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The Bartle-Dunford-Schwartz integral
II. $\mathcal{L}_p$-spaces, $1 \leq p < \infty$

BY

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The Bartle-Dunford-Schwartz integral II. $\mathcal{L}_p$-spaces, $1 \leq p < \infty$*

T.V. PANCHAPAGESAN

Abstract

The aim of the present part is to develop the theory of $\mathcal{L}_p$-spaces for the Bartle-Dunford-Schwartz integral with respect to a Banach space-valued σ-additive vector measure $\mathbf{m}$ dfined on a δ-ring of sets and obtain results analogous to those known for such spaces in the theory of the abstract Lebesgue and Bochner integrals. For this we adapt some of the techniques employed by Dobrakov in the study of integration with respect to operator valued measures. Though a few of these results are already there in the literature for $p = 1$ (sometimes with incorrect proofs as observed in Part I [P1])), they are treated here differently with simpler proofs.

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In the sequel, Definitions, Propositions, Theorems, Remarks, etc., of Part I([P1]) such as Definition 2.3, Proposition 2.10, Theorem 3.5, etc., will be referred to without any explicit reference to Part I. Moreover, the enumeration of sections will be continued from Part I. We use the same notation and terminology given in Part I.

5. THE SEMINORMS $\mathbf{m}_p^\bullet(\cdot, T)$ ON $\mathcal{L}_p\mathcal{M}(\mathbf{m})$, $1 \leq p < \infty$

Dobrakov introduced a seminorm $\hat{\mathbf{m}}(\cdot, T)$ in [Do2] to define and study exhaustively the $\mathcal{L}_1$-spaces associated with an operator valued measure $\mathbf{m}$ which is σ-additive in the strong operator topology on a δ-ring of sets and studied very briefly the corresponding $\mathcal{L}_p$-seminorms, $1 < p < \infty$, idicating the difficulties in developing analogous theory for the $\mathcal{L}_p$-spaces. Similar seminorms, for $p = 1$, were introduced in [KK], [MN], [Ri], etc., for studying the space of (KL) $\mathbf{m}$-integrable functions when $\mathbf{m}$ is a σ-additive vector measure. In the present section, we introduce the seminorms $\mathbf{m}_p^\bullet(\cdot, T)$, $1 \leq p < \infty$, similar to those in [Do2], define the spaces $\mathcal{L}_p\mathcal{M}(\mathbf{m})$, $\mathcal{I}_p(\mathbf{m})$ and $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ and study their basic properties. As in [Do1, Do2] we introduce various notions of convergence similar to those used in the theory of the abstract Lebesgue integral and study their interrelations.

Hereafter, most often, without mentioning Theorem 4.2 we shall directly apply Theorems 3.5 and 3.7 and Corollaries 3.8, 3.9 and 3.10 for functions m-integrable in T. (See Remark 4.3.)

Definition 5.1. Let $g : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ be an $\mathbf{m}$-measurable function. Let $1 \leq p < \infty$ and let $E \in \sigma(\mathcal{P})$. Then

$$\mathbf{m}_p^\bullet(g, E) = \sup\{|\int_E s d\mathbf{m}|^{\frac{1}{p}} : s \in \mathcal{I}_s,\ |s(t)| \leq |g(t)|^p \ \mathbf{m}\text{-a.e. in}\, E\}$$

is called the L_p-gauge of g on E. We define

$$\mathbf{m}_p^\bullet(g, T) = \sup_{E \in \sigma(\mathcal{P})} \mathbf{m}_p^\bullet(g, E).$$

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By Proposition 2.10, $\mathbf{m}_p^\bullet(g,E)$ and $\mathbf{m}_p^\bullet(g,T)$ are well defined and belong to $[0,\infty]$.

Lemma 5.2. Let $g: T \to \boldsymbol{K}$ or $[-\infty,\infty]$ be $\mathbf{m}$-measurable and let $1 \le p < \infty$. Let $x^* \in X^*$ and let $E \in \sigma(\mathcal{P})$. Then:

(i) $(x^*\mathbf{m})_p^\bullet(g,E) = \sup\left\{|\int_E s d(x^*\mathbf{m})|^{\frac{1}{p}} : s \in \mathcal{I}_s, |s(t)| \le |g(t)|^p \,\mathbf{m}\text{-a.e. in}\, E\right\}$.

(ii) $(x^*\mathbf{m})_p^\bullet(g,E) = \left(\int_E |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}}$.

Proof. (i) By Definition 5.1,

$$(x^*\mathbf{m})_p^\bullet(g,E) = \sup\left\{|\int_E s d(x^*\mathbf{m})|^{\frac{1}{p}} : s \in \mathcal{I}_s, |s(t)| \le |g(t)|^p \,(x^*\mathbf{m})\text{-a.e. in}\, E\right\}.$$

Let $\alpha = \sup\left\{|\int_E s d\mathbf{m}|^{\frac{1}{p}} : s \in \mathcal{I}_s, |s(t)| \le |g(t)|^p \,\mathbf{m}\text{-a.e. in}\, E\right\}$. Then clearly, $\alpha \le (x^*\mathbf{m})_p^\bullet(g,E)$. To prove the reverse inequality, let $0 \le c < (x^*\mathbf{m})_p^\bullet(g,E)$. Then there exist $s_0 \in \mathcal{I}_s$ and $M \in \sigma(\mathcal{P})$ with $v(x^*\mathbf{m})(M) = 0$ such that $|s_0(t)| \le |g(t)|^p$ for $t \in T\backslash M$ and $c < |\int_E s_0 d(x^*\mathbf{m})|^{\frac{1}{p}}$. If $s = s_0\chi_{E\backslash M}$, then $s \in \mathcal{I}_s$, $|s(t)| \le |g(t)|^p$ for $t \in E$ and hence $|s(t)| \le |g(t)|^p$ $\mathbf{m}$-a.e. in E and 'moreover, $c < |\int_E s d(x^*\mathbf{m})|^{\frac{1}{p}}$. Hence (i) holds.

(ii) Since $|\int_E s d(x^*\mathbf{m})| \le \int_E |g|^p dv(x^*\mathbf{m})$ for $s \in \mathcal{I}_s$ with $|s(t)| \le |g(t)|^p$ $(x^*\mathbf{m})$-a.e. in E, we have $(x^*\mathbf{m})_p^\bullet(g,E) \le \left(\int_E |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}}$. To prove the reverse inequality, let $0 \le c < \left(\int_E |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}}$. Then by Proposition 2.10 there exists $s = \sum_1^r a_i\chi_{E_i}$, $(E_i)_1^r \subset \mathcal{P}$, $E_i \cap E_j = \emptyset$ for $i \ne j$, with $|s(t)| \le |g(t)|^p$ $(x^*\mathbf{m})$-a.e. in E such that $c < (\sum_1^r |a_k| v(x^*\mathbf{m})(E\cap E_k))^{\frac{1}{p}}$. Hence there exist $(E_{kj})_{j=1}^{\ell_k} \subset \mathcal{P}$, pairwise disjoint, such that $\bigcup_{j=1}^{\ell_k} E_{kj} \subset E \cap E_k$, $k = 1,2,\ldots,r$ and such that

$$\begin{aligned} c^p &< \sum_{k=1}^r |a_k| \sum_{j=1}^{\ell_k} |(x^*\mathbf{m})(E_{kj})| \\ &= \int_E h d(x^*\mathbf{m}) \end{aligned}$$

where $h = \sum_{k=1}^r \sum_{j=1}^{\ell_k} |a_k| \operatorname{sgn}((x^*\mathbf{m})(E_{kj}))\chi_{E_{kj}}$. Then $h \in \mathcal{I}_s$ and $|h(t)| \le |g(t)|^p$ $(x^*\mathbf{m})$-a.e. in E. Hence (ii) holds.

Theorem 5.3. Let $g: T \to \boldsymbol{K}$ or $[-\infty,\infty]$ and let $1 \le p < \infty$. If g is $\mathbf{m}$-measurable and $E \in \sigma(\mathcal{P})$, then

$$\begin{aligned} \mathbf{m}_p^\bullet(g,E) &= \sup_{|x^*|\le 1}\left(\int_E |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} = \sup_{|x^*|\le 1}(x^*\mathbf{m})_p^\bullet(g,E) \\ &= \sup\left\{|\int_E f d\mathbf{m}|^{\frac{1}{p}} : f \in \mathcal{I}(\mathbf{m}),\, |f| \le |g|^p \ \mathbf{m}\text{-a.e. in}\ E\right\}. \end{aligned}$$

Consequently, $|\int_E |f|^p d\mathbf{m}|^{\frac{1}{p}} \le \mathbf{m}_p^\bullet(f,E)$ for $|f|^p \in \mathcal{I}(\mathbf{m})$ and for $E \in \sigma(\mathcal{P})$. Moreover,

$$|\int_E f d\mathbf{m}| \le \mathbf{m}_1^\bullet(f,E) \qquad (5.3.1)$$

for $f \in \mathcal{I}(\mathbf{m})$ and for $E \in \sigma(\mathcal{P})$.

Proof. By Definition 5.1 and Lemma 5.2(i) we have

$$\begin{aligned}\mathbf{m}^\bullet_p(g,E) &= \sup\left\{|\int_E s d\mathbf{m}|^{\frac{1}{p}} : s \in \mathcal{I}_s,\ |s(t)| \le |g(t)|^p\, \mathbf{m}\text{-a.e. in } E\right\}\\ &= \sup\left\{\left(\sup_{|x^*|\le 1} |\int_E s d(x^*\mathbf{m})|^{\frac{1}{p}}\right) : s \in \mathcal{I}_s,\ |s(t)| \le |g(t)|^p\, \mathbf{m}\text{-a.e. in } E\right\}\\ &= \sup_{|x^*|\le 1} (x^*\mathbf{m})^\bullet_p(g,E).\end{aligned}$$

The other equalities hold by Lemma 5.2(ii), by the fact that $\mathcal{I}_s \subset \mathcal{I}(\mathbf{m})$, and by Corollary 3.9. The second part is evident from the first.

Definition 5.4. Let $g : T \to \boldsymbol{K}$ or $[-\infty,\infty]$ be $\mathbf{m}$-measurable. Let $E \in \widetilde{\sigma(\mathcal{P})}$ (see Section 2, paragraphs preceding Notation 2.7), and let $1 \le p < \infty$. Then E is of the form $E = F \cup N$, $F \in \sigma(\mathcal{P})$, $N \subset M \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(M) = 0$. We define $\mathbf{m}^\bullet_p(g,E) = \mathbf{m}^\bullet_p(g,F)$. We also define $\int_E |g|^p dv(x^*\mathbf{m}) = \int_F |g|^p dv(x^*\mathbf{m})$. Then

$$\mathbf{m}^\bullet_p(g,E) = \sup_{|x^*|\le 1}\left(\int_E |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}}. \tag{5.4.1}$$

To verify that $\mathbf{m}^\bullet_p(g,E)$ and $\int_E |g|^p dv(x^*\mathbf{m})$ are well defined, let $E = F_1 \cup N_1 = F_2 \cup N_2$ with $F_i \in \sigma(\mathcal{P})$, $N_i \subset M_i \in \sigma(\mathcal{P})$ and $||\mathbf{m}||(M_i) = 0$ for $i = 1,2$. Let $M = M_1 \cup M_2$. Then $||\mathbf{m}||(M) = 0$ and $F_1 \cup M = F_2 \cup M$. Hence $\int_{F_1} s d\mathbf{m} = \int_{F_1\cup M} s d\mathbf{m} = \int_{F_2\cup M} s d\mathbf{m} = \int_{F_2} s d\mathbf{m}$ for $s \in \mathcal{I}_s$ and therefore, $\mathbf{m}^\bullet_p(g,E)$ is well defined. Moreover, $\int_{F_1} |g|^p dv(x^*\mathbf{m}) = \int_{F_1\cup M} |g|^p dv(x^*\mathbf{m}) = \int_{F_2\cup M} |g|^p dv(x^*\mathbf{m}) = \int_{F_2} |g|^p dv(x^*\mathbf{m})$ and hence $\int_E |g|^p dv(x^*\mathbf{m})$ is also well defined. Then (5.4.1) holds by Theorem 5.3.

Remark 5.5. When $\mathcal{P}$ is a σ-algebra $\mathcal{S}$, the gauge in Definition 5.1 above is given in [KK] for $p = 1$ and that too for (KL) $\mathbf{m}$-integrable $\mathcal{S}$-measurable functions only. Of course, there $\mathbf{m}$ has values in a real lcHs. The analogue of Theorem 5.3 for $p = 1$ is given in Lemma II.2.2 of [KK]. But our proof is more general and elementary than that of the said lemma in [KK] and is adaptable to the case of lcHs-valued vector measures defined on $\mathcal{P}$ (see Definition 13.1 and Theorem 13.2 of [P2]). A similar result for operator valued measures is given in Theorem 4' of [Do2] without proof.

Theorem 5.6. Let $1 \le p < \infty$. Let $f : T \to \boldsymbol{K}$ or $[-\infty,\infty]$ be $\mathbf{m}$-measurable and let $|f|^p$ be $\mathbf{m}$-integrable in T. Let $\gamma(\cdot) = \int_{(\cdot)} |f|^p d\mathbf{m}$. Then $\mathbf{m}^\bullet_p(f,E) = (||\gamma||(E))^{\frac{1}{p}}$, $E \in \sigma(\mathcal{P})$ and consequently, $\mathbf{m}^\bullet_p(f,T) = (||\gamma||(T))^{\frac{1}{p}} < \infty$. Moreover, $\mathbf{m}^\bullet_p(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$ (in the sense of Definition 2.1).

Proof. For $x^* \in X^*$ and for $E \in \sigma(\mathcal{P})$, $v(x^*\gamma)(E) = \int_E |f|^p dv(x^*\mathbf{m})$ by Theorem 3.5(ii) and Proposition 2.11. Therefore, by Theorem 5.3 and by (iii)(a) of Theorem 3.5 we have $\left(\mathbf{m}^\bullet_p(f,E)\right)^p = \sup_{|x^*|\le 1} v(x^*\gamma)(E) = ||\gamma||(E)$. Consequently, $\left(\mathbf{m}^\bullet_p(f,T)\right)^p = \sup_{E\in\sigma(\mathcal{P})}\left(\mathbf{m}^\bullet_p(f,E)\right)^p = \sup_{E\in\sigma(\mathcal{P})} ||\gamma||(E) = ||\gamma||(T) < \infty$ as γ is an X-valued σ-additive vector measure on $\sigma(\mathcal{P})$ by Theorem 3.5(ii). The continuity of $\mathbf{m}^\bullet_p(f,\cdot) = (||\gamma||(\cdot))^{\frac{1}{p}}$ is due to Proposition 2.3.

The converse of Theorem 5.6 is not true in general; i.e., if $f : T \to \boldsymbol{K}$ or $[-\infty,\infty]$ is $\mathbf{m}$-measurable with $\mathbf{m}^\bullet_p(f,T) < \infty$ for some p, $1 \le p < \infty$, then $|f|^p$ need not be $\mathbf{m}$-integrable in T. In fact, we have the following counter-example (see also p.31 of [KK]).

Counter-example 5.7. Let $T = \boldsymbol{N}$ and $\mathcal{S} = \sigma$-algebra of all subsets of T. Let $X = c_0$. Let $1 \leq p < \infty$ and let $f(t) = t^{\frac{1}{p}}$, $t \in T$. For $E \in \mathcal{S}$, let $\mathbf{m}(E) = (a_n) \in c_0$, where $a_n = \frac{1}{n}$ if $n \in E$ and $a_n = 0$ otherwise. Clearly $\mathbf{m}$ is σ-additive on $\mathcal{S}$. If $x^* \in c_0^* = l_1$, let $x^* = (x_n)$. Then $|x^*| = \sum_1^\infty |x_n| < \infty$. Now $\left(\mathbf{m}_p^\bullet(f,T)\right)^p = \sup_{|x^*|\leq 1} \int_T |f|^p dv(x^*\mathbf{m}) = \sup_{|x^*|\leq 1} \sum_1^\infty nv(x^*\mathbf{m})(\{n\})) = \sup_{|x^*|\leq 1} \sum_1^\infty n\frac{1}{n}|x_n| \leq 1 < \infty$. But $|f|^p$ is not $\mathbf{m}$-intergrable in T. In fact, on the contrary we would have $\int_{\{n\}} |f|^p d\mathbf{m} = e_n$ for $n \in T$, where $e_n = (\delta_{nj})_{j=1}^\infty$ and $\delta_{nj} = 1$ if $j = n$ and $= 0$ otherwise; and if $\boldsymbol{\gamma}(\cdot) = \int_{(\cdot)} |f|^p d\mathbf{m}$, then $\boldsymbol{\gamma}$ would be σ-additive on $\mathcal{S}$ with $\boldsymbol{\gamma}(T) \in c_0$. But, $\boldsymbol{\gamma}(T) = \sum_1^\infty \boldsymbol{\gamma}(\{n\}) = \sum_1^\infty e_n = (1,1,1,...) \notin c_0$. This contradiction shows that $f^p = |f|^p \notin \mathcal{I}(\mathbf{m})$.

