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Controllability of a Generalized Damped Wave Equation

Hugo Leiva

Abstract

In this paper we give a necessary and sufficient algebraic condition for the controllability of the following generalized damped wave equation on a Hilbert space X

$$\ddot{w} + \eta A^{\alpha}\dot{w} + \gamma A^{\beta} w = \begin{cases} d_1 u_1 + \cdots + d_m u_m, & \text{if } \alpha > 0 \\ u(t), & \text{if } \alpha = 0, \end{cases}$$

where $t \geq 0$, $\gamma > 0$, $\eta > 0$, $\beta \geq 0$ and $d_i \in X$; the scalar control functions $u_i \in L^2(0, t_1; I\!R)$; the distributed control $u \in L^2(0, t_1; X)$ and $A : D(A) \subset X \to X$ is a positive defined self-adjoint unbounded linear operator in X with compact resolvent. The equation $\ddot{w}+\eta A^{\alpha}\dot{w}+\gamma A^{\beta}w = 0$ can be written as a first order system in the space $D(A^{\beta/2}) \times X$ with corresponding linear operator $\mathcal{A}$. Then, we prove the following statements: I) $\mathcal{A}$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ such that for some positive constants $M(\eta, \gamma)$ and μ we have $\|T(t)\| \leq M(\eta, \gamma)e^{-\mu t}$, $t \geq 0$. II) If $2\alpha \geq \beta$, then $\{T(t)\}_{t\geq 0}$ is analytic in the space $D(A^{\alpha}) \times X$. III) If $2\alpha \geq \beta > \alpha$ or $2\alpha \leq \beta$, the system is approximatelly controllable on $[0, t_1]$. IV) If $2\alpha < \beta$, then $\{T(t)\}_{t\geq 0}$ is not analytic. V) If $\alpha = 0$, the system is exactly controllable on $[0, t_1]$. VI) If $\alpha \geq \beta > 0$, the question about the controllability of this system is opened.

Contents

1 Introduction

In this paper we give a necessary and sufficient algebraic condition for both, approximate and exact controllability for the following generalized damped wave equation on a Hilbert space X

$$\ddot{w} + \eta A^{\alpha}\dot{w} + \gamma A^{\beta} w = d_1 u_1 + \cdots + d_m u_m, \quad t \geq 0, \tag{1.1}$$

$$\ddot{w} + \eta \dot{w} + \gamma A^{\beta} w = u(t) \;\; t \geq 0, \tag{1.2}$$

$\gamma > 0$, $\eta > 0$, $\alpha > 0$, $\beta \geq 0$
$d_i \in X, \quad u_i \in L^2(0, t_1; I\!R); \quad i = 1, 2, \ldots, m$
$u \in L^2(0, t_1; X)$
$A : D(A) \subset X \to X$ is a positive defined self-adjoint unbounded linear operator in X with compact resolvent.

$$\mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A^{\beta} & -\eta A^{\alpha} \end{bmatrix}, \tag{1.3}$$

$$\ddot{w} + \eta A^{\alpha}\dot{w} + \gamma A^{\beta} w = 0$$

on the space

$$D(A^{\beta/2}) \times X.$$

.

I) $\mathcal{A}$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on $D(A^{\beta/2}) \times X$ such that $\|T(t)\| \leq M(\eta, \gamma)e^{-\mu t}, \quad t \geq 0.$
II) If $2\alpha \geq \beta$, then $\{T(t)\}_{t\geq 0}$ is analytic on the space $D(A^{\alpha}) \times X$.
III) If $2\alpha \geq \beta > \alpha$ or $2\alpha \leq \beta$ the system is approximatelly controllable on $[0, t_1]$.
IV) If $2\alpha < \beta$, then $\{T(t)\}_{t\geq 0}$ is not analytic
V) If $\alpha = 0$, the system is exactly controllable on $[0, t_1]$.
VI) If $\alpha \geq \beta > 0$, the question about the controllability of this system is opened.

$$\text{Rank}\left[P_j B \vdots A_j P_j B \vdots A_j^2 P_j B \vdots \cdots A_j^{2\gamma_j - 1} P_j B\right] = 2\gamma_j, \tag{1.4}$$

where $B : I\!R^m \to {}^2(\Omega, I\!R^2)$

$$BU = b_1U_1 + \cdots + b_mU_m, \quad b_i = \begin{bmatrix} 0 \\ d_i \end{bmatrix}, \quad A_j = \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_j^\beta & -\eta\lambda_j^\alpha \end{bmatrix} P_j,\, j \geq 1,$$

