(T)$, and $\sigma_{le}(T)$). The following results are obtained in similar way to the case of one variable.

Theorem 2.5. *Let $T \in \mathbb{A}^N(\mathcal{H})$, and let $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be the $w*$-continuous representation generated by T. In each of the following cases:*

1. *If $\sigma_{le}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and Φ is in the class $C_{0\cdot}$;*
2. *if $\sigma_{re}(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and Φ is in the class $C_{\cdot 0}$;*

$\mathcal{A}_T$ has the $X_{0,1/\|\Psi\|}$ property and therefore $T \in \mathbb{A}^N_{\aleph_0}(\mathcal{H})$ and $\mathcal{A}_T$ is reflexive.

Corollary 2.6. *Let $T \in \mathbb{A}^N(\mathcal{H})$, and let $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be the $w*$-continuous representation generated by T. In each of the following cases:*

1. *If $\sigma_l(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and Φ is in the class $C_{0\cdot}$;*
2. *if $\sigma_r(T) \cap \mathbb{D}^N$ is dominating for $\mathbb{T}^N$ and Φ is in the class $C_{\cdot 0}$;*

T has no trivial invariant subspaces.

3 The class $\mathbb{A}^N_{\aleph_0}(\mathcal{H})$ and the $X_{\theta,\gamma}$ property

In the case of one contraction T,it is well known (cf. [BFP85, Th. 6.3]), that $T \in \mathbb{A}_{\aleph_0}(\mathcal{H})$ if and only if $\mathcal{A}_T$ has the $X_{\theta,\lambda}$ property, for some $0 \le \theta < \lambda \le 1$. The following question was posed in [Oct95, Problem 4.3] for $N = 2$.

Problem 3.1. *Let $T = (T_1, \dots, T_N)$ be an N-tuple of commuting contractions that generate an isometric representation Φ of $H^\infty(\mathbb{D}^N)$. If T is in the class $\mathbb{A}^N_{\aleph_0}(\mathcal{H})$, does $\mathcal{A}_T(\mathcal{H})$ have the $X_{\theta,\gamma}$-property, for some $0 \le \theta < \gamma \le 1$?; is the $X_{0,1}$ property equivalent to the $X_{\theta,\gamma}$ property for every $0 \le \theta < \gamma \le 1$?*

In [Oct95] a partial answer to this question is given, but the proof of the result has a gap. On the other hand for one operator $T \in ACPB^1(\mathcal{H})$, C. Pearcy y W. S. Li posed the following question:

Problem 3.2. *[LP95, Problem 2.4] If $T \in ACPB^1(\mathcal{H})$ generate a representation Φ with norm M, and $T \in \mathbb{A}^1_{\aleph_0}(\mathcal{H})$ then, does $\mathcal{A}_T$ have the $X_{\theta,\gamma}$ property for some $0 \le \theta < \gamma \le 1/M$?*

This lend us to ask:

Problem 3.3. *If $T \in ACPB^N(\mathcal{H})$ generate a representation Φ with norm M, and $T \in \mathbb{A}^N_{\aleph_0}(\mathcal{H})$, does $\mathcal{A}_T$ has the $X_{\theta,\gamma}$ property for some $0 \le \theta < \gamma \le 1/M$?*

For $T \in ACPB^1(\mathcal{H}) \cap C_{00}$ in [LP95, Th. 2.14] an affirmative answer is obtain. In this Section we give a partial answer to the Problem 3.3 using techniques in [Oct95] and [Esc98]. We recall (cf. [BHGP88]) that if $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is a representation and $\mathcal{M} \subset \mathcal{H}$ is a closed subspace, $\mathcal{M}$ is *invariant* for Φ if $\Phi(u)(\mathcal{M}) \subset \mathcal{M}$ for every $u \in H^\infty(\mathbb{D}^N)$ and $\mathcal{M}$ is *semi-invariant* for Φ if we can write $\mathcal{M} = \mathcal{U} \ominus \mathcal{V}$, where $\mathcal{U} \supset \mathcal{V}$ are invariant subspaces for Φ. If $\mathcal{M}$ is semi-invariant for Φ, we will denote by $\Phi_{\mathcal{M}}$ the *compression* of Φ to $\mathcal{M}$, i.e.,

$$\Phi_{\mathcal{M}}(u) = P_{\mathcal{M}}\Phi(u)|\mathcal{M}, \qquad u \in H^\infty(\mathbb{D}^N),$$

where $P_{\mathcal{M}}$ denote the orthogonal projection onto the space $\mathcal{M}$. We recall that $\mathcal{M}$ is semi-invariant if and only if $\Phi_{\mathcal{M}}$ is multiplicative (cf. [Pis96, Th.1.7]).