However, when $c_0 \not\subset X$, we have the following characterization of $\mathbf{m}$-integrability of $|f|^p$.

Theorem 5.8. Let $c_0 \not\subset X$ and let $1 \leq p < \infty$. Then for an $\mathbf{m}$-measurable function f on T with values in $\boldsymbol{K}$ or $[-\infty,\infty]$, $|f|^p$ is $\mathbf{m}$-integrable in T if and only if $\mathbf{m}_p^\bullet(f,T) < \infty$.

Proof. In the light of Theorem 5.6, it is enough to show that the condition is sufficient. Let $\mathbf{m}_p^\bullet(f,T) < \infty$. Then $\sup_{|x^*|\leq 1} \int_T |f|^p dv(x^*\mathbf{m}) = (\mathbf{m}_p^\bullet(f,T))^p < \infty$ by Theorem 5.3. By Proposition 2.10 there exists a sequence $(s_n) \subset \mathcal{I}_s$ such that $0 \leq s_n \nearrow |f|^p$ $\mathbf{m}$-a.e. in T. If $u_n = s_n - s_{n-1}$, for $n \geq 1$, where $s_0 = 0$, then $\sum_1^\infty u_n = |f|^p$ $\mathbf{m}$-a.e. in T. Let $E \in \sigma(\mathcal{P})$. Then, for each $x^* \in X^*$, by the Beppo-Levi theorem for positive measures we have $\sum_1^\infty \int_E |u_n| dv(x^*\mathbf{m}) = \sum_1^\infty \int_E u_n dv(x^*\mathbf{m}) = \int_E |f|^p dv(x^*\mathbf{m}) < \infty$ and consequently, by Proposition 4, §8 of [Din1], $\sum_1^\infty |x^*(\int_E u_n d\mathbf{m})| < \infty$. As $c_0 \not\subset X$, by the Bessaga-Pelczyński theorem there exists a vector $x_E \in X$ such that $x_E = \sum_1^\infty \int_E u_n d\mathbf{m} = \lim_n \int_E s_n d\mathbf{m}$. Since this holds for each $E \in \sigma(\mathcal{P})$, by Definition 4.1, $|f|^p$ is $\mathbf{m}$-integrable in T.

Following [Do2] we give the following

Definition 5.9. Let $1 \leq p < \infty$. Then we define $\mathcal{L}_p\mathcal{M}(\mathbf{m}) = \{f : T \to \boldsymbol{K}, f\ \mathbf{m}\text{-measurable with } \mathbf{m}_p^\bullet(f,T) < \infty\}$; $\mathcal{I}_p(\mathbf{m}) = \{f : T \to \boldsymbol{K}, f\ \mathbf{m}\text{-measurable and } |f|^p \in \mathcal{I}(\mathbf{m})\}$ (so that $\mathcal{I}_1(\mathbf{m}) = \mathcal{I}(\mathbf{m})$ by Theorem 3.5(vii)); and $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \{f \in \mathcal{I}_p(\mathbf{m}) : \mathbf{m}_p^\bullet(f,T) < \infty\}$.

The following result is immediate from Theorems 5.6 and 5.8.

Theorem 5.10. Let $1 \leq p < \infty$. Then $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$. If $c_0 \not\subset X$, then $\mathcal{L}_p\mathcal{M}(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m})$.

As an immediate consequence of Definition 5.4 and Theorem 5.3 we have the following theorem, whose easy proof is omitted.

Theorem 5.11. Let $g : T \to \boldsymbol{K}$ or $[-\infty,\infty]$ be $\mathbf{m}$-measurable and let $E \in \widetilde{\sigma(\mathcal{P})}$. Let $1 \leq p < \infty$. Then the following assertions hold:

(i) $\mathbf{m}_p^\bullet(g,\cdot) : \widetilde{\sigma(\mathcal{P})} \to [0,\infty]$ is monotone and σ-subadditive and vanishes on $\emptyset$.

(ii) $\mathbf{m}_p^\bullet(ag,E) = |a|\mathbf{m}_p^\bullet(g,E)$ for $a \in \boldsymbol{K}$, where $0.\infty = 0$.

(iii) $(\inf_{t\in E} |g(t)|) \cdot (||\mathbf{m}||(E))^{\frac{1}{p}} \leq \mathbf{m}_p^\bullet(g,E) \leq (\sup_{t\in E} |g(t)|) \cdot (||\mathbf{m}||(E))^{\frac{1}{p}}$.

(iv) If h is $\mathbf{m}$-measurable and if $|h| \le |g|$ $\mathbf{m}$-a.e. in E, then $\mathbf{m}_p^\bullet(h, E) \le \mathbf{m}_p^\bullet(g, E)$.

(v) $\mathbf{m}_p^\bullet(g, E) = \mathbf{m}_p^\bullet(g, E \cap N(g))$. (See Notation 2.7.)

(vi) $\mathbf{m}_p^\bullet(g, E) = 0$ if and only if $||\mathbf{m}||(E \cap N(g)) = 0$.

Theorem 5.12. Let $g : T \to \boldsymbol{K}$ or $[-\infty, \infty]$ be $\mathbf{m}$-measurable, η a positive real and $E \in \widetilde{\sigma(\mathcal{P})}$. Let $1 \le p < \infty$. Then:

(i) **(The Tschebyscheff inequality).**

$$||\mathbf{m}||\left(\{t \in E : |g(t)| \ge \eta\}\right) \le \frac{1}{\eta^p}\left(\mathbf{m}_p^\bullet(g, E)\right)^p.$$

(ii) If $\mathbf{m}_p^\bullet(g, E) < \infty$, then g is finite $\mathbf{m}$-a.e. in E. Consequently, if $\mathbf{m}_p^\bullet(g, T) < \infty$, then g is finite $\mathbf{m}$-a.e. in T.

Proof.

(i) Let $F = \{t \in E : |g(t)| \ge \eta\}$. Then $F \in \widetilde{\sigma(\mathcal{P})}$ and by (i) and (iii) of Theorem 5.11 we have

$$\eta\left(||\mathbf{m}||(F)\right)^{\frac{1}{p}} \le \left(\inf_{t \in F} |g(t)|\right) \cdot \left(||\mathbf{m}||(F)\right)^{\frac{1}{p}} \le \mathbf{m}_p^\bullet(g, F) \le \mathbf{m}_p^\bullet(g, E)$$

and hence (i) holds.

(ii) Let $F = \{t \in E : |g(t)| = \infty\}$. Then $F \in \widetilde{\sigma(\mathcal{P})}$ and $F \subset F_n = \{t \in E : |g(t)| \ge n\} \in \widetilde{\sigma(\mathcal{P})}$, $n \in \boldsymbol{N}$. Now by (i), by Theorem 5.11(i) and by hypothesis, we have $||\mathbf{m}||(F) \le ||\mathbf{m}||(F_n) \le (\frac{1}{n})^p \cdot \left(\mathbf{m}_p^\bullet(g, E)\right)^p \to 0$ as $n \to \infty$ and hence g is finite $\mathbf{m}$-a.e. in E. Since $\mathbf{m}_p^\bullet(g, T) = \mathbf{m}_p^\bullet(g, N(g))$, the last part of (ii) also holds.

Theorem 5.13. Let f, g be $\mathbf{m}$-measurable on T with values in $\boldsymbol{K}$ and let $E \in \widetilde{\sigma(\mathcal{P})}$. Let $1 \le p < \infty$ and if $1 < p < \infty$, let $\frac{1}{p} + \frac{1}{q} = 1$. Then:

(i) $\mathbf{m}_p^\bullet(f + g, E) \le \mathbf{m}_p^\bullet(f, E) + \mathbf{m}_p^\bullet(g, E)$.

(ii) $\mathbf{m}_p^\bullet(f + g, T) \le \mathbf{m}_p^\bullet(f, T) + \mathbf{m}_p^\bullet(g, T)$.

(iii) Let $1 < p < \infty$. Then $\mathbf{m}_1^\bullet(fg, T) \le \mathbf{m}_p^\bullet(f, T) \cdot \mathbf{m}_q^\bullet(g, T)$. Consequently, if $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $g \in \mathcal{L}_q\mathcal{M}(\mathbf{m})$, then $fg \in \mathcal{L}_1\mathcal{M}(\mathbf{m})$. (The last part is improved in Theorem 9.1.)

Proof. In the light of Definition 5.4, without loss of generality we shall assume that $E \in \sigma(\mathcal{P})$. Then by Theorem 5.3 we have

$$\begin{aligned} \mathbf{m}_p^\bullet(f + g, E) &= \sup_{|x^*| \le 1}\left(\int_E |f + g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \\ &\le \sup_{|x^*| \le 1}\left(\int_E |f|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} + \sup_{|x^*| \le 1}\left(\int_E |g|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \\ &= \mathbf{m}_p^\bullet(f, E) + \mathbf{m}_p^\bullet(g, E). \end{aligned}$$

Consequently,

$$\begin{aligned} \mathbf{m}_p^\bullet(f + g, T) &= \sup_{E \in \sigma(\mathcal{P})} \mathbf{m}_p^\bullet(f + g, E) \\ &\le \sup_{E \in \sigma(\mathcal{P})} \mathbf{m}_p^\bullet(f, E) + \sup_{E \in \sigma(\mathcal{P})} \mathbf{m}_p^\bullet(g, E) \\ &= \mathbf{m}_p^\bullet(f, T) + \mathbf{m}_p^\bullet(g, T). \end{aligned}$$

Hence (i) and (ii) hold.

(iii) By Theorem 5.3 and by Hölder's inequality we have

$$\begin{aligned}\mathbf{m}_1^\bullet(fg,T) &= \sup_{|x^*|\le 1, E\in\sigma(\mathcal{P})}\int_E |fg|dv(x^*\mathbf{m})\\ &\le \sup_{|x^*|\le 1, E\in\sigma(\mathcal{P})}\left\{\left(\int_E |f|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}}\cdot\left(\int_E |g|^q dv(x^*\mathbf{m})\right)^{\frac{1}{q}}\right\}\\ &\le \mathbf{m}_p^\bullet(f,T)\cdot\mathbf{m}_q^\bullet(g,T).\end{aligned}$$

Consequently, if $f\in\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $g\in\mathcal{L}_q\mathcal{M}(\mathbf{m})$ then $fg\in\mathcal{L}_1\mathcal{M}(\mathbf{m})$.

Theorem 5.14. If $1\le p<\infty$, then $(\mathcal{L}_p\mathcal{M}(\mathbf{m}),\mathbf{m}_p^\bullet(\cdot,T))$ and $(\mathcal{L}_p\mathcal{I}(\mathbf{m}),\ \mathbf{m}_p^\bullet(\cdot,T))$ are seminomed spaces.

Proof. By Theorems 5.11(ii) and 5.13(ii), $(\mathcal{L}_p\mathcal{M}(\mathbf{m}),\mathbf{m}_p^\bullet(\cdot,T))$ is a seminormed space. Let $f,g\in\mathcal{L}_p\mathcal{I}(\mathbf{m})$ and let α be a scalar. Then $|f|^p\in\mathcal{I}(\mathbf{m})$ and $|g|^p\in\mathcal{I}(\mathbf{m})$. Since $|f+g|^p\le 2^p\max(|f|^p,|g|^p)\le 2^p(|f|^p+|g|^p)$, by (iv) and (vii) of Theorem 3.5 $|f+g|^p\in\mathcal{I}(\mathbf{m})$ and hence $f+g\in\mathcal{L}_p\mathcal{I}(\mathbf{m})$ by Theorem 5.10. Clearly, $|\alpha f|^p\in\mathcal{I}(\mathbf{m})$ and hence $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ is a vector space. Then by Theorem 5.10 and by the fact that $(\mathcal{L}_p\mathcal{M}(\mathbf{m}),\mathbf{m}_p(\cdot,T))$ is a seminormed space, it follows that $(\mathcal{L}_p\mathcal{I}(\mathbf{m},\mathbf{m}_p^\bullet(\cdot,T))$ is a seminormed space.

Notation 5.15. By $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathcal{L}_p\mathcal{I}(\mathbf{m})$, $1\le p<\infty$, we mean the seminormed spaces $\big(\mathcal{L}_p\mathcal{M}(\mathbf{m}),\mathbf{m}_p^\bullet(\cdot,T)\big)$ and $\big(\mathcal{L}_p\mathcal{I}(\mathbf{m}),\mathbf{m}_p^\bullet(\cdot,T)\big)$, respectively.

In [P2] we generalize Theorems 5.10 and 5.14 to lcHs-valued σ-additive $\mathbf{m}$ on $\mathcal{P}$. See Definition 13.6 and Theorem 13.7 of [P2].
Following [Do1, Do2] we give the following concepts of convergence for $\mathbf{m}$-measurable functions.

Definition 5.16. Let $f,(f_n)_{n\in\boldsymbol{N}},(f_\alpha)_{\alpha\in(D,\ge)}$ be $\mathbf{m}$-measurable on T with values in $\boldsymbol{K}$ where $(D,\ge)$ is a directed set. Then:

(i) The sequence $(f_n)_1^\infty$ is said to converge in measure to f in T (resp. to be Cauchy in measure in T) with respect to $\mathbf{m}$ (when there is no ambiguity of the measure in question we drop $\mathbf{m}$) if, for each $\eta>0$, $\lim_n ||\mathbf{m}||(\{t\in T:|f_n(t)-f(t)|\ge\eta\})=0$ (resp. $\lim_{n,r\to\infty}||\mathbf{m}||(\{t\in T:|f_n-f_r|\ge\eta\})=0$). Similarly we define convergence in measure to f in E and Cauchy in measure in E for $E\in\sigma(\mathcal{P})$. Similar definitions are given for the net $(f_\alpha)_{\alpha\in(D,\ge)}$.

(ii) The sequence $(f_n)_1^\infty$ is said to converge to f almost uniformly (resp. to be Cauchy for almost uniform convergence) in T (with respect to $\mathbf{m}$) if, given $\epsilon>0$, there exists a set $E_\epsilon\in\sigma(\mathcal{P})$ with $||\mathbf{m}||(E_\epsilon)<\epsilon$ such that the sequence (f_n) converges to f uniformly in $T\backslash E_\epsilon$ (resp. is Cauchy for uniform convergence in $T\backslash E_\epsilon$). Similarly we define almost uniform convergence and Cauchy for almost uniform convergence in $E\in\sigma(\mathcal{P})$.

(iii) Let $1\le p<\infty$. If we further assume that f,f_n, $n\in\boldsymbol{N}$ (resp. $f_\alpha,\alpha\in(D,\ge)$), are in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$, then the sequence (resp. the net) is said to converge to f in (mean^p) (with respect to $\mathbf{m}$) if $\lim_{n\to\infty}\mathbf{m}_p^\bullet(f_n-f,T)=0$ (resp. $\lim_\alpha\mathbf{m}_p^\bullet(f_\alpha,-f,T)=0$). It is said to be Cauchy in (mean^p) if $\lim_{n,r\to\infty}\mathbf{m}_p^\bullet(f_n-f_r,T)=0$ (resp. $\lim_{\alpha,\beta\to\infty}\mathbf{m}_p^\bullet(f_\alpha-f_\beta,T)=0$).

Definition 5.17. Two functions $f, g \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ are said to be $\mathbf{m}$-equivalent or simply, equivalent if $f = g$ $\mathbf{m}$-a.e. in T. In that case, we write $f \sim g$.