The same algebraic condition (1.4) hols for the exact controllablity of the system (1.2) if we change the operators B and A_j by:

$$B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad A_j = \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_j^\beta & -\eta \end{bmatrix} P_j,\, j \geq 1.$$

Also, condition (1.4) is equivalent that the operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ given by

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} BB^* e^{-A_j^* s} ds, \tag{1.5}$$

is invertible.

$$u(t) = B^*T^*(-t) \sum_{j=1}^{\infty} W_j^{-1}(t_1) P_j (T(-t_1)z_1 - z_0). \tag{1.6}$$

The uncontrolled equation has been studied by S. CHEN AND R. TRIGGIANI in [3] 1998.

$$\ddot{w} + B\dot{w} + Aw = 0 \ \text{ on } \ X, \tag{1.7}$$

B is positive self-adjoint operator with dense domain, and the following hypothesis holds:

There exists $0 < r < 1$ and $0 < \rho_1, \rho_2 < \infty$ such that

$$\rho_1 A^r \leq B \leq \rho_2 A^r. \tag{1.8}$$

The operator

$$\mathcal{A} = \begin{bmatrix} 0 & I_X \\ -A & -B \end{bmatrix}, \tag{1.9}$$

which corresponds to the equation $\ddot{w} + B\dot{w} + Aw = 0$ written as a first order system in the space $D(A^{1/2}) \times X$, generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ such that

i) $\|T(t)\| \leq 1, \quad t \geq 0$

ii) If $2\alpha \geq 1$, then $\{T(t)\}_{t\geq 0}$ is analytic.

iii) If $2\alpha < 1$, then $\{T(t)\}_{t\geq 0}$ is not analytic.

Results II) and IV) follow from this result if $\beta \geq \alpha$. But, if $\beta < \alpha$ condition (1.8) is not sastified.

In [10] (1998) I. Lasiecka and R. Triggiani study the exact **null** controllability of the following secon order equation

$$\ddot{w} + \rho A^r \dot{w} + Aw = u(t), \quad \rho > 0, \quad \frac{1}{2} \leq r \leq 1, \quad t \geq 0, \tag{1.10}$$

$u \in L^2(0, t_1; X)$. If $\frac{1}{2} \leq r < 1$, , then the system (1.10) is exactly **null** controllable on $[0, t_1]$, but if $\alpha = 1$, the system (1.10) is not exactly **null** controllable.

A particular case of equation (1.1) is the following Vibration of the Spring Equation

$$w_{tt} - 2\beta \Delta w_t + \Delta^2 w = a_1 u_1 + \cdots + a_m u_m, \quad t \geq 0, \quad \text{in } I\!R_+ \times \Omega \tag{1.11}$$

$$w = \Delta w = 0, \quad \text{on } I\!R_+ \times \partial\Omega. \tag{1.12}$$

Finally, our method can be applied to the following generalized thermoelastic plate equation

$$\begin{cases} \ddot{w} + hA^\alpha \ddot{w} + A^\beta w + \gamma A^\alpha \theta = & a_1 u_1 + \cdots + a_m u_m, \quad t \geq 0, \\ \dot{\theta} - \eta A^\alpha \theta + \Gamma \theta - \gamma A^\alpha \dot{w} = & d_1 u_1 + \cdots + d_m u_m, \quad t \geq 0, \end{cases}$$

Some notations for our work can be found in [11], [12], [8] and [13].

2 The Uncontrolled System

a) for all $x \in D(A)$ we have

$$Ax = \sum_{n=1}^{\infty} \lambda_n \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k} = \sum_{n=1}^{\infty} \lambda_n E_n x, \tag{2.13}$$

where $< \cdot, \cdot >$ is the inner product in X and

$$E_n x = \sum_{k=1}^{\gamma_n} < x, \phi_{n,k} > \phi_{n,k}. \tag{2.14}$$

So, $\{E_n\}$ is a family of complete orthogonal projections in X and
$x = \sum_{n=1}^{\infty} E_n x, \quad x \in X$.

b) the fraction power space X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} (\lambda_n)^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \quad x \in X^r,$$

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{2.15}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert Space with the norm and inner product given by:

$$\left\| \begin{bmatrix} w \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|u\|_r^2 + \|v\|^2, \quad < w, v >_r = < A^r w, A^r v > + < w, v > .$$

Now, making the following change of variable $w' = v$, we can write the second order equation (1.1) as first order system of ordinary differential equations in the Hilbert space $Z_{\beta/2} = D(A^{\beta/2}) \times X = X^{\beta/2} \times X$ as follows:

$$z' = \mathcal{A} z + Bu \quad z \in Z_{\beta/2}, \quad t \geq 0, \tag{2.16}$$

where the control $u \in L^2(0, t_1; I\!R^m)$ and

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad BU = b_1 U_1 + \cdots + b U_m, \quad b_i = \begin{bmatrix} 0 \\ d_i \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A^\beta & -\eta A^\alpha \end{bmatrix}, \tag{2.17}$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A^\beta) \times D(A^\alpha)$.