If $\{\lambda_n\}$ is a sequence in $\mathbb{D}^N$ and $\{e_n\}$ is a orthonormal basis for $\mathcal{H}$, the representation $\Psi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$, given for

$$\Psi(u)e_n = u(\lambda_n)e_n, \qquad u \in H^\infty(\mathbb{D}^N),\ n \in \mathbb{N},$$

is the *diagonal representation* associated with $\{\lambda_n\}$ and $\{e_n\}$. If $\{\lambda_n\}$ is dominating for $\mathbb{D}^N$, this representation has the following property (to see the proof of [BHGP88, Th. 3.1]): for all functional w^*-continuous L on $H^\infty(\mathbb{D}^N)$, there exists sequences $\{x_n\}$ y $\{y_n\}$ in the closed unit ball of $\mathcal{H}$, such that

$$\|L - x_n \otimes y_n\| \to 0,$$

and

$$\|x_n \otimes w\| + \|w \otimes y_n\| \to 0, \qquad w \in \mathcal{H}.$$

It is immediate of [BHGP88, Th.4.4] (to see also [Esc98]) the following result.

Proposition 3.4. *Let $T \in \mathbb{A}^N_{\aleph_0}(\mathcal{H})$ and let $\Phi :\ H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be the representation generated by T. For every sequence $\{\lambda_n\}$ in $\mathbb{D}^N$, there exists $\mathcal{M}$ a semi-invariant subspace Φ, and an orthonormal basis $\{e_n\}$ for $\mathcal{M}$ such that the compression $\Phi_{\mathcal{M}}$ of Φ to $\mathcal{M}$ is the diagonal representation associated to $\{\lambda_n\}$ and $\{e_n\}$.*

For every $\lambda = (\lambda_1, \ldots, \lambda_N)$ in $\mathbb{D}^N$, we fix $\phi_\lambda : \mathbb{D}^N \to \mathbb{D}^N$ the map defined by

$$\phi_\lambda(z) = (\phi_{\lambda_1}(z_1), \ldots, \phi_{\lambda_N}(z_N)),$$

where $\phi_{\lambda_i} : \mathbb{D} \to \mathbb{D}$, $(i = 1, \ldots, N)$, is the Möbius transformation

$$\phi_{\lambda_i}(z) = \frac{z - \lambda_i}{1 - \overline{\lambda_i} z}, \qquad z \in \mathbb{D}.$$

We write besides

$$R_\lambda : H^\infty(\mathbb{D}^N) \to H^\infty(\mathbb{D}^N), \qquad f \to f \circ \phi_\lambda.$$

Observe that R_λ is an isomorphism of dual algebras, and we can write $S_\lambda^* = R_\lambda$, for a bounded operator $S_\lambda : \mathcal{Q} \to \mathcal{Q}$. We see that S_λ is a isometry, and that $S_\lambda(\mathcal{E}_\lambda) = \mathcal{E}_0$.

If $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ is a representation w^* continuous, we will write $\Phi_\lambda = \Phi \circ R_\lambda$. Clearly Φ_λ is also a representation w^*-continuous. We will write $x \otimes_\lambda y$ for the functional

$$H^\infty(\mathbb{D}^N) \to \mathbb{C}, \qquad u \to \langle \Phi_\lambda x, y \rangle,$$

with $x, y \in \mathcal{H}$. We see that $S_\lambda(x \otimes y) = x \otimes_\lambda y$. We will use T_λ by denote the N-tuple $(\Psi_\lambda(\chi_1), \ldots, \Psi_\lambda(\chi_N))$.

Finally, we state our main result.

Theorem 3.5. *Let T be a bounded polynomially N-tuple of commuting contractions and let $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ be the representation w^*-continuous with norm M. If T is in the class $\mathbb{A}_{\aleph_0}^{N}(\mathcal{H})$ and for some indexes $i, j \in \{1, \ldots, N\}$, T_i^* y T_j are in the class $C_{0\cdot}$, then Φ satisfies the $X_{0,1/M}$ property.*

Proof. 1. *For every $\lambda \in \mathbb{D}^N$, we will consider the w^*-continuous functional*

$$\mathcal{E}_\lambda \colon H^\infty(\mathbb{D}^N) \to \mathbb{C}, \qquad f \to f(\lambda).$$

Let $\{\lambda_n\}$ be a dominating sequence for $\mathbb{T}^N$. By the Proposition 3.4, there exists a semi-invariant subspace $\mathcal{M}$ of $\mathcal{H}$ for Φ, and an orthonormal basis $\{e_n\}$ of $\mathcal{M}$, such that the compression $\Phi_{\mathcal{M}}$ of Φ to $\mathcal{M}$ is just the diagonal representation associated to $\{\lambda_n\}$ and $\{e_n\}$. Because $\{\lambda_n\}$ is dominating for $\mathbb{D}^N$, by the observations before the Proposition 3.4, for every w^*-continuous functional L on $H^\infty(\mathbb{D}^N)$, with $\|L\| \leq 1$ and every $\epsilon > 0$, there exist vectors $x, y \in \mathcal{H}$ with $\|x\|, \|y\| \leq 1$, such that

$$\|L - x \otimes y\| < \epsilon.$$

In particular, we consider the w^*-continuous functionals

$$\mathcal{E}_0^{(k)}\colon H^\infty(\mathbb{D}^N) \to \mathbb{C}, \qquad f \to \left(\frac{\partial^{2k} f}{\partial z_i^k \partial z_j^k}\right)(0)/(k!)^2.$$