Theorem 5.18. Let $1 \le p < \infty$. Let $f, g, (f_n)_{n\in\boldsymbol{N}}$ and the net $(f_\alpha)_{\alpha\in(D,\ge)}$ be $\mathbf{m}$-measurable scalar functions on T. Then:

(i) $'\sim'$ is an equivalence relation on $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and we denote $\mathcal{L}_p\mathcal{M}(\mathbf{m})/\sim$ by $L_p\mathcal{M}(\mathbf{m})$.
(ii) For $f, g \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$, $f \sim g$ if and only if $\mathbf{m}_p^\bullet(f - g, T) = 0$.
(iii) For $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$, $\mathbf{m}_p^\bullet(f, T) = 0$ if and only if $||\mathbf{m}||(N(f)) = 0$ and hence if and only if $f = 0$ $\mathbf{m}$-a.e. in T.
(iv) If (f_n) (resp. $(f_\alpha)_{\alpha\in(D,\ge)}$) converges in measure to f and g in T, then $f = g$ $\mathbf{m}$-a.e. in T. If (f_n) (resp. (f_α)) converges in measure to f, then (f_n) (resp. (f_α)) is Cauchy in measure.
(v) Let $(f_n)_1^\infty \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $(f_\alpha)_{\alpha\in(D,\ge)} \subset \mathcal{L}_p(\mathbf{m})$). If (f_n) (resp. $(f_\alpha)_{\alpha\in(D,\ge)}$) converges in (mean^p) to f and g, then $f = g$ $\mathbf{m}$-a.e. in T. If (f_n) (resp. (f_α)) converges to f in (mean^p), then it is Cauchy in (mean^p).
(vi) Let $(f_n)_1^\infty \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$ (resp. $(f_\alpha)_{\alpha\in(D,\ge)} \subset \mathcal{L}_p(\mathbf{m})$). If (f_n) (resp. $(f_\alpha)_{\alpha\in(D,\ge)}$) converges to f in (mean^p), then it converges to f in measure in T.
(vii) If $f_n \to f$ in measure in T (resp. $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$ and if f_n converges to f in (mean^p)), then every subsequence of (f_n) also converges to f in measure in T (resp. in (mean^p)).
(viii) **(Generalized Egoroff theorem).** Let $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and let $(h_n)_1^\infty$ be $\mathbf{m}$-measurable scalar functons on T and let $h : T \to \boldsymbol{K}$ If $h_n \to h$ $\mathbf{m}$-a.e. in T, then h is $\mathbf{m}$-measurable, $h_n \to h$ in measure in T and also almost uniformly in T.

Proof. The easy verification of (i)-(iii) is left to the reader. Since $||\mathbf{m}||$ is non negative, monotone and subadditive, the classical proof holds here to prove (iv). For example, see the proof of Theorem 22C of [H].

(v) Since $\mathbf{m}_p^\bullet(\cdot, T)$ is subadditive by Theorem 5.13(ii), $\mathbf{m}_p^\bullet(f - g, T) \le \mathbf{m}_p^\bullet(f - f_n, T) + \mathbf{m}_p^\bullet(f_n - g, T) \to 0$ as $n \to \infty$ (resp. $\le \mathbf{m}_p^\bullet(f - f_\alpha, T) + \mathbf{m}_p^\bullet(f_\alpha - g, T) \to 0$ as $\alpha \to \infty$). Hence the first part holds by (ii). Similarly the second part is proved.

(vi) is immediate from the hypothesis and the Tschebyscheff inequality (see Theorem 5.12(i)) while (vii) follows from the respective definitions of convergence.

(viii) Clearly, h is $\mathbf{m}$-measurable. By hypothesis there exists $N \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(N) = 0$ such that $h_n(t) \to h(t)$ for $t \in T\backslash N$ and such that $h_n\chi_{T\backslash N}$, $h\chi_{T\backslash N}$ are $\sigma(\mathcal{P})$-measurable, since h_n is equal to a $\sigma(\mathcal{P})$-measurable function $\mathbf{m}$-a.e. in T. (See the proof of Proposition 2.10.) Then, given $\eta > 0$, let $E_n = \{t \in T\backslash N : |h_n(t) - h(t)| \ge \eta\}$. As $h_n \to h$ pointwise in $T\backslash N$, $\lim_n E_n = \emptyset$ and hence by the hypothesis that $\mathcal{P}$ is a σ-ring $\mathcal{S}$ and by Proposition 2.3, $\lim_n ||\mathbf{m}||(E_n) = 0$. If $E_n(\eta) = \{t \in T : |h_n(t) - h(t)| \ge \eta\}$, then $E_n(\eta) \subset E_n \cup N$ and hence $\lim_n ||\mathbf{m}||\,(E_n(\eta)) = 0$. Therefore, $h_n \to h$ in measure in T. Since $||\mathbf{m}||$ is continuous and σ-subadditive on $\mathcal{S}$, the proof of Theorem 1 in §21 of [Be] can be adapted here to show that $h_n \to h$ almost uniformly in T. Hence (viii) holds.

As $||\mathbf{m}||$ is non negative, monotone and σ-subadditive on $\sigma(\mathcal{P})$, the following theorem can be proved by an argument similar to the numerical case (for example, see [H, pp. 92-94]). The proof is left to the reader.

Theorem 5.19. Let $(f_n)_1^\infty$ be a sequence of $\mathbf{m}$-measurable scalar functions on T. If it is Cauchy in measure in T, then there exists a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_1^\infty$ such that $(f_{n_k})_{k=1}^\infty$ is almost uniformly Cauchy in T and thus there exists an $\mathbf{m}$-measurable scalar function f on T such that $\lim_k f_{n_k}(t) = f(t)$ almost uniformly in T and hence $f_n \to f$ in measure in T and $f_{n_k} \to f$ $\mathbf{m}$-a.e. in T.

6. COMPLETENESS OF $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ AND $\mathcal{L}_p\mathcal{I}(\mathbf{m}), 1 \le p \le \infty$ AND OF $\mathcal{L}_\infty(\mathbf{m})$

We first give a generalized Fatou's lemma in Theorem 6.1 and then use it to show that the seminormed space $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ is complete for $1 \le p < \infty$ (Theorem 6.3) . Then, following Dobrakov [Do2], we introduce the spaces $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ and $\mathcal{L}_p(\mathbf{m})$ for $1 \le p < \infty$ and show that $\mathcal{L}_p(\mathbf{m})$ is complete. By proving that $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$, we deduce that $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ is complete for $1 \le p < \infty$. When $p = 1$ we also give an alternative proof for the completeness of $\mathcal{L}_1(\mathbf{m})$; this proof is based on inequality (5.3.1) and is similar to the proof of the corresponding part in Theorem 7 of [Do2]. The proof of Theorem 6.3 is analogous to that of the Riesz-Fischer theorem given in [Ru] and can be adapted to give an alternative simpler proof of Theorem 7 of [Do2]. Finally we define $(\mathcal{L}_\infty(\mathbf{m}), ||\cdot||_\infty)$ and show that it is a complete seminormed space.

Theorem 6.1. Let f, f_n, $n \in \boldsymbol{N}$, be $\mathbf{m}$-measurable on T with values in $\boldsymbol{K}$ or in $[-\infty, \infty]$. Let $1 \le p < \infty$. Then:

(i)**(The Fatou property of $\mathbf{m}_p^\bullet(\cdot, E)$).** If $|f_n| \nearrow |f|$ $\mathbf{m}$-a.e. in $E \in \widetilde{\sigma(\mathcal{P})}$, then $\mathbf{m}_p^\bullet(f, E) = \sup_n \mathbf{m}_p^\bullet(f_n, E) = \lim_n \mathbf{m}_p^\bullet(f_n, E)$.

(ii) **(Generalized Fatou's lemma).** $\mathbf{m}_p^\bullet(\liminf_n |f_n|, E) \le \liminf_n \mathbf{m}_p^\bullet(f_n, E)$ for $E \in \widetilde{\sigma(\mathcal{P})}$.

Proof. In the light of Definition 5.4, without loss of generality we shall assume that $E \in \sigma(\mathcal{P})$.

(i) By hypothesis there exists $M \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(M) = 0$ such that $|f_n(t)|\chi_{E\setminus M}(t) \nearrow |f(t)|\chi_{E\setminus M}(t)$ for $t \in T$ and $f_n\chi_{E\setminus M}$, $n \in \boldsymbol{N}$, and $f\chi_{E\setminus M}$ are $\sigma(\mathcal{P})$-measurable (see the proof of Proposition 2.10). Now, by Theorem 5.3 and by MCT for positive measures we have

$$\begin{aligned}
\mathbf{m}_p^\bullet(f, E) &= \sup_{|x^*|\le 1} \left(\int_E |f|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \\
&= \sup_{|x^*|\le 1} \sup_n \left(\int_E |f_n|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \\
&= \sup_n \sup_{|x^*|\le 1} \left(\int_E |f_n|^p dv(x^*\mathbf{m})\right)^{\frac{1}{p}} \\
&= \sup_n \mathbf{m}_p^\bullet(f_n, E) = \lim_n \mathbf{m}_p^\bullet(f_n, E)
\end{aligned}$$

as $\mathbf{m}_p^\bullet(f_n, E) \nearrow$.

(ii) Let $g_n = \inf_{k \ge n} |f_k|$. Then $g_n \nearrow$ and $\liminf_n |f_n| = \lim_n g_n = \sup_n g_n$. Therefore, by (i) we have

$$\mathbf{m}_p^\bullet(\liminf_n |f_n|, E) = \mathbf{m}_p^\bullet(\lim_n g_n, E) = \lim_n \mathbf{m}_p^\bullet(g_n, E) \le \liminf_n \mathbf{m}_p^\bullet(f_n, E)$$

as $g_n = |g_n| \le |f_n|$, $n \in \boldsymbol{N}$

Remark 6.2. When $p = 1$, the generalized Fatou's lemma for operator valued measures is stated without proof in Lemma 1 on p.614 of [Do4].

Theorem 6.3. Let $1 \le p < \infty$. Then:

(i) $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ is a complete seminormed space.

(ii) If $c_0 \not\subset X$, then $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ is a complete seminormed space.

Proof. (i) Let $(f_n)_1^\infty$ be Cauchy in (mean^p) in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$. Then we can choose a subsequence (f_{n_k}) of (f_n) such that $\mathbf{m}_p^\bullet(f_{n_{k+1}} - f_{n_k}) < \frac{1}{2^k}$, $k \in \mathbb{N}$. Let $h_k = f_{n_k}$, $k \in \mathbb{N}$. Let

$$g_k = \sum_{i=1}^{k} |h_{i+1} - h_i| \text{ and } g = \sum_{i=1}^{\infty} |h_{i+1} - h_i|.$$

Then by Theorem 5.13(ii), $\mathbf{m}_p^\bullet(g_k, T) \le \sum_{i=1}^\infty \mathbf{m}_p^\bullet(h_{i+1} - h_i, T) < 1$ for $k \in \mathbb{N}$ and $g_k \nearrow g$ in T. Hence by the Fatou property we have $\mathbf{m}_p^\bullet(g, T) = \sup_k \mathbf{m}_p^\bullet(g_k, T) \le 1$. Then by Theorem 5.12(ii), g is finite $\mathbf{m}$-a.e. in T. By this fact and by Definition 3.1, we can choose $N \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(N) = 0$ such that g is finite in $T \backslash N$ and such that $h_i \chi_{T\backslash N}$, $i \in \mathbb{N}$, are $\sigma(\mathcal{P})$-measurable. Then the series $\sum_{i=1}^\infty (h_{i+1} - h_i)$ is absolutely convergent in $T\backslash N$. Let $h_0 = 0$. Define

$$f(t) = \begin{cases} \sum_{k=0}^\infty (h_{k+1}(t) - h_k(t)) = \lim_k h_k(t), & \text{for } t \in T\backslash N \\ 0, & \text{for } t \in N. \end{cases}$$

Then f is finite in T, is $\sigma(\mathcal{P})$-measurable and is $\mathbf{m}$-a.e. pointwise limit of $(h_k)_1^\infty$ in T. Let $\epsilon > 0$. By hypothesis there exists n_0 such that $\mathbf{m}_p^\bullet(f_n - f_r, T) < \epsilon$ for $n, r \ge n_0$ and hence $\mathbf{m}_p^\bullet(h_k - h_\ell, T) < \epsilon$ for $n_k, n_\ell \ge n_0$. Let $F = \bigcup_{k=1}^\infty N(h_k) \cap (T\backslash N)$. Then $F \in \sigma(\mathcal{P})$ and $N(f) \subset F$. Now by the generalized Fatou's lemma we have

$$\begin{aligned} \mathbf{m}_p^\bullet(f - h_k, T) &= \mathbf{m}_p^\bullet(|f - h_k|, T) \\ &= \mathbf{m}_p^\bullet\left(\lim_\ell |h_\ell - h_k|, T\right) \\ &= \mathbf{m}_p^\bullet\left(\liminf_{\ell\to\infty} |h_\ell - h_k|, T\right) \\ &= \mathbf{m}_p^\bullet\left(\liminf_{\ell\to\infty} |h_\ell - h_k|, F\right) \\ &\le \liminf_{\ell\to\infty} \mathbf{m}_p^\bullet(h_\ell - h_k, F) \\ &< \epsilon \end{aligned}$$

for $n_k \ge n_0$, since $\liminf_{\ell\to\infty} |h_\ell - h_k| = 0$ in $T\backslash F$. Then by the triangular inequality, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and moreover, $\lim_k \mathbf{m}_p^\bullet(f - h_k, T) = 0$. Consequently, $\mathbf{m}_p^\bullet(f - f_n, T) \le \mathbf{m}_p^\bullet(f - f_{n_k}, T) + \mathbf{m}_p^\bullet(f_{n_k} - f_n, T) \to 0$ as $n_k, n \to \infty$ and hence $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Thus (i) holds.

(ii) As $c_0 \not\subset X$, by Theorem 5.10 $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and hence the result holds by (i).

The following corollary is immediate from the last part of the proof of Theorem 6.3(i).

Corollary 6.4. Let $1 \le p < \infty$. If $(f_n)_1^\infty \subset \mathcal{L}_p\mathcal{M}(\mathbf{m})$ is Cauchy in (mean^p) and if there exist a scalar function f on T and a subsequence (f_{n_k}) of (f_n) such that $f_{n_k}(t) \to f(t)$ $\mathbf{m}$-a.e. in T, then f is $\mathbf{m}$-measurable, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathbf{m}_p^\bullet(f_n - f, T) \to 0$ as $n \to \infty$.

Following the terminology adopted by Dobrakov [Do2], we define two more spaces as below.

Definition 6.5. Let $1 \le p < \infty$. Then we define $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) =$ the closure of $\mathcal{I}_s$ in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ with respect to $\mathbf{m}_p^\bullet(\cdot, T)$; and $\mathcal{L}_p(\mathbf{m}) = \{f \in \mathcal{L}_p\mathcal{M}(\mathbf{m}) : \mathbf{m}_p^\bullet(f, \cdot) \text{ is continuous on } \sigma(\mathcal{P})\}$. If we consider equivalence classes of functions, then we shall write L_p instead of $\mathcal{L}_p$.

Clearly, $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ is a linear subspace of $\mathcal{L}_p\mathcal{M}(\mathbf{m})$. If $f, g \in \mathcal{L}_p(\mathbf{m})$, then by Theorem 13(i), $\mathbf{m}_p^\bullet(f+g,\cdot) \le \mathbf{m}_p^\bullet(f,\cdot) + \mathbf{m}_p^\bullet(g,\cdot)$ and hence $f+g \in \mathcal{L}_p(\mathbf{m})$. Obviously, $\alpha f \in \mathcal{L}_p(\mathbf{m})$ for any scalar α. Hence $\mathcal{L}_p(\mathbf{m})$ is also a linear subspace of $\mathcal{L}_p\mathcal{M}(\mathbf{m})$.

Notation 6.6. By $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ and $\mathcal{L}_p(\mathbf{m})$ we mean the seminormed spaces $\left(\mathcal{L}_p\mathcal{I}_s(\mathbf{m}), \mathbf{m}_p^\bullet(\cdot, T)\right)$ and $\left(\mathcal{L}_p(\mathbf{m}), \mathbf{m}_p^\bullet(\cdot, T)\right)$, respectively.

Theorem 6.7. Let $1 \le p < \infty$. If $f : T \to \boldsymbol{K}$ is $\mathbf{m}$-measurable and $\mathbf{m}_p^\bullet(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$, then $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and hence $f \in \mathcal{L}_p(\mathbf{m})$. Moreover, $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$.

Proof. Let $\mathbf{m}_p^\bullet(f,\cdot)$ be continuous on $\sigma(\mathcal{P})$. By Proposition 2.10 there exists a sequence $(s_n) \subset \mathcal{I}_s$ such that $0 \le s_n(t) \nearrow |f(t)|^p$ $\mathbf{m}$-a.e. in T. Let $\gamma_n(\cdot) = \int_{(\cdot)} s_n d\mathbf{m}$, $n \in \boldsymbol{N}$. Then $||\gamma_n||(\cdot) = \sup_{|x^*|\le 1}\left(\int_{(\cdot)} s_n dv(x^*\mathbf{m})\right) \le \sup_{|x^*|\le 1}\left(\int_{(\cdot)} |f|^p dv(x^*\mathbf{m})\right) = \left(\mathbf{m}_p^\bullet(f,\cdot)\right)^p$ by Proposition 2.11 and by Theorem 5.3 and hence $||\gamma_n||, n \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$. Then, by Proposition 2.5, $\gamma_n, n \in \boldsymbol{N}$, are uniformly σ-additive on $\sigma(\mathcal{P})$. Consequently, by Theorem 4.4, $|f|^p$ is $\mathbf{m}$-integrable in T. Then by Theorem 5.6, $\mathbf{m}_p^\bullet(f, T) < \infty$ and hence $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$. Then by hypothesis and Definition 6.5, $f \in \mathcal{L}_p(\mathbf{m})$. Moreover, as $|f|^p \in \mathcal{I}(\mathbf{m})$, by Theorem 5.10 $f \in \mathcal{L}_p\mathcal{I}(\mathbf{m})$. This also proves that $\mathcal{L}_p(\mathbf{m}) \subset \mathcal{L}_p\mathcal{I}(\mathbf{m})$.