In same way the equation (1.2) can be written as

$$z' = \mathcal{A} z + Bu \quad z \in Z_{\beta/2}, \quad t \geq 0, \tag{2.18}$$

where the control $u \in L^2(0, t_1; X)$ and

$$z = \begin{bmatrix} w \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ I_X \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A^\beta & -\eta I_X \end{bmatrix}. \tag{2.19}$$

Through this work we will assume the following condition:

$$\eta^2 \neq 4\gamma \lambda_n^{\beta - 2\alpha}, \quad n = 1, 2, \ldots.$$

Theorem 2.1 *The operator $\mathcal{A}$ given by (2.17), is the infinitesimal generator of an analytic semigroup $\{T(t)\}_{t \geq 0}$ given by*

$$T(t) z = \sum_{n=1}^{\infty} e^{A_n t} P_n z, \quad z \in Z_{\beta/2}, \quad t \geq 0 \tag{2.20}$$

where $\{P_n\}_{n\geq 0}$ *is a complete family of orthogonal projections in the Hilbert space* $Z_{\beta/2}$ *given by*

$$P_n = diag\,[E_n, E_n]\ ,\ n \geq 1\ , \tag{2.21}$$

and

$$A_n = B_n P_n, \quad B_n = \begin{bmatrix} 0 & 1 \\ -\gamma\lambda_n^{\beta} & -\eta\lambda_n^{\alpha} \end{bmatrix},\ n \geq 1. \tag{2.22}$$

This semigroup decays exponentially to zero. In fact, we have the following estimate

$$\|T(t)\| \leq M(\eta,\gamma)e^{-\mu t}, \quad t \geq 0, \tag{2.23}$$

where

$$\mu = \lambda_1^{\alpha} \inf_{n\geq 1}\left\{ Re\left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma\lambda_n^{\beta-2\alpha}}}{2}\right)\right\}$$

and

$$\frac{M(\eta,\gamma)}{2\sqrt{2}} = \sup_{n\geq 1}\left\{ 2\left|\frac{\eta \pm \sqrt{\eta^2 - 4\gamma}}{2\sqrt{\eta^2 - 4\gamma\lambda^{\beta-2\alpha}}}\right|, \left|2\gamma\sqrt{\frac{\lambda_n^{\beta-2\alpha}}{\eta^2 - 4\gamma\lambda_n^{\beta-2\alpha}}}\right|\right\}.$$

Moreover,

I) If $2\alpha \geq \beta$*, then* $\{T(t)\}_{t\geq 0}$ *is analytic on the space* $Z_\alpha = X^\alpha \times X$.

II) If $2\alpha < \beta$*, then* $\{T(t)\}_{t\geq 0}$ *is not analytic.*

Proof Let us compute $\mathcal{A}z$:

$$\begin{aligned}
\mathcal{A}z &= \begin{bmatrix} 0 & I \\ -\gamma A^{\beta} & -\eta A^{\alpha} \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} \\
&= \begin{bmatrix} v \\ -\gamma A^{\beta}w - \eta A^{\alpha}v \end{bmatrix} \\
&= \begin{bmatrix} \sum_{n=1}^{\infty} E_n v \\ -\gamma\sum_{n=1}^{\infty}\lambda_n^{\beta}E_n w - \eta\sum_{n=1}^{\infty}\lambda_n^{\alpha}E_n v \end{bmatrix} \\
&= \sum_{n=1}^{\infty}\begin{bmatrix} E_n v \\ -\gamma\lambda_n^{\beta}E_n w - \eta\lambda_n^{\alpha}E_n v \end{bmatrix} \\
&= \sum_{n=1}^{\infty}\begin{bmatrix} 0 & 1 \\ -\gamma\lambda_n^{\beta} & -\eta\lambda^{\alpha} \end{bmatrix}\begin{bmatrix} E_n & 0 \\ 0 & E_n \end{bmatrix}\begin{bmatrix} w \\ v \end{bmatrix} \\
&= \sum_{n=1}^{\infty} A_n P_n z.
\end{aligned}$$