Since $\|\mathcal{E}_0^{(k)}\| \le 1$, there exist sequences $\{u_k\}$ and $\{v_k\}$, in the closed unit ball of $\mathcal{H}$, such that

$$\|\mathcal{E}_0^{(k)} - u_k \otimes v_k\| \to 0, \qquad k \in \mathbb{N}.$$

We define $x_k = T_i^k u_k$ and $y_k = \left(T_j^*\right)^k v_k$. Then $\|x_k\|, \|y_k\| \le 1$ and for every $w \in \mathcal{H}$

$$\langle \Phi(f)x_k, w\rangle = \langle \Phi(f)u_k, (T_i^*)^k w\rangle \to 0,$$

and

$$\langle \Phi(f)w, y_k\rangle = \langle T_j^k w, \Phi(f)^* v_k\rangle \to 0,$$

if $k \to \infty$ uniformly for f in the closed unit ball in $H^\infty(\mathbb{D}^N)$. So we have that $\|x_k \otimes w\| \to 0$ and $\|w \otimes y_k\| \to 0$ for all $w \in \mathcal{H}$.

In addition, since

$$\begin{aligned} x_k \otimes y_k(f) &= \langle \Phi(f)x_k, y_k\rangle \\ &= \langle \Phi(f)T_i u_k, \left(T_j^*\right)^k v_k\rangle \\ &= \langle \Phi(\pi_i^k \pi_j^k f)u_k, v_k\rangle \\ &= u_k \otimes v_k(\pi_i^k \pi_j^k f), \end{aligned}$$

and $\mathcal{E}_0^{(k)}(\pi_i^k \pi_j^k f) = \mathcal{E}_0(f)$ for all $f \in H^\infty(\mathbb{D}^N)$, we see that

$$|\mathcal{E}_0(f) - x_k \otimes y_k(f)| = |\mathcal{E}_0^{(k)}(\pi_i^k \pi_j^k f) - u_k \otimes v_k(\pi_i^k \pi_j^k f)| \le \|\mathcal{E}_0^{(k)} - u_k \otimes v_k\|,$$

for every $f \in H^\infty(\mathbb{D}^N)$, with $\|f\|_\infty \le 1$. Hence, $\|\mathcal{E}_0 - x_k \otimes y_k\| \to 0$ as $k \to \infty$.

Let now $\lambda \in \mathbb{D}^N$ be an arbitrary point in $\mathbb{D}^N$. Clearly, the N-tuple T_λ satisfies the hypothesis of our theorem and so using the results proved before, we see that there exists sequences $\{x_k\}$ and $\{y_k\}$, in the closed unit ball of $\mathcal{H}$, such that

$$\begin{gathered} \|\mathcal{E}_0 - x_k \otimes_\lambda y_k\| \to 0, \\ \|x_k \otimes_\lambda w\| \to 0, \qquad w \in \mathcal{H} \text{ and,} \\ \|w \otimes_\lambda y_k\| \to 0, \qquad w \in \mathcal{H}. \end{gathered}$$

But then $x_k \otimes w = S_\lambda^{-1}(x_k \otimes_\lambda w) \to 0$ *for all* $w \in \mathcal{H}$, $w \otimes y_k = S_\lambda^{-1}(w \otimes_\lambda y_k) \to 0$ *for all* $w \in \mathcal{H}$ *and*

$$\|\mathcal{E}_\lambda - x_k \otimes y_k\| = \|S_\lambda^{-1}(\mathcal{E}_0 - x_k \otimes_\lambda y_k)\| \to 0.$$

It is immediate now that $\mathcal{A}_T$ *has the* $X_{0,1/M}$ *property.*

Corollary 3.6. *Let* T *a polynomially bounded* N*-tuple of commuting contractions and let* $\Phi : H^\infty(\mathbb{D}^N) \to \mathcal{L}(\mathcal{H})$ *be the* w^**-continuous representation generated by* T *with norm* M*. If* Φ *is in the class* $\mathbb{A}_{\aleph_0}^N(\mathcal{H})$ *and for some indexes* $i, j \in \{1, \ldots, N\}$, T_i^* *and* T_j *are in the class* $C_{0\cdot}$*; then* $\mathcal{A}_T$ *satisfies the* $X_{\theta,\gamma}$ *property, for some* $0 \le \theta < \gamma \le M$*, if and only if* $\mathcal{A}_T$ *satisfies the* $X_{0,1/M}$ *property.*

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