Convesely, let $f \in \mathcal{L}_p\mathcal{I}(\mathbf{m}) (= \mathcal{I}_p(\mathbf{m}))$. Then $|f|^p$ is $\mathbf{m}$-integrable in T and hence by Theorem 5.6, not only $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ but also $\mathbf{m}_p^\bullet(f,\cdot)$ is continuous on $\sigma(\mathcal{P})$. Hence $f \in \mathcal{L}_p(\mathbf{m})$. Thus $\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$.

Theorem 6.8 (Analogue of the Riesz-Fischer theorem). Let $1 \le p < \infty$. Then $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and hence $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ $(= \mathcal{L}_p(\mathbf{m}))$ is a complete seminormed space. If $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) = \{f \in \mathcal{L}_p(\mathbf{m} : f\ \sigma(\mathcal{P})$-measur
able$\}$, then $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ is complete. Consequently, $L_p(\mathbf{m})$ and $L_p(\sigma(\mathcal{P}), \mathbf{m})$ are Banach spaces and are treated as function spaces in which two functions which are equal $\mathbf{m}$-a.e. in T are identified.

Proof. Let $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$ and let $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$ for some $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$. Let $\epsilon > 0$. Then there exists n_0 such that $\mathbf{m}_p^\bullet(f_n - f, T) < \frac{\epsilon}{2}$ for $n \ge n_0$. Let $(E_k)_1^\infty \searrow \emptyset$ in $\sigma(\mathcal{P})$. As $f_{n_0} \in \mathcal{L}_p(\mathbf{m})$, by Definition 6.5 $\mathbf{m}_p^\bullet(f_{n_0},\cdot)$ is continuous on $\sigma(\mathcal{P})$ and hence there exists k_0 such that $\mathbf{m}_p^\bullet(f_{n_0}, E_k) < \frac{\epsilon}{2}$ for $k \ge k_0$. Consequently,

$$\mathbf{m}_p^\bullet(f, E_k) \le \mathbf{m}_p^\bullet(f - f_{n_0}, E_k) + \mathbf{m}_p^\bullet(f_{n_0}, E_k) \le \mathbf{m}_p^\bullet(f - f_{n_0}, T) + \mathbf{m}_p^\bullet(f_{n_0}, E_k) < \epsilon$$

for $k \ge k_0$. Hence $f \in \mathcal{L}_p(\mathbf{m})$ and thus $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$. Now by Theorems 6.3(i) and 6.7, $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ is a complete seminormed space.

Now, let $(f_n)_1^\infty \subset \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ be Cauchy in (mean^p). As $\mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m}) \subset \mathcal{L}_p(\mathbf{m})$ and as $\mathcal{L}_p(\mathbf{m})$ is complete, there exists $f \in \mathcal{L}_p(\mathbf{m})$ such that $\mathbf{m}_p^\bullet(f_n - f, T) \to 0$ as $n \to \infty$. As f is $\mathbf{m}$-measurable, there exists $N \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(N) = 0$ such that $f\chi_{T\setminus N}$ is $\sigma(\mathcal{P})$-measurable and clearly $f\chi_{T\setminus N} \in \mathcal{L}_p(\sigma(\mathcal{P}), \mathbf{m})$ and $\lim_n (\mathbf{m})_p^\bullet(f_n - f\chi_{T\setminus N}, T) = 0$. Hence $\mathcal{L}_P(\sigma(\mathcal{P}), \mathbf{m})$ is complete. Then the last part is immediate from the previous parts.

For $p = 1$, we give below an alternative proof similar to that in the proof of Theorem 7 of [Do2].
By Theorem 5.10, $\mathcal{L}_1\mathcal{I}(\mathbf{m}) = \mathcal{I}(\mathbf{m}) \subset \mathcal{L}_1\mathcal{M}(\mathbf{m})$. Let $(f_n)_1^\infty \subset \mathcal{I}(\mathbf{m})$ and let $f \in \mathcal{L}_1\mathcal{M}(\mathbf{m})$ be such that $\lim_n \mathbf{m}_1^\bullet(f_n - f, T) = 0$. We shall show that $f \in \mathcal{I}(\mathbf{m})$. By the proof of Theorem 6.3(i) there exist a subsequence $(f_{n_k})_{k=1}^\infty$ of $(f_n)_1^\infty$ and a function $g \in \mathcal{L}_1\mathcal{M}(\mathbf{m})$ such that $f_{n_k} \to g$ $\mathbf{m}$-a.e.

in T as $k \to \infty$ and $\lim_n \mathbf{m}_1^\bullet(f_n - g, T) = 0$. Then, by Theorem 5.18(v), $f = g$ $\mathbf{m}$-a.e. in T. Moreover, by inequality (5.3.1),

$$\left| \int_E f_{n_k} d\mathbf{m} - \int_E f_{n_\ell} d\mathbf{m} \right| \le \mathbf{m}_1^\bullet(f_{n_k} - f_{n_\ell}, T) \to 0 \text{ as } k, \ell \to \infty$$

for $E \in \sigma(\mathcal{P})$. Thus $(\int_E f_{n_k} d\mathbf{m})$ is Cauchy and hence converges to a vector x_E (say) in X for each $E \in \sigma(\mathcal{P})$. Consequently, by Theorem 4.8, f is $\mathbf{m}$-integrable in T and hence $f \in \mathcal{I}(\mathbf{m})$.

Remark 6.9. When $\mathcal{P}$ is a σ-algebra Σ and X is a real Banach space, the last part of Theorem 6.8 for $p = 1$ follows from Theorems IV.7.1 and IV.4.1 of [KK], where the proof is indirect and complicated. A somewhat direct proof of the completeness of $\mathcal{L}_1(\sigma(\mathcal{P}), \mathbf{m})$ when $\mathcal{P}$ is a σ-algebra Σ and when $\mathbf{m}$ has values in a complex Frechét space is given in [FNR]. Under the σ-algebra hypothesis, [Ri] gives a nice proof of the last part of Theorem 6.8 for $p = 1$, and the Rybakov theorem plays a vital role in his proof. In our case we cannot use his argument as the Rybakov theorem is not available for σ-additive vector measures defined on δ-rings. The last part of Theorem 6.8 for $p = 1$ is treated in Theorem 4.7 of [MN] for the δ-ring case, and his proof is based on Lemma 3.4 and Theorem 3.5 of [L]. As noted in Remark 3.12, the proofs of these results as given in [L] are incorrect. However, we have provided a correct proof of these results in the said remark and hence Theorem 4.7 of [MN] is restored. The proof of the first part of Theorem 6.3 for $p = 1$ is similar to that of Lemma 3.13(a) of [MN]. Now let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive where X is an lcHs. When X is a Frechét space Theorem 6.3(i) is generalized in Theorem 14.2 of [P2]; the second part of Definition 6.5 is generalized in Definition 14.4 in [P2] for lcHs-valued $\mathbf{m}$ on $\mathcal{P}$ and for such $\mathbf{m}$, Theorem 6.7 (resp. Theorem 6.8) is generalized in Theorems 14.5 and 14.7 (resp. in Theorem 14.8) of [P2].

Definition 6.10. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. We define $\mathcal{L}_\infty(\mathbf{m}) = \{f : T \to \boldsymbol{K}, f \ \mathbf{m}\text{-essentially bounded in} T\}$ and $||f||_\infty = \text{ess sup}_{t \in T} |f(t)|$ for $f \in \mathcal{L}_\infty(\mathbf{m})$.

Theorem 6.11. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then $(\mathcal{L}_\infty(\mathbf{m}), ||\cdot||_\infty)$ or simply, $\mathcal{L}_\infty(\mathbf{m})$ is a complete seminormed space. Consequently, $L_\infty(\mathbf{m}) = \mathcal{L}_\infty(\mathbf{m})/\sim$ is a Banach space.

Proof. Since $||\mathbf{m}||$ is a σ-subadditive submeasure on $\sigma(\mathcal{P})$, the last part of the proof of Theorem 3.11 of [Ru] holds here and hence the theorem.

Remark 6.12. Relations between the spaces $\mathcal{L}_p(\mathbf{m})$, $1 \le p \le \infty$, are studied in Section 9 below.

7. CHARACTERIZATIONS OF $\mathcal{L}_p\mathcal{I}(\mathbf{m})$, $1 \le p < \infty$

Let $1 \le p < \infty$. If $(f_n) \subset \mathcal{L}_p(\mathbf{m})$, $f : T \to \boldsymbol{K}$ and if $f_n \to f$ $\mathbf{m}$-a.e. in T, then we first obtain a characterization in Theorem 7.1 for (f_n) to converge to f in (mean^p). Then we deduce in Theorem 7.2 the analogue of an improved version of the Vitali convergence theorem in Halmos ([H], Theorem 26C) for $\mathcal{L}_p(\mathbf{m})$. From Theorems 7.1 and 7.2 we deduce a.e. convergence (mean^p)-version as well as convergence in measure (mean^p)-versions of LDCT and LBCT for $\mathcal{L}_p(\mathbf{m})$. Using the present a.e. convergence (mean^p)-version of LDCT we prove that an $\mathbf{m}$-measurable scalar function f on T belongs to $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ if and only if f belongs to $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$ (see Definition 6.5). In other words, by this result and by Theorems 5.10 and 6.7 we have $\mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$ for $p \in [1, \infty)$ and this is the main result of this section. As in the cases of the abstract Lebesgue and Bochner integrals (see [DS],[H]), we also show that the class of $\mathbf{m}$-integrable functions f can be defined in terms of convergence in measure to f of a sequence $(s_n) \subset \mathcal{I}_s$ with $\mathbf{m}_1^\bullet(s_n - s_k, T) \to 0$ as $n, k \to \infty$.

This result does not hold for the general Dobrakov integral. See Remark 7.13.

To obtain the analogue of an improved version of the Vitali convergence theorem in Halmos ([H], Theorem 26C) for $\mathcal{L}_p(\mathbf{m})$ we prove the following

Theorem 7.1. Let $1 \le p < \infty$. Let $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$ and let $f : T \to \boldsymbol{K}$ Suppose (f_n) converges to f $\mathbf{m}$-a.e. in T. Then $\mathbf{m}_p^\bullet(f_n - f, T) \to 0$ as $n \to \infty$ if and only if $\mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \boldsymbol{N}$ are uniformly continuous on $\sigma(\mathcal{P})$. In that case, $f \in \mathcal{L}_p(\mathbf{m})$ and further, when $p = 1$,

$$\int_E f d\mathbf{m} = \lim_n \int_E f_n d\mathbf{m}, \quad E \in \sigma(\mathcal{P})$$

and the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. Clearly f is $\mathbf{m}$-measurable. If $\mathbf{m}_p^\bullet(f_n - f, T) \to 0$, then, given $\epsilon > 0$, there exists n_0 such that $\mathbf{m}_p^\bullet(f_n - f_\ell, T) < \frac{\epsilon}{2}$ for $n, \ell \ge n_0$. Let $E_k \searrow \emptyset$ in $\sigma(\mathcal{P})$. Then by Definition 6.5, there exists k_0 such that $\mathbf{m}_p^\bullet(f_n, E_k) < \frac{\epsilon}{2}$ for $n = 1, 2, ..., n_0$ and for $k \ge k_0$. Consequently, $\mathbf{m}_p^\bullet(f_n, E_k) \le \mathbf{m}_p^\bullet(f_n - f_{n_0}, E_k) + \mathbf{m}_p^\bullet(f_{n_0}, E_k) \le \mathbf{m}_p^\bullet(f_n - f_{n_0}, T) + \mathbf{m}_p^\bullet(f_{n_0}, E_k) < \epsilon$ for $n \ge n_0$ and for $k \ge k_0$. Hence $\mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$.

Conversely, let $\mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \boldsymbol{N}$ be uniformly continuous on $\sigma(\mathcal{P})$. Let $E_k \searrow \emptyset$ in $\sigma(\mathcal{P})$ and let $\epsilon > 0$. By hypothesis there exists k_0 such that $\mathbf{m}_p^\bullet(f_n, E_k) < \epsilon$ for all $n \in \boldsymbol{N}$ and for $k \ge k_0$. Consequently, by the generalized Fatou's lemma (Theorem 6.1(ii)), $\mathbf{m}_p^\bullet(f, E_k) = \mathbf{m}_p^\bullet(\liminf_n |f_n|, E_k) \le \liminf_{n\to\infty} \mathbf{m}_p^\bullet(f_n, E_k) < \epsilon$ for $k \ge k_0$. Hence $\mathbf{m}_p^\bullet(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$ and consequently, by Theorem 6.7, $f \in \mathcal{L}_p(\mathbf{m})$.

By Theorems 5.10 and 6.7, $|f_n|^p$, $n \in \boldsymbol{N}$ and $|f|^p$ are $\mathbf{m}$-integrable in T. Now, let $\gamma_n(\cdot) = \int_{(\cdot)} |f_n|^p d\mathbf{m}$ for $n \in \boldsymbol{N}$ and let $\gamma_0(\cdot) = \int_{(\cdot)} |f|^p d\mathbf{m}$. By Theorem 5.6, $(||\gamma_n||(\cdot))^{\frac{1}{p}} = \mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \boldsymbol{N}$ and $(||\gamma_0||(\cdot))^{\frac{1}{p}} = \mathbf{m}_p^\bullet(f, \cdot)$. Then by hypothesis and by the fact that $f \in \mathcal{L}_p(\mathbf{m})$, $||\gamma_n||$, $n \in \boldsymbol{N}\bigcup\{0\}$, are uniformly continuous on $\sigma(\mathcal{P})$. Therefore, by Proposition 2.5, γ_n, $n \in \boldsymbol{N}\bigcup\{0\}$, are uniformly σ-additive on $\sigma(\mathcal{P})$. Then by Proposition 2.6 there exists a σ-additive control measure $\mu : \sigma(\mathcal{P}) \to [0, \infty)$ such that $||\gamma_n||, n \in \boldsymbol{N} \cup \{0\}$, are uniformly μ-continuous. Thus, given $\epsilon > 0$, there exists $\delta > 0$ such that $||\gamma_\mathbf{n}||(E) < (\frac{\epsilon}{3})^p$, for $n \in \boldsymbol{N}\bigcup\{0\}$, whenever $\mu(E) < \delta$. By hypothesis there exists $M \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(M) = 0$ such that $f_n(t) \to f(t)$ for $t \in T\backslash M$ and such that $f_n\chi_{T\backslash M}$, $n \in \boldsymbol{N}$ and $f\chi_{T\backslash M}$ are $\sigma(\mathcal{P})$-measurable (see the proof of Proposition 2.10). Let $F = \bigcup_{n=1}^\infty N(f_n) \cap (T\backslash M)$. Clearly, $F \in \sigma(\mathcal{P})$. Then by the Egoroff-Lusin theorem there exist a set $N \in \sigma(\mathcal{P}) \cap F$ with $\mu(N) = 0$ and a sequence $(F_k) \subset \mathcal{P}$ with $F_k \nearrow F\backslash N$ such that $f_n \to f$ uniformly in each F_k. As $F\backslash N\backslash F_k \searrow \emptyset$, there exists k_0 such that $\mu(F\backslash N\backslash F_{k_0}) < \delta$. Then $||\gamma_n||(F\backslash N\backslash F_{k_0}) < (\frac{\epsilon}{3})^p$ for $n \in \boldsymbol{N}\bigcup\{0\}$. Choose n_0 such that $||f_n - f||_{F_{k_0}} \cdot (||\mathbf{m}||(F_{k_0}))^{\frac{1}{p}} < \frac{\epsilon}{3}$ for $n \ge n_0$. Note that $||\gamma_n||(M \cup N) = 0$ for $n \in \boldsymbol{N}\cup\{0\}$. Then by (i) and (iii) of Theorem 5.11, by (i) of Theorem 5.13 and by Theorem 5.6 we have $\mathbf{m}_p^\bullet(f_n - f, T) \le \mathbf{m}_p^\bullet(f_n - f, F_{k_0}) + \mathbf{m}_p^\bullet(f_n - f, F\backslash N\backslash F_{k_0}) + \mathbf{m}_p^\bullet(f_n, N) + \mathbf{m}_p^\bullet(f, N) + \mathbf{m}_p^\bullet(f, M) + \mathbf{m}_p^\bullet(f_n, M) \le ||f_n - f||_{F_{k_0}} \cdot (||\mathbf{m}||(F_{k_0}))^{\frac{1}{p}} + (||\gamma_n||(F\backslash N\backslash F_{k_0}))^{\frac{1}{p}} + (||\gamma_0||(F\backslash N\backslash F_{k_0}))^{\frac{1}{p}} + (||\gamma_n||(N))^{\frac{1}{p}} + (||\gamma_0||(N))^{\frac{1}{p}} + (||\gamma_0||(M))^{\frac{1}{p}} + (||\gamma_n||(M))^{\frac{1}{p}} < \epsilon$ for $n \ge n_0$. Hence $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then by the triangular inequality, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and consequently, by Theorem 6.8, $f \in \mathcal{L}_p(\mathbf{m})$. The last part is immediate from inequality (5.3.1) since $\mathcal{L}_1(\mathbf{m}) = \mathcal{L}_1\mathcal{I}(\mathbf{m}) = \mathcal{I}(\mathbf{m})$ by Theorems 6.7 and 5.10, respectively.