It is clear that $A_nP_n = P_nA_n$. Now, we need to check condition (4.60) from Lemma 4.1. To this end, we have to compute the spectrum of the matrix B_n. The characteristic equation of B_n is given by

$$\lambda^2 + \eta\lambda_n^\alpha\lambda + \gamma\lambda_n^\beta = 0,$$

and the roots of it are given by

$$\lambda = -\lambda_n^\alpha \left(\frac{\eta \pm \sqrt{\eta^2 - 4\gamma\lambda_n^{\beta-2\alpha}}}{2} \right), \quad n = 1, 2, \ldots.$$

On the other hand, $e^{A_nt} = e^{B_nt}P_n$ and e^{B_nt} is given by:

$$e^{B_nt} = \begin{bmatrix} \frac{\rho_2}{\rho_2-\rho_1}e^{-\lambda_n^\alpha\rho_1 t} + \frac{\rho_1}{\rho_1-\rho_2}e^{-\lambda_n^\alpha\rho_2 t} & \frac{1}{\lambda_n^\alpha(\rho_2-\rho_1)}e^{-\lambda_n^\alpha\rho_1 t} + \frac{1}{\lambda_n^\alpha(\rho_1-\rho_2)}e^{-\lambda_n^\alpha\rho_2 t} \\ S(n)\lambda_n^{\frac{\beta}{2}}e^{-\lambda_n^\alpha\rho_1 t} - S(n)\lambda_n^{\frac{\beta}{2}}e^{-\lambda_n^\alpha\rho_2 t} & \frac{\rho_1-\eta}{\rho_2-\rho_1}e^{-\lambda_n^\alpha\rho_1 t} + \frac{\rho_2-\eta}{\rho_1-\rho_2}e^{-\lambda_n^\alpha\rho_2 t} \end{bmatrix},$$

where ρ_1 and ρ_2 are given by:

$$\rho_1 = \frac{\eta + \sqrt{\eta^2 - 4\gamma\lambda_n^{\beta-2\alpha}}}{2}, \quad \rho_2 = \frac{\eta - \sqrt{\eta^2 - 4\gamma\lambda_n^{\beta-2\alpha}}}{2}, \quad S = \gamma\sqrt{\frac{\lambda_n^{\beta-2\alpha}}{\eta^2 - 4\gamma\lambda_n^{\beta-2\alpha}}}$$

Now, consider $z = (z_1, z_2)^T \in Z_{\beta/2}$ such that $\|z\|_{Z_{\beta/2}} = 1$. Then,

$$\|z_1\|_{\beta/2}^2 = \sum_{j=1}^{\infty} \lambda_j^\beta \|E_jz_1\|^2 \leq 1 \quad \text{and} \quad \|z_2\|_X^2 = \sum_{j=1}^{\infty} \|E_jz_2\|^2 \leq 1.$$

Therefore, $\lambda_j^{\beta/2}\|E_jz_1\| \leq 1$, $\|E_jz_2\| \leq 1$, $\quad j = 1, 2, \ldots.$

Then,

$$
\begin{aligned}
\|e^{A_n t}z\|_Z^2 &= \left\| \begin{bmatrix} \frac{\rho_2}{\rho_2-\rho_1}e^{-\lambda_n^\alpha \rho_1 t}E_n z_1 + \frac{\rho_1}{\rho_1-\rho_2}e^{-\lambda_n^\alpha \rho_2 t}E_n z_1 \\ S(n)\lambda_n^{\frac{\beta}{2}}e^{-\lambda_n^\alpha \rho_1 t}E_n z_1 - S(n)\lambda_n^{\frac{\beta}{2}}e^{-\lambda_n^\alpha \rho_2 t}E_n z_1 \end{bmatrix} \right. \\
&+ \left. \begin{bmatrix} \frac{1}{\lambda_n^\alpha(\rho_2-\rho_1)}e^{-\lambda_n^\alpha \rho_1 t}E_n z_2 + \frac{1}{\lambda_n^\alpha(\rho_1-\rho_2)}e^{-\lambda_n^\alpha \rho_2 t}E_n z_2 \\ \frac{\rho_1-\eta}{\rho_2-\rho_1}e^{-\lambda_n^\alpha \rho_1 t}E_n z_2 + \frac{\rho_2-\eta}{\rho_1-\rho_2}e^{-\lambda_n^\alpha \rho_2 t}E_n z_2 \end{bmatrix} \right\|_Z^2 \\
&= \left\| \begin{bmatrix} a(n)E_n z_1 + \frac{b(n)}{\lambda_n^\alpha}E_n z_2 \\ c(n)\lambda_n^{\frac{\beta}{2}}E_n z_1 + d(n)E_n z_2 \end{bmatrix} \right\|_Z^2 \\
&= \|a(n)E_n z_1 + \frac{b(n)}{\lambda_n^\alpha}E_n z_2\|_{\frac{\beta}{2}}^2 + \|c(n)\lambda_n^{\frac{\beta}{2}}E_n z_1 + d(n)E_n z_2\|_X^2 \\
&= \sum_{j=1}^{\infty} \lambda_j^\beta \|E_j\left(a(n)E_n z_1 + \frac{b(n)}{\lambda_n^\alpha}E_n z_2\right)\|^2 \\
&+ \sum_{j=1}^{\infty} \|E_j\left(c(n)\lambda_n^{\frac{\beta}{2}}E_n z_1 + d(n)E_n z_2\right)\|^2 \\
&= \lambda_n^\beta \|a(n)E_n z_1 + \frac{b(n)}{\lambda_n^\alpha}E_n z_2\|^2 + \|c(n)\lambda_n^{\frac{\beta}{2}}E_n z_1 + d(n)E_n z_2\|^2 \\
&\leq (|a(n)| + |\frac{\lambda^{\frac{\beta}{2}}}{\lambda_n^\alpha}b(n)|)^2 + (|c(n)| + |d(n)|)^2,
\end{aligned}
$$

where

$$
\begin{aligned}
a(n) &= \frac{\rho_2}{\rho_2-\rho_1}e^{-\lambda_n^\alpha \rho_1 t} + \frac{\rho_1}{\rho_1-\rho_2}e^{-\lambda_n^\alpha \rho_2 t} \\
b(n) &= \frac{1}{(\rho_2-\rho_1)}e^{-\lambda_n^\alpha \rho_1 t} + \frac{1}{(\rho_1-\rho_2)}e^{-\lambda_n^\alpha \rho_2 t} \\
c(n) &= S(n)e^{-\lambda_n^\alpha \rho_1 t} - S(n)e^{-\lambda_n^\alpha \rho_2 t} \\
d(n) &= \frac{\rho_1-\eta}{\rho_2-\rho_1}e^{-\lambda_n^\alpha \rho_1 t} + \frac{\rho_2-\eta}{\rho_1-\rho_2}e^{-\lambda_n^\alpha \rho_2 t} \\
|\frac{\lambda^{\frac{\beta}{2}}}{\lambda_n^\alpha}b(n)| &= \left|\sqrt{\frac{\lambda_n^{\beta-2\alpha}}{\eta^2-4\gamma\lambda_n^{\beta-2\alpha}}}\right|.
\end{aligned}
$$

Then, if we put

$$
\begin{aligned}
\mu &= \lambda_1^\alpha \sup_{n\geq 1}\left\{ \operatorname{Re}\left(\frac{\eta \pm \sqrt{\eta^2-4\gamma\lambda_n^{\beta-2\alpha}}}{2}\right)\right\}, \\
\frac{M(\eta,\gamma)}{2\sqrt{2}} &= \sup_{n\geq 1}\left\{ 2\left|\frac{\eta\pm\sqrt{\eta^2-4\gamma}}{\sqrt{\eta^2-4\gamma\lambda^{\beta-2\alpha}}}\right|, \left|2\gamma\sqrt{\frac{\lambda_n^{\beta-2\alpha}}{\eta^2-4\gamma\lambda_n^{\beta-2\alpha}}}\right|\right\},
\end{aligned}
$$

we get that

$$
\|e^{A_n t}\| \leq M(\eta,\gamma)e^{-\mu t}, \quad t\geq 0 \quad n = 1, 2, \ldots.
$$

Hence, applying Lemma 4.1 we obtain that $\mathcal{A}$ generates a strongly continuous semigroup given by (2.1). Next, we prove this semigroup decays exponentially to zero. In fact,

$$\begin{aligned}\|T(t)z\|^2 &\leq \sum_{n=1}^{\infty} \|e^{A_n t} P_n z\|^2 \\ &\leq \sum_{n=1}^{\infty} \|e^{A_n t}\|^2 \|P_n z\|^2 \\ &\leq M^2(\eta,\gamma) e^{-2\mu t} \sum_{n=1}^{\infty} \|P_n z\|^2 \\ &= M^2(\eta,\gamma) e^{-2\mu t} \|z\|^2.\end{aligned}$$