Theorem 7.2 (Analogue of an improved version of the Vitali convergence theorem in Halmos ([H], Theorem 26C) for $\mathcal{L}_p(\mathbf{m})$). Let $1 \le p < \infty$ and let $f : T \to \boldsymbol{K}$be $\mathbf{m}$-measurable. A sequence $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$ converges to f in (mean^p) if and only if (f_n) converges to f in measure in T, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathbf{m}_p^\bullet(f_n, \cdot), n \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$. In that case, $f \in \mathcal{L}_p(\mathbf{m})$. Moreover, for $p = 1$, results in the last part of Theorem 7.1 hold here verbatim.

Proof. Let $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then by the triangular inequality $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and then by Theorem 5.18(vi), $f_n \to f$ in measure in T. Moreover, by the proof of the necessity part of Theorem 7.1, $\mathbf{m}_p^\bullet(f_n, \cdot), n \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$.

Conversely, let $f_n \to f$ in measure in T, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathbf{m}_p^\bullet(f_n, \cdot), n \in \boldsymbol{N}$, be uniformly continuous on $\sigma(\mathcal{P})$. If possible, let $\mathbf{m}_p^\bullet(f_n - f, T) \not\to 0$ as $n \to \infty$. Then there would exist an $\epsilon > 0$ and a subsequence (g_n) of (f_n) such that $\mathbf{m}_p^\bullet(g_n - f, T) > \epsilon$ for $n \in \boldsymbol{N}$. On the other hand, by Theorem 5.18(iv), by the first part of Theorem 5.18(vii) and by Theorem 5.19 there would exist a subsequence (g_{n_k}) of (g_n) such that $g_{n_k} \to f$ $\mathbf{m}$-a.e. in T. Then, as $\mathbf{m}_p^\bullet(g_{n_k}, \cdot)$, $k \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$, by Theorem 7.1 there would exist k_0 such that $\mathbf{m}_p^\bullet(g_{n_k} - f, T) < \epsilon$ for all $k \ge k_0$. This is a contradiction and hence $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$.

Remark 7.3. In the light of Theorem 7.2, Theorem 26C of [H] can be improved as below.

Let Σ be a σ-ring of subsets of T and let $\mu : \Sigma \to [0, \infty]$ be a positive measure. Let $\mathcal{P} = \{E \in \Sigma : \mu(E) < \infty\}$. Then $\mathcal{P}$ is a δ-ring and a Σ-measurable function is μ-integrable in T if and only if it is $\sigma(\mathcal{P})$-measurable and is $\mu|_\mathcal{P}$-integrable in T since $N(f)$ is of σ-finite measure whenever f is μ-integrable in T. Consequently, by Theorem 7.2, **a sequence (f_n) of μ-integrable scalar functions on T converges in the mean to the integrable function f if and only if (f_n) converges in measure to f in T and $(\nu_n(\cdot))_1^\infty$ are equicontinuous from above at 0 (in the sense of p.108 of Halmos [H]), where** $\nu_n(\cdot) = \int_{(\cdot)} |f_n| d\mu$. (Note that in our terminology, $\nu_n(\cdot)$ is the same as $\mu_1^\bullet(f_n, \cdot)$ and equiconitnuity of $(\nu_n)_1^\infty$ from above at 0 is the same as uniform continuity of $(\mu_1^\bullet(f_n, \cdot))_1^\infty$ on $\sigma(\mathcal{P})$.) Clearly, this is an improved version of Theorem 26C of Halmos [H].

The following versions of LDCT and LBCT are immediate from Theorems 7.1 and 7.2.

Theorem 7.4 (a.e. convergence and convergence in measure versions of LDCT and LBCT for $\mathcal{L}_p(\mathbf{m})$). Let $1 \le p < \infty$. Let f_n, $n \in \boldsymbol{N}$, be $\mathbf{m}$-measurable scalar functions on T and let $g \in \mathcal{L}_p(\mathbf{m})$ such that $|f_n(t)| \le |g(t)|$ $\mathbf{m}$-a.e. in T (resp. let $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and let M be a finite constant such that $|f_n(t)| \le M$ $\mathbf{m}$-a.e. in T) for all n. If $f_n(t) \to f(t)$ $\mathbf{m}$-a.e. in T where f is a scalar function on T or if f is an $\mathbf{m}$-measurable scalar function on T and if $f_n \to f$ in measure in T, then f, f_n, $n \in \boldsymbol{N}$, belong to $\mathcal{L}_p(\mathbf{m})$ and $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. When $p = 1$,

$$\lim_n \int_E f_n d\mathbf{m} = \int_E f d\mathbf{m}, \;\; E \in \sigma(\mathcal{P}) \;\; (\text{resp.}\, E \in \mathcal{S})$$

where the limit is uniform with respect to $E \in \sigma(\mathcal{P})$ (resp. $E \in \mathcal{S}$).

Proof. As $g \in \mathcal{L}_p(\mathbf{m})$, $\mathbf{m}_p^\bullet(g, \cdot)$ is continuous on $\sigma(\mathcal{P})$. By hypothesis and by Theorem 5.11(i), $\mathbf{m}_p^\bullet(f_n, \cdot) \le \mathbf{m}_p^\bullet(g, \cdot)$ for all n and hence $\mathbf{m}_p(f_n, \cdot), n \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$. Then LDCT holds by Theorem 7.1 (resp. by Theorem 7.2) if $f_n \to f$ $\mathbf{m}$-a.e. in T (resp. if $f_n \to f$ in measure in T).

Now let $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and let $|f_n(t)| \le M$ $\mathbf{m}$-a.e. in T for $n \in \boldsymbol{N}$. As $\mathcal{L}_p(\mathbf{m}) = \mathcal{I}_p(\mathbf{m})$ by Theorems 5.10 and 6.7, and as $\mathcal{S}$ is a σ-ring, by the last part of Theorem 3.5(v) the constant function $M \in \mathcal{L}_p(\mathbf{m})$ and hence the present versions of LBCT follow from the corresponding versions of LDCT obtained above.

Using Theorem 7.4 we prove the following main theorem of the present section.

Theorem 7.5 (Characterizations of $\mathcal{L}_p\mathcal{I}(\mathbf{m})$). Let $1 \le p < \infty$ and let $f : T \to \boldsymbol{K}$ be $\mathbf{m}$-measurable. Then the following statements are equivalent:

(i) $f \in \mathcal{I}_p(\mathbf{m})$.

(ii) $\mathbf{m}_p^\bullet(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$ (so that $f \in \mathcal{L}_p(\mathbf{m})$ by Theorem 6.7).

(iii) **(Simple Function Approximation)**. There exists a sequence $(s_n) \subset \mathcal{I}_s$ such that $s_n \to f$ $\mathbf{m}$-a.e. in T and $\lim_n \mathbf{m}_p^\bullet(s_n - f, T) = 0$.

Consequently,

$$\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m}).$$

If $c_0 \not\subset X$, then

$$\mathcal{L}_p\mathcal{M}(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m}).$$

Proof. (i)$\Leftrightarrow$(ii) by Theorems 5.10 and 6.7.

(ii)$\Rightarrow$(iii) By Proposition 2.10 there exists $(s_n) \subset \mathcal{I}_s$ such that $s_n \to f$ $\mathbf{m}$-a.e. in T and $|s_n| \nearrow |f|$ $\mathbf{m}$-a.e. in T. As $f \in \mathcal{L}_p(\mathbf{m})$ by hypothesis and by Theorem 6.7, Theorem 7.4 implies $\lim_n \mathbf{m}_p^\bullet(f - s_n, T) = 0$. (For $p = 1$, one can use (i) and Theorem 3.7.)

(iii)$\Rightarrow$(ii) Let $\epsilon > 0$ and let $E_k \searrow \emptyset$ in $\sigma(\mathcal{P})$. By hypothesis, there exists n_0 such that $\mathbf{m}_p^\bullet(s_n - f, T) < \frac{\epsilon}{2}$ for $n \ge n_0$. By Theorem 5.6, $\mathbf{m}_p^\bullet(s_{n_0}, \cdot)$ is continuous on $\sigma(\mathcal{P})$ and hence there exists k_0 such that $\mathbf{m}_p^\bullet(s_{n_0}, E_k) < \frac{\epsilon}{2}$ for $k \ge k_0$. Consequently, $\mathbf{m}_p^\bullet(f, E_k) \le \mathbf{m}_p^\bullet(f - s_{n_0}, E_k) + \mathbf{m}_p^\bullet(s_{n_0}, E_k) \le \mathbf{m}_p^\bullet(f - s_{n_0}, T) + \mathbf{m}_p^\bullet(s_{n_0}, E_k) < \epsilon$ for $k \ge k_0$. Hence $\mathbf{m}_p^\bullet(f, \cdot)$ is continuous on $\sigma(\mathcal{P})$.

Thus (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii).

Since $I_s \subset \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}(\mathbf{m})$ by Theorem 5.10, $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) \subset$ closure of $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ in $\mathcal{L}_p\mathcal{M}(\mathbf{m}) =$ closure of $\mathcal{L}_p(\mathbf{m})$ in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ (by Theorem 6.7) . But by Theorem 6.8, $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and hence $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) \subset \mathcal{L}_p(\mathbf{m})$. On the other hand, $\mathcal{I}_s(\mathbf{m})$ is dense in $\mathcal{L}_p(\mathbf{m})$ by (iii) and hence $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) \supset \mathcal{L}_p(\mathbf{m})$. Therefore, $\mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m})$. Consequently, by Theorems 5.10 and 6.7 we have

$$\mathcal{L}_p\mathcal{I}(\mathbf{m}) = \mathcal{I}_p(\mathbf{m}) = \mathcal{L}_p\mathcal{I}_s(\mathbf{m}) = \mathcal{L}_p(\mathbf{m}).$$

If $c_0 \not\subset X$, use the second part of Theorem 5.10 along with the previous part.

Notation 7.6. In the light of Theorem 7.5, **we shall hereafter use the symbol $\mathcal{L}_p(\mathbf{m})$ not only to denote the space given in the second part of Definition 6.5 but also any-one of the spaces $\mathcal{L}_p\mathcal{I}_s(\mathbf{m})$, $\mathcal{L}_p\mathcal{I}(\mathbf{m})$ or $\mathcal{I}_p(\mathbf{m})$.** The quotient $\mathcal{L}_p(\mathbf{m})/\sim$ is denoted by $L_p(\mathbf{m})$, and is treated as a function space in which two functions which are equal $\mathbf{m}$-a.e. in T are identified.

The following theorem is immediate from Theorem 7.5.

Theorem 7.7. $\mathcal{I}_s$ is dense in $\mathcal{L}_p(\mathbf{m})$ for $1 \le p < \infty$.

Remark 7.8. For $p = 1$, Theorem 7.7 subsumes Theorem 4.5 of [MN], which is assumed valid by Theorem 3.5 in [L]. But the proof of the said theorem in [L] is incorrect as observed in Remark 3.12, where a correct proof for the said theorem of [L] (with the vector measure having values in a sequentially complete lcHs) is also given. Also see Remark 6.9.

Remark 7.9. Let X be an lcHs and let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Definition 6.5 is generalized to such $\mathbf{m}$ in Definitions 14.4 and 15.4 of [P2] where Theorems 7.1, 7.2, 7.4 and 7.5 are generalized in Theorems 15.1, 15.2, 15.3 and 15.5, respectively, when X is quasicomplete (resp. sequentially complete (for $\sigma(\mathcal{P})$-measurable functions)).

Convention 7.10. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. If $f \in \mathcal{L}_\infty(\mathbf{m})$, then there exists $M \in \sigma(\mathcal{P})$ such that $||\mathbf{m}||(M) = 0$ and such that $||f||_\infty = \sup_{t \in T\setminus M} |f(t)|$. If we define $g(t) = f(t)$ for $t \in T\setminus M$ and $g(t) = 0$ for $t \in M$, then $f = g$ $\mathbf{m}$-a.e. in T and $||f||_\infty = \sup_{t\in T} |g(t)|$. Thus, for the equivalence class $\tilde{f}$ determined by $f \in \mathcal{L}_\infty(\mathbf{m})$, there exists a bounded $\mathbf{m}$-measurable function $g_{\tilde{f}} \in \tilde{f}$ such that $\sup_{t\in t} |g_{\tilde{f}}(t)| = ||f||_\infty$ and hence we make the convention to define $L_\infty(\mathbf{m}) = \{g_{\tilde{f}} :$ only one representative from $\tilde{f}$ for $f \in \mathcal{L}_\infty(\mathbf{m})\}$. Thus, for $f \in L_\infty(\mathbf{m})$, $||f||_\infty = \sup_{t\in T} |f(t)|$. Similarly, the spaces $L_p(\mathbf{m})$, $1 \le p < \infty$, are treated as function spaces (see the last part of Notation 7.6).

Theorem 7.11. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Let $\mathcal{L}_p^r(\mathbf{m}) = \{f \in \mathcal{L}_p(\mathbf{m}),\ f \text{ real valued}\}$ for $1 \le p \le \infty$. Then $L_p^r(\mathbf{m})$, $1 \le p \le \infty$, are Banach lattices under the partial order $f \le g$ defined by $f(t) \le g(t)$ $\mathbf{m}$-a.e. in T.

Proof. Under the given partial order, clearly $L_\infty^r(\mathbf{m})$ is a vector lattice. If $|f| \le |g|$, $f, g \in L_\infty^r(\mathbf{m})$, then $||f||_\infty = \sup_{t\in t} |f(t)| \le \sup_{t\in T} |g(t)| = ||g||_\infty$ (see Convention 7.10). Then by Theorem 6.11, $L_\infty^r(\mathbf{m})$ is a Banach lattice.

Let $1 \le p < \infty$. Then $L_p^r(\mathbf{m})$ is a lattice by the facts that $L_p^r(\mathbf{m})$ is a vector space and that $\max(f, g) = \frac{1}{2}(f + g + |f - g|)$ and $\min(f, g) = \frac{1}{2}(f + g - |f - g|)$. (see Theorem 3.5(vii)). Clearly, $f \le g$ implies $f + h \le g + h$ and $\alpha f \le \alpha g$ for $f, g, h \in L_p^r(\mathbf{m})$ and $\alpha \ge 0$. Hence $L_p^r(\mathbf{m})$ is a vector lattice. Consequently, by Theorems 5.11(iv) and 6.8, $L_p^r(\mathbf{m})$ is a Banach lattice.

Using the convergence in measure of a sequence of simple scalar (resp. vector) functions which are Cauchy in mean, the abstract Lebesgue integral (resp. the Bochner integral) is defined in [H] (resp. in [DS]). The following theorem asserts that an analogous result holds for $\mathbf{m}$-integrable functions.

Theorem 7.12. Let $f : T \to \boldsymbol{K}$ be $\mathbf{m}$-measurable and let $1 \le p < \infty$. Then $f \in \mathcal{L}_p(\mathbf{m})$ if and only if there exists a sequence $(s_n) \subset \mathcal{I}_s$ (resp. $(f_n) \subset \mathcal{L}_p(\mathbf{m})$) such that $s_n \to f$ (resp. $f_n \to f$) in measure in T and (s_n) (resp. (f_n)) is Cauchy in (mean^p). In that case, $\lim_n \mathbf{m}_p^\bullet(s_n - f, T) = 0$ (resp. $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$). When $p = 1$, $f \in \mathcal{L}_1(\mathbf{m})$, and

$$\int_E f d\mathbf{m} = \lim_n \int_E s_n d\mathbf{m} \text{ (resp. } \lim_n \int_E f_n d\mathbf{m}), \ E \in \sigma(\mathcal{P})$$

the limit being uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. Let $f \in \mathcal{L}_p(\mathbf{m})$. Then by Theorem 7.5 there exists a sequence $(s_n) \subset \mathcal{I}_s \subset \mathcal{L}_p(\mathbf{m})$ such that $s_n \to f$ $\mathbf{m}$-a.e. in T and $\mathbf{m}_p^\bullet(s_n - f, T) \to 0$ as $n \to \infty$. Then by Theorem 5.18(vi), (s_n)

converges to f in measure in T. Clearly, (s_n) is Cauchy in (mean^p).