Therefore,

$$\|T(t)\| \leq M(\eta,\gamma) e^{-\mu t}, \quad t \geq 0.$$

Proof of the analyticity:

We have the following situation:

a) $\mathrm{Re}(\rho_1(n)) > 0, \quad \mathrm{Re}(\rho_2(n)) > 0, \quad n = 1, 2, \ldots$

b) if $2\alpha = \beta$, then $\rho_1(n), \rho_2(n)$ are constants.

c) if $2\alpha > \beta$ then

$$\lim_{n\to\infty} \mathrm{Re}(\rho_1(n)) = \eta \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Re}(\rho_2(n)) = 0 \tag{2.24}$$

d) if $2\alpha < \beta$, then

$$\lim_{n\to\infty} \mathrm{Re}(\rho_1(n)) = \lim_{n\to\infty} \mathrm{Re}(\rho_2(n)) = \frac{\eta}{2} \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Im}(\rho_1(n)) = \infty. \tag{2.25}$$

Therefore, for $2\alpha < \beta$ the operator $-\mathcal{A}$ can not be sectorial which implies that the semigroup $\{T(t)\}_{t\geq 0}$ can never be analytic.

Claim 1. If $2\alpha \geq \beta$, then $\mathcal{A}$ generates a semigroup $\{T(t)\}_{t\geq 0}$ on the space $Z_\alpha = X^\alpha \times X$ given by (2.20). In fact, we can apply Lemma 4.1 to prove this claim. To this end we shall find a uniform bound for $\|e^{A_n t}\|_{L(X^\alpha \times X)}$.

Now, consider $z = (z_1, z_2)^T \in Z_\alpha$ such that $\|z\|_{X^\alpha \times X} = 1$. Then,

$$\|z_1\|_\alpha^2 = \sum_{j=1}^{\infty} \lambda_j^{2\alpha} \|E_j z_1\|^2 \leq 1 \quad \text{and} \quad \|z_2\|_X^2 = \sum_{j=1}^{\infty} \|E_j z_2\|^2 \leq 1.$$

Therefore, $\lambda_j^\alpha \|E_j z_1\| \leq 1, \quad \|E_j z_2\| \leq 1, \quad j = 1, 2, \ldots$, and using the foregoing notation we obtain the following estimate

$$\begin{aligned}
\|e^{A_n t}z\|^2_{X^\alpha\times X} &= \left\|\begin{bmatrix} a(n)E_nz_1+\frac{b(n)}{\lambda_n^\alpha}E_nz_2 \\ c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1+d(n)E_nz_2\end{bmatrix}\right\|^2_{Z_\alpha} \\
&= \|a(n)E_nz_1+\frac{b(n)}{\lambda_n^\alpha}E_nz_2\|^2_\alpha+\|c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1+d(n)E_nz_2\|^2_X \\
&= \sum_{j=1}^\infty \lambda_j^{2\alpha}\|E_j\left(a(n)E_nz_1+\frac{b(n)}{\lambda_n^\alpha}E_nz_2\right)\|^2 \\
&+ \sum_{j=1}^\infty \|E_j\left(c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1+d(n)E_nz_2\right)\|^2 \\
&= \lambda_n^{2\alpha}\|a(n)E_nz_1+\frac{b(n)}{\lambda_n^\alpha}E_nz_2\|^2+\|c(n)\lambda_n^{\frac{\beta}{2}}E_nz_1+d(n)E_nz_2\|^2 \\
&\leq (|a(n)|+|b(n)|)^2+(\lambda_n^{\beta/2}\|E_nz_1\||c(n)|+|d(n)|)^2,
\end{aligned}$$

Now, since $\alpha \geq \frac{\beta}{2}$, then $X^\alpha \subset X^{\beta/2}$ is a continuous inclusion. Therefore, there exists a constant $R_{\alpha\beta}>0$ such that

$$\|z\|_{\beta/2} \leq R_{\alpha\beta}\|z\|_\alpha, \quad z\in X^\alpha.$$

Hence,

$$\|e^{A_nt}z\|^2_{X^\alpha\times X} \leq (|a(n)|+|b(n)|)^2+(|c(n)|R_{\alpha\beta}+|d(n)|)^2.$$