Conversely, let $(s_n)_1^\infty \subset \mathcal{I}_s$ (resp. $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$) satisfy the hypothesis. Let $u_n = s_n$ for all n or $u_n = f_n$ for all n, as the case may be. By hypothesis, $u_n \to f$ in measure in T and (u_n) is Cauchy in (mean^p). Then by Theorem 5.19 there exist a subsequence $(u_{n_k})_{k=1}^\infty$ of $(u_n)_1^\infty$ and an $\mathbf{m}$-measurable scalar function g on T such that $u_{n_k} \to g$ $\mathbf{m}$-a.e. in T and $u_{n_k} \to g$ in measure in T. Then by (vii) and (iv) of Theorem 5.18, $f = g$ $\mathbf{m}$-a.e. in T. Consequently, by Corollary 6.4, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and $\mathbf{m}_p^\bullet(u_n - f, T) \to 0$ as $n \to \infty$. As $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ by Theorem 6.8, $f \in \mathcal{L}_p(\mathbf{m})$.

Now let $p = 1$. Given $\epsilon > 0$, there exists n_0 such that $\mathbf{m}_1^\bullet(u_n - f, T) < \epsilon$ for $n \geq n_0$. Then, by inequality (5.3.1) we have

$$\left| \int_E u_n d\mathbf{m} - \int_E f d\mathbf{m} \right| \leq \mathbf{m}_1^\bullet(u_n - f, T) < \epsilon$$

for $n \geq n_0$ and for all $E \in \sigma(\mathcal{P})$. Hence the last part also holds.

Remark 7.13. The above theorem (for $p = 1$) fails for the Dobrakov integral of vector functions. See p. 530 of [Do1].

8. OTHER CONVERGENCE THEOREMS FOR $\mathcal{L}_p(\mathbf{m})$, $1 \leq p < \infty$

In this section we give two versions of the Vitali convergence theorem analogous to Theorems III.3.6 and III.6.15 of [DS]. We also give LDCT for nets analogous to Theorem III.3.7 of [DS]. Finally we include some results of Dobrakov [Do3,Do4] specialized to vector measures.

Notation 8.1. For $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$, $1 \leq p < \infty$, $\mathbf{m}_p^\bullet(f, T \backslash A) = \mathbf{m}_p^\bullet(f, N(f) \backslash A)$ for $A \in \sigma(\mathcal{P})$. (See Definition 5.4.)

For $p = 1$, the following lemma is essentially the Banach space version of Lemma 3.4 of [L2] whose proof is corrected in Remark 3.12.

Lemma 8.2. Let $1 \leq p < \infty$. If $f \in \mathcal{L}_p(\mathbf{m})$, then, for each $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{P}$ such that $\sup_{|x^*| \leq 1} \int_{T \backslash A_\epsilon} |f|^p dv(x^*\mathbf{m}) < \epsilon$ or equivalently, $\mathbf{m}_p^\bullet(f, T \backslash A_\epsilon) < \epsilon^{\frac{1}{p}}$.

Proof. Let $\hat{f}$ be a $\sigma(\mathcal{P})$-measurable function such that $\hat{f} = f$ $\mathbf{m}$-a.e. in T. Then $\hat{f} \in \mathcal{L}_p(\mathbf{m})$ and $\int_E |\hat{f}|^p d\mathbf{m} = \int_E |f|^p d\mathbf{m}$ for $E \in \sigma(\mathcal{P})$. Let $\gamma : \sigma(\mathcal{P}) \to X$ be given by $\gamma(\cdot) = \int_{(\cdot)} |\hat{f}|^p d\mathbf{m}$. As $N(\hat{f}) \in \sigma(\mathcal{P})$, there exists an increasing sequence $(E_n) \subset \mathcal{P}$ such that $N(\hat{f}) = \bigcup_1^\infty E_n$. By Theorem 3.5(ii), γ is an X-valued σ-additive vector measure on $\sigma(\mathcal{P})$, and hence by Proposition 2.3, $||\gamma|| \left(N(\hat{f}) \backslash E_n \right) \searrow 0$ as $n \to \infty$. Thus, there exists n_0 such that $||\gamma|| \left(N(\hat{f}) \backslash E_n \right) < \epsilon$ for $n \geq n_0$. Let $A_\epsilon = E_{n_0}$. Then by Definition 5.4, Notation 8.1 and Theorem 5.3, we have

$$\mathbf{m}_p^\bullet(f, T \backslash A_\epsilon) = \mathbf{m}_p^\bullet(f, N(f) \backslash A_\epsilon) = \sup_{|x^*| \leq 1} \left(\int_{N(f) \backslash A_\epsilon} |f|^p dv(x^*\mathbf{m}) \right)^{\frac{1}{p}} =$$

$$\sup_{|x^*| \leq 1} \left(\int_{T \backslash A_\epsilon} |f|^p dv(x^*\mathbf{m}) \right)^{\frac{1}{p}} = \sup_{|x^*| \leq 1} \left(\int_{N(\hat{f}) \backslash A_\epsilon} |\hat{f}|^p dv(x^*\mathbf{m}) \right)^{\frac{1}{p}} =$$

$$\left(||\gamma||(N(\hat{f})\backslash A_\epsilon)\right)^{\frac{1}{p}} < \epsilon^{\frac{1}{p}}$$

and hence

$$\sup_{|x^*|\le 1} \int_{T\backslash A_\epsilon} |f|^p dv(x^*\mathbf{m}) < \epsilon.$$

Definition 8.3. A set function $\lambda : \sigma(\mathcal{P}) \to [0,\infty]$ is said to be $\mathbf{m}$-continuous on $\sigma(\mathcal{P})$ (in symbols, $\lambda \ll \mathbf{m}$) if, given $\epsilon > 0$, there exists $\delta > 0$ such that $\lambda(E) < \epsilon$ whenever $||\mathbf{m}||(E) < \delta$, $E \in \sigma(\mathcal{P})$.

Lemma 8.4. Let $1 \le p < \infty$. For $f \in \mathcal{L}_p(\mathbf{m})$, $\mathbf{m}_p^\bullet(f,\cdot) \ll \mathbf{m}$ on $\sigma(\mathcal{P})$.

Proof. By Theorem 7.5 there exists a sequence $(s_n) \subset \mathcal{I}_s$ such tha $s_n \to f$ $\mathbf{m}$-a.e. in T and such that $\mathbf{m}_p^\bullet(f - s_n, T) \to 0$ as $n \to \infty$. Thus, given $\epsilon > 0$, there exists n_0 such that $\mathbf{m}_p^\bullet(f - s_{n_0}, T) < \frac{\epsilon}{2}$. Let $s = s_{n_0} = \sum_1^r a_i \chi_{E_i}$ with $(E_i)_1^r \subset \mathcal{P}$, and let $M = ||s||_T$. Let $A_\epsilon = \bigcup_1^r E_i$. Let $E \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(E) < (\frac{\epsilon}{2(M+1)})^p$. Then by Theorem 5.11(iii) we have $\mathbf{m}_p^\bullet(s,E) \le ||s||_E \cdot (||\mathbf{m}||(E))^{\frac{1}{p}} \le \frac{M\epsilon}{2(M+1)} < \frac{\epsilon}{2}$. Consequently, by Theorems 5.11(i) and 5.13(ii) we have

$$\mathbf{m}_p^\bullet(f,E) \le \mathbf{m}_p^\bullet(f - s, E) + \mathbf{m}_p^\bullet(s,E) \le \mathbf{m}_p^\bullet(f - s_{n_0}, T) + \mathbf{m}_p^\bullet(s,E) < \epsilon.$$

Hence $\mathbf{m}_p^\bullet(f,\cdot) \ll \mathbf{m}$.

The following theorem is an analogue of Theorem III.3.6 of [DS] for $\mathcal{L}_p(\mathbf{m})$.

Theorem 8.5 (Analogue of the convergence in measure version of the Vitali convergence theorem of [DS] for $\mathcal{L}_p(\mathbf{m})$). Let $1 \le p < \infty$. Let $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$ and let $f : T \to \boldsymbol{K}$ be $\mathbf{m}$-measurable. Then $f \in \mathcal{L}_p(\mathbf{m})$ and $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$ if and only if the following conditions hold:

(i) $f_n \to f$ in measure in each $E \in \mathcal{P}$.
(ii) $\mathbf{m}_p^\bullet(f_n,\cdot)$, $n \in \boldsymbol{N}$ are uniformly $\mathbf{m}$-continuous on $\sigma(\mathcal{P})$, in the sense that, given $\epsilon > 0$, there exists $\delta > 0$ such that $\mathbf{m}_p^\bullet(f_n, E) < \epsilon$ for all $n \in \boldsymbol{N}$ whenever $E \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(E) < \delta$.
(iii) For each $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{P}$ such that $\mathbf{m}_p^\bullet(f_n, T\backslash A_\epsilon) < \epsilon$ for all $n \in \boldsymbol{N}$ (See Notation 8.1.)

In such case, when $p = 1$, $\int_E f d\mathbf{m} = \lim_n \int_E f_n d\mathbf{m}$ for $E \in \sigma(\mathcal{P})$ and the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. Let $f \in \mathcal{L}_p(\mathbf{m})$ and let $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then by Theorem 5.18(vi), $f_n \to f$ in measure in T and hence (i) holds. Let $\epsilon > 0$ and let $f_0 = f$. By hypothesis there exists n_0 such that $\mathbf{m}_p^\bullet(f_n - f_0, T) < \frac{\epsilon}{2}$ for $n \ge n_0$. By Lemma 8.4 there exists $\delta > 0$ such that $\mathbf{m}_p^\bullet(f_i, E) < \frac{\epsilon}{2}$ for $i = 0,1,2,...,n_0$, whenever $E \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(E) < \delta$. Consequently, for such E we also have

$$\mathbf{m}_p^\bullet(f_n, E) \le \mathbf{m}_p^\bullet(f_n - f_0, E) + \mathbf{m}_p^\bullet(f_0, E) \le \mathbf{m}_p^\bullet(f_n - f_0, T) + \mathbf{m}_p^\bullet(f_0, E) < \epsilon$$

for $n \ge n_0$. Hence (ii) holds. By Lemma 8.2 there exists $A_\epsilon \in \mathcal{P}$ such that $\mathbf{m}_p^\bullet(f_n, T\backslash A_\epsilon) < \frac{\epsilon}{2}$ for $n = 0,1,2,...,n_0$. For $n \ge n_0$, $\mathbf{m}_p^\bullet(f_n, T\backslash A_\epsilon) \le \mathbf{m}_p^\bullet(f_n - f_0, T\backslash A_\epsilon) + \mathbf{m}_p^\bullet(f_0, T\backslash A_\epsilon) \le \mathbf{m}_p^\bullet(f_n - f_0, T) + \mathbf{m}_p^\bullet(f_0, T\backslash A_\epsilon) < \epsilon$, by Definition 5.4 and Theorems 5.13(i) and 5.11(i). Hence (iii) holds.

Conversely, let (i), (ii) and (iii) hold and let $\epsilon > 0$. By (iii) there exists $A_\epsilon \in \mathcal{P}$ such that $\mathbf{m}_p^\bullet(f_n, T\backslash A_\epsilon) < \frac{\epsilon}{6}$ for $n \in \boldsymbol{N}$. By (ii) there exists $\delta > 0$ such that $\mathbf{m}_p^\bullet(f_n, E) < \frac{\epsilon}{6}$ for $n \in \boldsymbol{N}$whenever $E \in \sigma(\mathcal{P})$ with $||\mathbf{m}||(E) < \delta$. Let $\delta_0 = \frac{\epsilon}{3(||\mathbf{m}||(A_\epsilon)+1)^{\frac{1}{p}}}$. Let $E_n = \{t \in A_\epsilon : |f_n(t) - f(t)| > \delta_0\}$.

By (i), $f_n \to f$ in measure in A_ϵ and hence there exists n_0 such that $||\mathbf{m}||(E_n) < \delta$ for $n \ge n_0$. Therefore, $\mathbf{m}_p^\bullet(f_k, E_n) < \frac{\epsilon}{6}$ for $k \in \boldsymbol{N}$ and for $n \ge n_0$. Since $||f_n - f||_{A_\epsilon \backslash E_n} \le \delta_0$, by Theorem 5.11(iii) we have

$$\mathbf{m}_p^\bullet(f_n - f, A_\epsilon \backslash E_n) \le \delta_0(||\mathbf{m}||(A_\epsilon \backslash E_n)^{\frac{1}{p}} < \frac{\epsilon}{3}$$

for all $n \in \boldsymbol{N}$ Then by Definition 5.4 and by Theorem 5.13(i) we have

$$\mathbf{m}_p^\bullet(f_n - f, T) \le \mathbf{m}_p^\bullet(f_n - f, T\backslash A_\epsilon) + \mathbf{m}_p^\bullet(f_n - f, A_\epsilon \backslash E_n) + \mathbf{m}_p^\bullet(f_n - f, E_n) < \epsilon$$

for $n \ge n_0$ and hence $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then by the triangular inequality, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and as $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_P\mathcal{M}(\mathbf{m})$ by Theorem 6.8, $f \in \mathcal{L}_p(\mathbf{m})$.

The last part is due to the first, Theorems 4.2 and 5.11(i) and inequality (5.3.1).

The following theorem is an analogue of Theorem III.6.15 od [DS] for $\mathcal{L}_p(\mathbf{m})$.

Theorem 8.6 (Analogue of the a.e. convergence version of the Vitali convergence theorem of [DS] for $\mathcal{L}_p(\mathbf{m})$). Let $1 \le p < \infty$. Let $(f_n) \subset \mathcal{L}_p(\mathbf{m})$ and let $f : T \to \boldsymbol{K}$ Suppose $f_n \to f$ $\mathbf{m}$-a.e. in T. Then $f \in \mathcal{L}_p(\mathbf{m})$ and $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$ if and only if the following conditions are satisfied:

(a) $\mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \boldsymbol{N}$, are uniformly $\mathbf{m}$-continuous on $\sigma(\mathcal{P})$.

(b) For each $\epsilon > 0$, there exists $A_\epsilon \in \mathcal{P}$ such that $\mathbf{m}_p^\bullet(f_n, T\backslash A_\epsilon) < \epsilon$ for all n.

In such case, for $p = 1$, $\int_E f d\mathbf{m} = \lim_n \int_E f_n d\mathbf{m}$, for $E \in \sigma(\mathcal{P})$ and the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. Let $f \in \mathcal{L}_p(\mathbf{m})$ and let $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then (a) and (b) hold by Theorem 8.5.

Conversely, let (a) and (b) hold. By hypothesis, f is $\mathbf{m}$-measurable. Let $\epsilon > 0$. By (b) there exists $A \in \mathcal{P}$ such that

$$\mathbf{m}_p^\bullet(f_n, T\backslash A) < \frac{\epsilon}{3} \qquad (8.6.1)$$

for all n. Let $\Sigma = \sigma(\mathcal{P}) \cap A$. Then Σ is a σ-algebra of subsets of A and by hypothesis $f_n \to f$ $\mathbf{m}$-a.e. in A. Therefore, by Theorem 5.18(viii), $f_n \to f$ in measure in A. Moreover, (a) implies that $\mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \boldsymbol{N}$, are uniformly $\mathbf{m}|_\Sigma$-continuous on Σ. Hence conditions (i) and (ii) of Theorem 8.5 are satisfied with $\mathcal{P}$ and $\sigma(\mathcal{P})$ being replaced by Σ. Further, as $||\mathbf{m}||(A) < \infty$, condition (iii) of the said theorem also holds with $A_\epsilon = A$. Hence by Theorem 8.5, there exists n_0 such that $\mathbf{m}_p^\bullet(f_n - f, A) < \frac{\epsilon}{3}$ for $n \ge n_0$. By (8.6.1) and by the generalaized Fatou's lemma (Theorem 6.1(ii)), $\mathbf{m}_p^\bullet(f, T\backslash A) = \mathbf{m}_p^\bullet(\liminf_n f_n, T\backslash A) \le \liminf_n \mathbf{m}_p^\bullet(f_n, T\backslash A) < \frac{\epsilon}{3}$. Consequently, $\mathbf{m}_p^\bullet(f_n - f, T) \le \mathbf{m}_p^\bullet(f_n - f, A) + \mathbf{m}_p^\bullet(f_n, T\backslash A) + \mathbf{m}_p^\bullet(f, T\backslash A) < \epsilon$ for $n \ge n_0$ and hence $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then by the triangular inequality, $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$. As $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ by Theorem 6.8, it follows that $f \in \mathcal{L}_p(\mathbf{m})$. The last part is due to the first part, Theorems 4.2 and 5.11(i) and inequality (5.3.1).

Theorem 8.7. LDCT and LBCT as given in Theorem 7.4 are deducible from Theorems 8.5 and 8.6.

Proof. For the dominating function g in LDCT, Lemmas 8.2 and 8.4 hold and hence (ii) and (iii) of Theorem 8.5 (resp. (a) and (b) of Theorem 8.6) hold. Thus, if $f_n \to f$ in measure in T (resp. $\mathbf{m}$-a.e. in T), then LDCT holds by Theorem 8.5 (resp. by Theorem 8.6). If $\mathcal{P}$ is a σ-ring, then constant functions belong to $\mathcal{L}_p(\mathbf{m})$ and hence both the versions of LBCT follow from the corresponding versions of LDCT.

Now we define a translation invariant pseudo metric ρ (similar to that in Section 2, Chapter III of [DS]) in the set of all $\mathbf{m}$-measurable scalar functions. Then the following lemma says that $f_n \to f$ in measure in T if and only if $\rho(f_n, f) \to 0$. Using this lemma, we obtain an analogue of Theorem III.3.7 of [DS] in Theorem 8.10 below.

Definition 8.8. Let $\mathcal{M}(\widetilde{\sigma(\mathcal{P})})$ be the set of all $\widetilde{\sigma(\mathcal{P})}$-measurable (i.e. $\mathbf{m}$-measurable) scalar functions on T. For $f \in \mathcal{M}(\widetilde{\sigma(\mathcal{P})})$ and $c > 0$, c real, let $\{t \in T : |f(t)| > c\} = T(|f| > c)$. Let φ be a continuous strictly increasing real function on $[0, \infty)$ such that $\varphi(0) = 0$, $\varphi(x+y) \le \varphi(x) + \varphi(y)$ if $0 \le x \le y$ and $\varphi(\infty) = \lim_{x\to\infty} \varphi(x)$ exists as a real number. (For example, φ given by $\varphi(x) = \frac{x}{1+x}$ satisfies these conditions.) For $f, g \in \mathcal{M}(\widetilde{\sigma(\mathcal{P})})$, we define $\rho(f,g) = \inf_{c>0}\{c + \varphi(||\mathbf{m}||(T(|f-g| > c)))\}$.

It is easy to verify that ρ is a translation invariant pseudo metric on $\mathcal{M}(\widetilde{\sigma(\mathcal{P})})$. (See p.102 of [DS].)

Lemma 8.9. Let $(f_\alpha)_{\alpha\in(D,\ge)}$ be a net of $\mathbf{m}$-measurable scalar functions on T and let $f : T \to \boldsymbol{K}$ be $\mathbf{m}$-measurable. Then $f_\alpha \to f$ in measure in T if and only if $\rho(f_\alpha, f) \to 0$ as $\alpha \to \infty$.

Proof. Let $f_\alpha \to f$ in measure in T. Let $\epsilon > 0$. By the continuity of φ in $t = 0$, there exists $\delta > 0$ such that $0 \le \varphi(t) < \epsilon$ if $0 \le t < \delta$. As $f_\alpha \to f$ in measure in T (see Definition 5.16(i)), there exists α_0 such that $||\mathbf{m}||(T(|f_\alpha - f| > \epsilon)) < \delta$ for $\alpha \ge \alpha_0$ so that $\varphi(||\mathbf{m}||(T|f_\alpha - f| > \epsilon)) < \epsilon$ for $\alpha \ge \alpha_0$. Thus $\rho(f_\alpha, f) \le \epsilon + \varphi(||\mathbf{m}||(T(|f_\alpha - f| > \epsilon))) < 2\epsilon$ for $\alpha \ge \alpha_0$. This shows that $\lim_\alpha \rho(f_\alpha, f) = 0$.

Conversely, let $\lim_\alpha \rho(f_\alpha, f) = 0$. Let $\epsilon > 0$ and let $\varphi(||\mathbf{m}||(T(|f_\alpha - f| > \epsilon))) > \delta > 0$. Then, for $0 < \eta \le \epsilon$, $T(|f_\alpha - f| > \eta) \supset T(|f_\alpha - f| > \epsilon)$ and hence $\varphi(||\mathbf{m}||(T(|f_\alpha - f| > \eta))) > \delta$ and thus $\eta + \varphi(||\mathbf{m}||(T(|f_\alpha - f| > \eta))) > \delta$ for $0 < \eta \le \epsilon$. If $\eta > \epsilon$, then $\eta + \varphi(||\mathbf{m}||(T(|f_\alpha - f| > \eta))) \ge \eta > \epsilon$. Thus $\rho(f_\alpha, f) \ge \min\{\epsilon, \delta\}$. If (f_α) does not converge to f in measure in T, then there would exist $\epsilon > 0$, $\delta > 0$ such that for each $\alpha \in (D, \ge)$ there would exist $\beta_\alpha \in (D, \ge)$ with $\beta_\alpha \ge \alpha$ such that $\varphi(||\mathbf{m}||(T(|f_{\beta_\alpha} - f| > \epsilon))) > \delta$. Then by the above argument we would have $\rho(f_{\beta_\alpha}, f) \ge \min\{\epsilon, \delta\}$. This is a contradiction as there exists an $\alpha_0 \in (D, \ge)$ such that $\rho(f_\alpha, f) < \min\{\epsilon, \delta\}$ for all $\alpha \ge \alpha_0$. Hence the lemma holds.

Theorem 8.10 (LDCT for nets-convergence in measure version for $\mathcal{L}_p(\mathbf{m})$). Let $1 \le p < \infty$. Let $(f_\alpha)_{\alpha\in(D,\ge)}$ be a net of $\mathbf{m}$-measurable scalar functions on T and let $f : T \to \boldsymbol{K}$ be $\mathbf{m}$-measurable. Let $g \in \mathcal{L}_p(\mathbf{m})$. If $|f_\alpha(t)| \le |g(t)|$ $\mathbf{m}$-a.e. in T for each α, then $f_\alpha \to f$ in measure in T if and only if $f \in \mathcal{L}_p(\mathbf{m})$ and $\lim_{\alpha\in(D,\ge)} \mathbf{m}_p^\bullet(f_\alpha - f, T) = 0$. In such case, for $p = 1$, we have $f \in \mathcal{L}_1(\mathbf{m})$ and

$$\int_E f d\mathbf{m} = \lim_\alpha \int_E f_\alpha d\mathbf{m}, \quad E \in \sigma(\mathcal{P})$$

where the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. First let us consider the case of a sequence (f_n) satisfying $|f_n(t)| \le |g(t)|$ $\mathbf{m}$-a.e. in T, where $g \in \mathcal{L}_p(\mathbf{m})$. If $f_n \to f$ in measure in T, then by Theorem 7.4 or 8.7, $f \in \mathcal{L}_p(\mathbf{m})$ and $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Conversely, if $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$, then by Theorem 5.18(vi), $f_n \to f$ in measure in T. Hence the first part of the theorem holds for sequences.

Now let us pass on to the case of nets. Let $(f_\alpha)_{\alpha\in(D,\ge)}$ satisfy the hypothesis of domination and let $f_\alpha \to f$ in measure in T. Then $\mathbf{m}_p^\bullet(f_\alpha, T) \le \mathbf{m}_p^\bullet(g, T)$ for all α and hence $(f_\alpha)_{\alpha\in(D,\ge)} \subset \mathcal{L}_p(\mathbf{m})$

by Theorem 3.5(vii). If $\mathbf{m}_p^\bullet(f_\alpha - f, T) \not\to 0$, then there would exist an $\epsilon > 0$ such that for each $\alpha \in (D, \geq)$ there would exist $\beta_\alpha \geq \alpha$ in $(D, \geq)$ such that $\mathbf{m}_p^\bullet(f_{\beta_\alpha} - f, T) > \epsilon$. As $f_\alpha \to f$ in measure in T, (f_{β_α}) also converges to f in measure in T. Then by Lemma 8.9, for each $n \in \mathbb{N}$ there exists $f_{\beta_{\alpha_n}}$ with $\rho(f_{\beta_{\alpha_n}}, f) < \frac{1}{n}$ and hence $\rho(f_{\beta_{\alpha_n}}, f) \to 0$ as $n \to \infty$. Therefore, again by Lemma 8.9, $f_{\beta_{\alpha_n}} \to f$ in measure in T and consequently, by the previous case of sequences $\mathbf{m}_p^\bullet(f_{\beta_{\alpha_n}} - f, T) \to 0$, which is a contradiction. Hence

$$\lim_\alpha \mathbf{m}_p^\bullet(f_\alpha - f, T) = 0. \tag{8.10.1}$$

As $(f_\alpha) \subset \mathcal{L}_p(\mathbf{m})$, by (8.10.1) and by the triangular inequality we have $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$. As $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ by Theorem 6.8, it follows from (8.10.1) that $f \in \mathcal{L}_p(\mathbf{m})$. Thus the necessity part of the theorem holds.

Conversely, if $f \in \mathcal{L}_p(\mathbf{m})$ and if $\mathbf{m}_p^\bullet(f_\alpha - f, T) \to 0$ as $\alpha \to \infty$, then by Theorem 5.18(vi) $f_\alpha \to f$ in measure in T as $\alpha \to \infty$.

Let $p = 1$. Then $f \in \mathcal{L}_1(\mathbf{m})$ and given $\epsilon > 0$, there exists α_0 such that $\mathbf{m}_1^\bullet(f_\alpha - f, T) < \epsilon$ for $\alpha \geq \alpha_0$. Then, by inequality (5.3.1) and by Theorems 4.2 and 5.11(i), we have $|\int_E f d\mathbf{m} - \int_E f_\alpha d\mathbf{m}| \leq \mathbf{m}_1^\bullet(f - f_\alpha, E) \leq \mathbf{m}_1^\bullet(f - f_\alpha, T) < \epsilon$ for all $\alpha \geq \alpha_0$ and for all $E \in \sigma(\mathcal{P})$. Hence the last part also holds.

Corollary 8.11 (LBCT for nets-convergence in measure version for $\mathcal{L}_p(\mathbf{m})$) . Let $1 \leq p < \infty$. Let $\mathcal{P}$ be a σ-ring $\mathcal{S}$ and let $0 < M < \infty$. Let $(f_\alpha)_{\alpha \in (D, \geq)}$ be a net of $\mathbf{m}$-measurable functions on T with values in $\mathbb{K}$and let $f : T \to \mathbb{K}$ be $\mathbf{m}$-measurable. If $|f_\alpha(t)| \leq M$ $\mathbf{m}$-a.e. in T for each α, then $f_\alpha \to f$ in measure in T if and only if $f \in \mathcal{L}_p(\mathbf{m})$ and $\lim_\alpha \mathbf{m}_p^\bullet(f_\alpha - f, T) = 0$. In such case, for $p = 1$, we have $f \in \mathcal{L}_1(\mathbf{m})$ and

$$\int_E f d\mathbf{m} = \lim_\alpha \int_E f_\alpha d\mathbf{m}, \;\; E \in \mathcal{S}$$

where the limit is uniform with respect to $E \in \mathcal{S}$.

Proof. As $\mathcal{P}$ is a σ-ring $\mathcal{S}$, by the last part of Theorem 3.5(v) constant functions are in $\mathcal{I}_p(\mathbf{m})$ ($=\mathcal{L}_p(\mathbf{m})$ by Theorems 5.10 and 6.7) and hence the result is immediate from Theorem 8.10.

Remark 8.12. Theorems III.3.6 and III.3.7 of [DS] hold for any complex valued or extended real valued finitely additive set function μ defined on a σ-algebra of sets Σ, even though the spaces $\mathcal{L}_p(\mu)$, $1 \leq p < \infty$, are not complete. But, our proofs of the analogues of these theorems for $\mathbf{m}$ are based on the facts that $\mathcal{L}_p(\mathbf{m})$ is closed in $\mathcal{L}_p\mathcal{M}(\mathbf{m})$ and that $\mathbf{m}$ is σ-additive. When X is an lcHs and $\mathbf{m} : \mathcal{P} \to X$ is σ-additive, Theorems 8.5, 8.6, 8.7 and 8.10 and Corollary 8.11 are generalized in Theorems 15.12 of [P2] for quasicomplete X (resp. sequential complete X (for $\sigma(\mathcal{P})$-measurable scalar functions)).

If $c_0 \not\subset X$, then the hypothesis that $\mathcal{P}$ is a σ-ring in LBCT can be weakened as follows.

Theorem 8.13 (LBCT). Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and let $||\mathbf{m}||(T) < \infty$. If $c_0 \not\subset X$, then the versions of LBCT in Theorem 7.4 and Corollary 8.11 hold.

Proof. By hypothesis and by Theorems 4.2, 5.11(iii) and 5.8, constant functions are $\mathbf{m}$-integrable in T and hence the results hold. (See the proofs of Theorem 7.4 and Corollary 8.11).

The following result is an analogue of Theorem 1 of [Do4] for $\mathcal{L}_p(\mathbf{m})$, $1 \le p < \infty$.

Theorem 8.14 (Extended Vitali convergence theorem). Let $1 \le p < \infty$ and let $(f_n)_1^\infty \subset \mathcal{L}_p(\mathbf{m})$. Suppose $f_n \to f$ $\mathbf{m}$-a.e. in T where f is a scalar function on T, or suppose there exists an $\mathbf{m}$-measurable scalar function f on T such that $f_n \to f$ in measure in T. Then the following statements are equivalent:

(i) $\mathbf{m}_p^\bullet(f_n - f, T) \to 0$ as $n \to \infty$.

(ii) $f \in \mathcal{L}_p(\mathbf{m})$ and $\mathbf{m}_p^\bullet(f_n, E) \to \mathbf{m}_p^\bullet(f, E)$ as $n \to \infty$, for each $E \in \sigma(\mathcal{P})$.

(iii) $\mathbf{m}_p^\bullet(f_n, \cdot)$, $n \in \mathbb{N}$ are uniformly continuous on $\sigma(\mathcal{P})$.

If anyone of the above statements holds for $p = 1$, then

$$\lim_n \int_E f_n d\mathbf{m} = \int_E f d\mathbf{m}, \quad E \in \sigma(\mathcal{P})$$

and the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof.(i)$\Rightarrow$(ii) Let $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. Then by Theorem 5.13(ii), $f \in \mathcal{L}_p\mathcal{M}(\mathbf{m})$ and consequently, by Theorem 6.8, $f \in \mathcal{L}_p(\mathbf{m})$. Moreover, by Theorems 5.13(i) and 5.11(i) we have

$$|\mathbf{m}_p^\bullet(f_n, E) - \mathbf{m}_p^\bullet(f, E)| \le \mathbf{m}_p^\bullet(f_n - f, E) \le \mathbf{m}_p^\bullet(f_n - f, T) \to 0 \text{ as } n \to \infty$$

for each $E \in \sigma(\mathcal{P})$.

(ii)$\Rightarrow$(iii) Let $E_k \searrow \emptyset$ in $\sigma(\mathcal{P})$ and let $\epsilon > 0$. As $f \in \mathcal{L}_p(\mathbf{m})$, by Definition 6.5 there exists k_0 such that $\mathbf{m}_p^\bullet(f, E_k) < \frac{\epsilon}{2}$ for $k \ge k_0$. By hypothesis, there exists n_0 such that $|\mathbf{m}_p^\bullet(f_n, E_{k_0}) - \mathbf{m}_p^\bullet(f, E_{k_0})| < \frac{\epsilon}{2}$ for $n \ge n_0$. Hence by Theorem 5.13(i), $\mathbf{m}_p^\bullet(f_n, E_{k_0}) < \epsilon$ for $n \ge n_0$. As $f_n \in \mathcal{L}_p(\mathbf{m})$ for $n = 1, 2, ..., n_0$, there exists $k_1 > k_0$ such that $\mathbf{m}_p^\bullet(f_n, E_k) < \epsilon$ for $n = 1, 2, ..., n_0$ and for $k \ge k_1$. Then by Theorem 5.11(i), (iii) holds.

(iii)$\Rightarrow$(i) by Theorem 7.1 if $f_n \to f$ $\mathbf{m}$-a.e in T and by Theorem 7.2 if $f_n \to f$ in measure in T.

The last part is immediate from (i), inequality (5.3.1) and Theorem 5.11(i).

The following result is an analogue of Theorem 2 of [Do4] for $\mathcal{L}_p(\mathbf{m})$, $1 \le p < \infty$.

Theorem 8.15 (Monotone convergence theorem for $\mathcal{L}_p(\mathbf{m})$). Let $c_0 \not\subset X$ and let $1 \le p < \infty$. Let f_n, $n \in \mathbb{N}$ be $\mathbf{m}$-measurable scalar functions on T and let $f : T \to \mathbb{K}$ Suppose $f_n \to f$ $\mathbf{m}$-a.e. in T and $|f_n| \nearrow |f|$ $\mathbf{m}$-a.e. in T. Then the following statements are equivalent:

(i) $\sup_n \mathbf{m}_p^\bullet(f_n, T) < \infty$.

(ii) $f \in \mathcal{L}_p(\mathbf{m})$.