Then, there exists a constant $\overline{M}(\eta,\gamma)>0$ such that

$$\|e^{A_nt}\| \leq \overline{M}(\eta,\gamma)e^{-\mu t}, \quad t\geq 0 \quad n=1,2,\ldots,$$

and

$$\|T(t)\| \leq \overline{M}(\eta,\gamma)e^{-\mu t}, \quad t\geq 0.$$

To prove the analyticity of $\{T(t)\}_{t\geq 0}$ on $X^\alpha\times X$, we apply Lemma 4.2 to prove that $-\mathcal{A}$ is a sectorial operator. From the first part of the proof we know that the spectrum of $A_n:\mathcal{R}(P_n)\to\mathcal{R}(P_n), \quad n=1,2,\ldots$ is given by

$$\begin{aligned}
\sigma(A_n) &= \left\{-\lambda_n^\alpha\left(\frac{\eta\pm\sqrt{\eta^2-4\gamma\lambda_n^{\beta-2\alpha}}}{2}\right)\right\} \\
&= -\lambda_n^\alpha\left\{\left(\frac{\eta\pm\sqrt{\eta^2-4\gamma\lambda_n^{\beta-2\alpha}}}{2}\right)\right\}.
\end{aligned}$$

Then,

$$-\frac{1}{\lambda_n^\alpha}\sigma(A_n)=\left\{\left(\frac{\eta\pm\sqrt{\eta^2-4\gamma\lambda_n^{\beta-2\alpha}}}{2}\right)\right\}.$$

Since $2\alpha > \beta$, then there exists a bounded set S in the complex plane such that $\mathrm{Re}(S) > 0$ and

$$-\frac{1}{\lambda_n^\alpha}\sigma(A_n) \subset S, \quad n = 1, 2, \ldots.$$

Then, Lemma 4.2 can be applied. □

Remark 2.1 *The analyticity of the operator $-\mathcal{A}$ given by the foregoing Theorem, can be proved directly by constracting a sector where it is analytic. This constraction gives us some ideas to prove the exact controllability of the equation (2.18) and for that and others purpose we will give this other poof.*

Indeed, consider the following 2×2 matrices

$$\overline{K}_n = \begin{bmatrix} 1 & 1 \\ \sigma_1(n) & \sigma_2(n) \end{bmatrix}, \quad \overline{K}_n^{-1} = \frac{1}{\sigma_2(n) - \sigma_1(n)} \begin{bmatrix} \sigma_2(n) & -1 \\ -\sigma_1(n) & 1 \end{bmatrix}, \tag{2.26}$$

where

$$\sigma_1(n) = -\lambda_n^\alpha \rho_1(n) \quad \textit{and} \quad \sigma_1(n) = -\lambda_n^\alpha \rho_2(n), n = 1, 2, \ldots. \tag{2.27}$$

Then,

$$B_n = \overline{K}_n^{-1} \overline{J}_n \overline{K}_n, n = 1, 2, 3, \ldots, \tag{2.28}$$

with

$$\overline{J}_n = \begin{bmatrix} \sigma_1(n) & 0 \\ 0 & \sigma_2(n) \end{bmatrix}.$$

Next, we define the following two linear bounded operators

$$K_n : X \times X \to X^\alpha \times X, \quad K_n^{-1} : X^\alpha \times X \to X \times X, \tag{2.29}$$

as follows $K_n = \overline{K}_n^{-1} P_n$ and $K_n = \overline{K}_n^{-1} P_n$.

Let us find bounds for $\|K_n^{-1}\|$ and $\|K_n\|$. Consider $z = (z_1, z_2)^T \in Z = X^\alpha \times X$, such that $\|z\|_Z = 1$. Then,

$$\|z_1\|_\alpha^2 = \sum_{j=1}^\infty \lambda_j^\alpha \|E_j z_1\|^2 \leq 1 \quad \textit{and} \quad \|z_2\|_X^2 = \sum_{j=1}^\infty \|E_j z_2\|^2 \leq 1.$$

Therefore, $\lambda_j^\alpha \|E_j z_1\| \leq 1, \quad \|E_j z_2\| \leq 1, \quad j = 1, 2, \ldots.$