If (i) or (ii) holds, then f, f_n, $n \in \mathbb{N}$ belong to $\mathcal{L}_p(\mathbf{m})$ and $\lim_n \mathbf{m}_p^\bullet(f_n - f, T) = 0$. In such case, for $p = 1$,

$$\lim_n \int_E f_n d\mathbf{m} = \int_E f d\mathbf{m}, \quad E \in \sigma(\mathcal{P})$$

and the limit is uniform with respect to $E \in \sigma(\mathcal{P})$.

Proof. By hypothesis and by Theorem 6.1(i),

$$\sup_n \mathbf{m}_p^\bullet(f_n, T) = \mathbf{m}_p^\bullet(f, T). \tag{8.15.1}$$

Then (i)$\Rightarrow$(ii) by (8.15.1) and by Theorem 5.8; (ii)$\Rightarrow$(i) by Theorem 5.8 and (8.15.1). If (i) or (ii) holds, then then (ii) holds and hence the last part holds by Theorem 7.4.

The hypothesis that $c_0 \not\subset X$ in the above theorem cannot be omitted as shown in the following counter-example.

Counter-example 8.16. Let T, S, X, $\mathbf{m}$ and f be as in Counter-example 5.7. Let $f_n = f\chi_{\{t \le n\}}$ for $n \in \boldsymbol{N}$. Then $f_n(t) = |f_n(t)| \nearrow |f(t)| = f(t)$ for $t \in T$. As shown in the discussion of the said counter-example, $\mathbf{m}_p(f,T) < \infty$ and by Theorem 6.1(i), $\sup_n \mathbf{m}_p^\bullet(f_n, T) = \mathbf{m}_p^\bullet(f,T)$. But, by Counter-example 5.7, $f \notin \mathcal{L}_p(\mathbf{m}) = \mathcal{I}_p(\mathbf{m})$ for $1 \le p < \infty$.

The following result is an analogue of Theorem 6 of [Do3] for $\mathcal{L}_p(\mathbf{m})$, $1 \le p < \infty$.

Theorem 8.17. Let $1 \le p < \infty$. Suppose $\mathbf{m}_n : \mathcal{P} \to X$ is σ-additive for $n \in \boldsymbol{N}$. Let $(f_k)_{k=1}^\infty$ be $\mathbf{m}_n$-measurable scalar functions for $n \in \boldsymbol{N}$, let f_0 be a scalar function on T and let $\lim_k f_k(t) = f_0(t)$ for $t \in T \backslash M$, where $M \in \sigma(\mathcal{P})$ with $\sup_n ||\mathbf{m}_n||(M) = 0$. Suppose $\sup_n ||\mathbf{m}_n||(E) < \infty$ for each $E \in \mathcal{P}$. Then:

(i) If $(\mathbf{m}_n)_p^\bullet(f_k, \cdot)$, k, $n \in \boldsymbol{N}$, are uniformly continuous on $\sigma(\mathcal{P})$, then

$$\lim_{k\to\infty} \sup_n (\mathbf{m}_n)_p^\bullet(f_k - f_0, T) = 0. \tag{8.17.1}$$

When $p = 1$,

$$\lim_k \int_E f_k d\mathbf{m}_n = \int_E f_0 d\mathbf{m}_n, \quad E \in \sigma(\mathcal{P})$$

and the limit is uniform with respect to $n \in \boldsymbol{N}$ and $E \in \sigma(\mathcal{P})$.

(ii) (**Extended LDCT for $\mathcal{L}_p(\mathbf{m})$**). Let $g \in \bigcap_{n=1}^\infty \mathcal{L}_p(\mathbf{m}_n)$ be such that $\mathbf{m}_n^\bullet(g, \cdot)$,$n \in \boldsymbol{N}$, are unfiformly continuous on $\sigma(\mathcal{P})$. If, for each $n \in \boldsymbol{N}$, $|f_k(t)| \le |g(t)|$ $\mathbf{m}_n$-a.e. in T for all $k \in \boldsymbol{N}$, then the conclusions of (i) hold.

Proof. (i) In the light of Definition 3.1, without loss of generality we shall assume that the functions $(f_k)_1^\infty$ are $\sigma(\mathcal{P})$-measurable. Then by hypothesis and by Theorems 6.7 and 7.1, $(f_k)_{k=0}^\infty \subset \mathcal{L}_p(\mathbf{m}_n)$ for each $n \in \boldsymbol{N}$. Let $\gamma_{n,k}(\cdot) = \int_{(\cdot)} |f_k|^p d\mathbf{m}_n$ for $k \in \boldsymbol{N} \cup \{0\}$. Then by Theorem 5.6 and by hypothesis,

$$||\gamma_{n,k}||(\cdot) = ((\mathbf{m}_n)_p^\bullet(f_k, \cdot))^p, \ \ k \in \boldsymbol{N} \cup \{0\} \tag{8.17.2}$$

are uniformly continuous on $\sigma(\mathcal{P})$ for each $n \in \boldsymbol{N}$ and hence, in particular, by Proposition 2.5, $(\gamma_{n,k})_{k=0}^\infty$ are uniformly σ-additive on $\sigma(\mathcal{P})$ for each $n \in \boldsymbol{N}$. Then, for each $n \in \boldsymbol{N}$, by Proposition 2.6 there exists a control measure $\mu_n : \sigma(\mathcal{P}) \to [0, \infty)$ for $(\gamma_{n,k})_{k=0}^\infty$. Let $K_n = \sup_{E \in \sigma(\mathcal{P})} \mu_n(E)$. Let

$$\mu(E) = \sum_{n=1}^\infty \frac{1}{2^n} \left(\frac{\mu_n(E)}{1 + K_n} \right) \text{ for } E \in \sigma(\mathcal{P}).$$

Then $\mu : \sigma(\mathcal{P}) \to [0, \infty)$ is σ-additive, and $\mu(N) = 0$ implies

$$\sup_{n \in \boldsymbol{N}, k \in \boldsymbol{N} \cup \{0\}} ||\gamma_{n,k}||(N) = 0$$

so that, by (8.17.2), $(\mathbf{m}_n)_p^\bullet(f_k, N) = 0$ (8.17.3) for all $n \in \boldsymbol{N}$ and for all $k \in \boldsymbol{N} \cup \{0\}$.

Let $F = \bigcup_{k=1}^\infty N(f_k) \cap (T \backslash M)$. Then $F \in \sigma(\mathcal{P})$. By hypothesis, $f_k \chi_{T\backslash M} \to f_0 \chi_{T \backslash M}$ pointwise in T. Then by the Egoroff-Lusin theorem there exist $N \in \sigma(\mathcal{P}) \cap F$ with $\mu(N) = 0$ and a sequence

$(F_\ell)_1^\infty \subset \mathcal{P}$ with $F_l \nearrow F\backslash N$ such that $f_k \to f_0$ uniformly in each F_ℓ.

Let $\epsilon > 0$. Since $F\backslash N\backslash F_\ell \searrow \emptyset$, by hypothesis there exists ℓ_0 such that

$$\sup_{n\in\mathbb{N}, k\in\mathbb{N}\cup\{0\}} (\mathbf{m}_n)_p^\bullet(f_k, F\backslash N\backslash F_\ell) < \frac{\epsilon}{3}$$

for $\ell \geq \ell_0$. By hypothesis, there exists a finite constant K such that $\sup_n ||\mathbf{m}_n||(F_{\ell_0}) \leq K$. As $f_k \to f_0$ uniformly in F_{ℓ_0}, there exists k_0 such that $||f_k - f_0||_{F_{\ell_0}} \cdot K^{\frac{1}{p}} < \frac{\epsilon}{3}$ for all $k \geq k_0$. Then, by Theorems 5.11 and 5.13 and by (8.17.3), we have

$$\begin{aligned}
&(\mathbf{m}_n)_p^\bullet(f_k - f_0, T)\\
&= (\mathbf{m}_n)_p^\bullet(f_k - f_0, F) = (\mathbf{m}_n)_p^\bullet(f_k - f_0, F\backslash N)\\
&\leq (\mathbf{m}_n)_p^\bullet(f_k - f_0, F_{\ell_0}) + (\mathbf{m}_n)_p^\bullet(f_k - f_0, F\backslash N\backslash F_{\ell_0})\\
&< ||f_k - f_0||_{F_{\ell_0}} \cdot (||\mathbf{m}_n||(F_{\ell_0}))^{\frac{1}{p}} + (\mathbf{m}_n)_p^\bullet(f_k, F\backslash N\backslash F_{\ell_0}) + (\mathbf{m}_n)_p^\bullet(f_0, F\backslash N\backslash F_{\ell_0})\\
&< \epsilon
\end{aligned}$$

for $k \geq k_0$ and for all $n \in \mathbb{N}$. Hence (8.17.1) holds. The last part of (i) is due to (8.17.1) and inequality (5.3.1).

(ii) is immediate from (i).

9. RELATIONS BETWEEN THE SPACES $\mathcal{L}_p(\mathbf{m})$

In this section we obtain results analogous to those in Section 5, §12 of [Din] for the spaces $\mathcal{L}_p(\mathbf{m})$. The following theorem palys a key role in this section.

Theorem 9.1 (Hölder's inequality). Let $\mathbf{m} \to X$ be σ-additive. Let $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. If $f \in \mathcal{L}_p(\mathbf{m})$ and $g \in \mathcal{L}_q(\mathbf{m})$, then $fg \in \mathcal{L}_1(\mathbf{m})$ and

$$\mathbf{m}_1^\bullet(fg, T) \leq \mathbf{m}_p^\bullet(f, T) \cdot \mathbf{m}_q^\bullet(g, T). \tag{9.1.1}$$

Proof. By Proposition 2.6 there exist $(s_n)_1^\infty$, $(\omega_n)_1^\infty \subset \mathcal{I}_s$ such that $s_n \to f$ and $|s_n| \nearrow |f|$ $\mathbf{m}$-a.e. in T and $\omega_n \to g$ and $|\omega_n| \nearrow |g|$ $\mathbf{m}$-a.e. in T. Then by LDCT (Theorem 7.4), $\lim_n \mathbf{m}_p^\bullet(f - s_n, T) = 0$ and $\lim_n \mathbf{m}_q^\bullet(g - \omega_n, T) = 0$. Hence, given $\epsilon > 0$, there exists n_0 such that $\mathbf{m}_p^\bullet(s_n - s_r, T) \cdot \mathbf{m}_q^\bullet(g, T) < \frac{\epsilon}{2}$ and $\mathbf{m}_q^\bullet(\omega_n - \omega_r, T) \cdot \mathbf{m}_p(f, T) < \frac{\epsilon}{2}$ for $n, r \geq n_0$. Let $E \in \sigma(\mathcal{P})$. Then by Theorems 5.3 and 5.13(iii) we have

$$\begin{aligned}
&|\int_E s_n\omega_n d\mathbf{m} - \int_E s_r\omega_r d\mathbf{m}|\\
&\leq \sup_{|x^*|\leq 1} \int_E |s_n(\omega_n - \omega_r)| dv(x^*\mathbf{m}) + \sup_{|x^*|\leq 1} \int_E |\omega_r(s_n - s_r)| dv(x^*\mathbf{m})\\
&\leq \mathbf{m}_1^\bullet(s_n(\omega_n - \omega_r), T) + \mathbf{m}_1^\bullet(\omega_r(s_n - s_r), T)\\
&\leq \mathbf{m}_p^\bullet(s_n, T) \cdot \mathbf{m}_q^\bullet(\omega_n - \omega_r, T) + \mathbf{m}_q^\bullet(\omega_r, T) \cdot \mathbf{m}_p^\bullet(s_n - s_r, T)\\
&\leq \mathbf{m}_p^\bullet(f, T) \cdot \mathbf{m}_q^\bullet(\omega_n - \omega_r, T) + \mathbf{m}_q^\bullet(g, T) \cdot \mathbf{m}_p^\bullet(s_n - s_r, T)\\
&< \epsilon
\end{aligned}$$

for $n, r \geq n_0$ and this holds for all $E \in \sigma(\mathcal{P})$. Since $(s_n\omega_n)_1^\infty \subset \mathcal{I}_s$ and since $s_n\omega_n \to fg$ $\mathbf{m}$-a.e. in T such that $\lim_n \int_E s_n\omega_n d\mathbf{m} = x_E(\text{say}) \in X$ for $E \in \sigma(\mathcal{P})$, by Definition 4.1 $fg \in \mathcal{L}_1(\mathbf{m})$. Inequality

(9.1.1) holds by Theorems 5.10 and 5.13(iii).

Theorem 9.2. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive. Then:

(i) If $1 \le r < p < s \le \infty$, then $\mathcal{L}_r(\mathbf{m}) \bigcap \mathcal{L}_s(\mathbf{m}) \subset \mathcal{L}_p(\mathbf{m})$.
(ii) If $f : T \to \boldsymbol{K}$is $\mathbf{m}$-measurable, then the set $\mathcal{I}_f = \{p : 1 \le p \le \infty, f \in \mathcal{L}_p(\mathbf{m})\}$ is either void or an interval, where singletons are considered as intervals.
(iii) For f in (ii) with $\mathcal{I}_f \ne \emptyset$, the function $p \to \log \mathbf{m}_p^\bullet(f,T)$ is convex on $\mathcal{I}_f$ and the function $p \to \mathbf{m}_p^\bullet(f,T)$ is continuous on $Int\,\mathcal{I}_f$.

Proof. The proof of Proposition 21, §12 of [Din] holds here in virtue of Theorems 3.5(vi), 4.2, 5.3 and 9.1. Details are left to the reader.

Theorem 9.3. Let $\mathbf{m} : \mathcal{P} \to X$ be σ-additive and let $A \in \widetilde{\sigma(\mathcal{P})}$ such that χ_A es $\mathbf{m}$-integrable in T. Then the set $\mathcal{I}_f(A) = \{p : 1 \le p \le \infty,\ f\chi_A \in \mathcal{L}_p(\mathbf{m})\}$ is either void or an interval containing 1 ($\mathcal{I}_f(A) = \{1\}$ is permitted) and the function $p \to \mathbf{m}_p^\bullet(f\chi_A, T) \cdot (||\mathbf{m}||(A))^{-\frac{1}{p}}$ is incresing in $\mathcal{I}_f(A)$, where $\mathbf{m}_\infty^\bullet(f\chi_A, T) = ||f\chi_A||_\infty$.

Proof. In view of (vi) and (vii) (Domination principle) of Theorem 3.5, and Theorems 4.2, 5.3 and 9.1, the proof of Proposition 22, §12 of [Din] holds here verbatim and the details are left to the reader.

Corollary 9.4. Let $\mathbf{m} : \mathcal{S} \to X$ be σ-additive, where $\mathcal{S}$ is a σ-ring of subsets of T. Then:

(i) If $1 \le r < s \le \infty$, then $\mathcal{L}_s(\mathbf{m}) \subset \mathcal{L}_r(\mathbf{m})$ and the topology of $\mathcal{L}_s(\mathbf{m})$ is finer than that of $\mathcal{L}_r(\mathbf{m})$.
(ii) If $f : T \to \boldsymbol{K}$is $\mathbf{m}$-measurable, then the set $\mathcal{I}_f = \{p : 1 \le p \le \infty,\ f \in \mathcal{L}_p(\mathbf{m})\}$ is either void or an interval containing 1 ($\mathcal{I}_f = \{1\}$ is permitted).
(iii) If $\mathcal{I}_f \ne \emptyset$, then the function $p \to \mathbf{m}_p^\bullet(f,T) \cdot (||\mathbf{m}||(N(f))^{-\frac{1}{p}}$ is an increasing function on $\mathcal{I}_f$.

Proof. By hypothesis on $\mathcal{S}$ and by the last part of Theorem 3.5(v), $\mathcal{L}_\infty(\mathbf{m}) \subset \mathcal{L}_p(\mathbf{m})$ for $1 \le p \le \infty$. Let $s > r$ and let $f \in \mathcal{L}_s(\mathbf{m})$. If $E = \{t \in T : |f(t)| > 1\}$ and $F = \{t \in N(f) : |f(t) \le 1\}$, then E and F belong to $\widetilde{\sigma(\mathcal{P})}$, $|f\chi_E|^r \le |f\chi_E|^s \le |f|^s \in \mathcal{L}_1(\mathbf{m})$ and $f\chi_F|^r \le \chi_F \in \mathcal{L}_1(\mathbf{m})$. Hence by Theorems 4.2 and 3.5(vii) (Domination principle), $f\chi_E$ and $f\chi_F$ belong to $\mathcal{L}_r(\mathbf{m})$ and hence $f \in \mathcal{L}_r(\mathbf{m})$. Therfore, the first part of (i) holds. The second part of (i) holds by Theorem 9.3, as $||\mathbf{m}||(N(f)) < \infty$.

(ii) is due to Theorem 9.2(ii) and (i) of the present corollary.

(iii) is immediate if we take $A = N(f)$ in Theorem 9.3.

Remark 9.5. Theorems 9.2 and 9.3 and Corollary 9.4 are generalized to an lcHs-valued $\mathbf{m}$ in Theorem 15.13 of [P2].

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