Then,

$$\begin{aligned}
\|K_n^{-1}z\|^2_{X\times X} &= \frac{1}{\lambda_n^{2\alpha}|\rho_2-\rho_1|^2}\left\|\begin{bmatrix}\sigma_2(n)E_nz_1-E_nz_2\\ \sigma_1(n)E_nz_1+E_nz_2\end{bmatrix}\right\|^2_{X\times X}\\
&= \frac{1}{\lambda_n^{2\alpha}|\rho_2-\rho_1|^2}\left\{\|\sigma_2(n)E_nz_1-E_nz_2\|^2+\|\sigma_1(n)E_nz_1+E_nz_2\|^2\right\}\\
&\le \frac{1}{\lambda_n^{2\alpha}|\rho_2-\rho_1|^2}\left\{(|\rho_2(n)|\|\lambda_n^{\alpha}E_nz_1\|+\|E_nz_2\|)^2\right\}\\
&+ \frac{1}{\lambda_n^{2\alpha}|\rho_2-\rho_1|^2}\left\{(|\rho_1(n)|\|\lambda_n^{\alpha}E_nz_1\|+\|E_nz_2\|)^2\right\}\\
&\le \frac{1}{\lambda_n^{2\alpha}}\cdot\frac{(|\rho_2(n)|+1)^2+(|\rho_1(n)|+1)^2}{|\rho_2-\rho_1|^2}\\
&\le \frac{\Gamma_1^2(\eta,\gamma)}{\lambda_n^{2\alpha}}.
\end{aligned}$$

Therefore,

$$\|K_n^{-1}\|_{L(X^\alpha\times X,X\times X)}\le\frac{\Gamma_1(\eta,\gamma)}{\lambda_n^{\alpha}}. \tag{2.30}$$

Now, we will find a bound for $\|K_n\|_{L(X\times X,X^\alpha\times X)}$. *To this end we consider* $z=(z_1,z_2)^T\in Z=X\times X$, *such that* $\|z\|_Z=1$. *Then,*

$$\|z_1\|^2=\sum_{j=1}^{\infty}\lambda_j^{2\alpha}\|E_jz_1\|^2\le 1 \quad and \quad \|z_2\|^2_X=\sum_{j=1}^{\infty}\|E_jz_2\|^2\le 1.$$

Therefore, $\|E_jz_1\|\le 1,\quad \|E_jz_2\|\le 1,\quad j=1,2,\ldots.$

Then,

$$\begin{aligned}
\|K_nz\|^2_{X^\alpha\times X} &= \left\|\begin{bmatrix}E_nz_1+E_nz_2\\ \sigma_1(n)E_nz_1+\sigma_2(n)E_nz_2\end{bmatrix}\right\|^2_{X^\alpha\times X}\\
&= \lambda_n^{2\alpha}\|E_nz_1+E_nz_2\|^2+\|\sigma_1(n)E_nz_1+\sigma_2(n)E_nz_2\|^2\\
&\le \lambda_n^{2\alpha}\left\{4+(|\rho_1(n)|+|\rho_2(n)|)^2\right\}\\
&\le \Gamma_2^2(\eta,\gamma)\lambda_n^{2\alpha}.
\end{aligned}$$

Therefore,

$$\|K_n\|_{L(X\times X,X^\alpha\times X)}\le\Gamma_2(\eta,\gamma)\lambda_n^{\alpha}. \tag{2.31}$$

Now, to prove that $\mathcal{A}$ *is sectorial, we first prove that the operator*

$\mathcal{A}_\epsilon = -\mathcal{A} + \epsilon$ is sectorial for $\epsilon > 0$. With this purpose, we consider the 2×2 matrices

$$\begin{aligned} \overline{J}_{n\epsilon} &= -\overline{J}_n + \epsilon = diag\,[\lambda_n^\alpha \rho_1(n) + \epsilon, \lambda_n^\alpha \rho_2(n) + \epsilon] && (2.32)\\ &= (\lambda_n^\alpha \rho_1(n) + \epsilon) \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + (\lambda_n^\alpha \rho_2(n) + \epsilon) \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} && (2.33)\\ &= (\lambda_n^\alpha \rho_1(n) + \epsilon)\, q_1 + (\lambda_n^\alpha \rho_2(n) + \epsilon)\, q_2, && (2.34) \end{aligned}$$

and the operators $J_{n\epsilon} = \overline{J}_{n\epsilon} P_n : Z \to Z_\alpha$

Let S_θ be the following sector:

$$S_\theta = \{\lambda \in \boldsymbol{C} : \theta \le |arg(\lambda)| \le \pi, \quad \lambda \ne 0\}, \tag{2.35}$$

where

$$\sup_{n \ge 1}\{|arg(\rho_1(n))|\} < \theta < \frac{\pi}{2}.$$

If $\lambda \in S_\theta$, then λ is distinct than $\lambda_n^\alpha \rho_i(n), i = 1, 2$. Therefore,

$$\| \left(\lambda - \overline{J}_{n\epsilon}\right)^{-1} y\|^2 = \frac{1}{(\lambda - (\lambda_n^\alpha \rho_1(n) + \epsilon))^2}\|q_1 y\|^2 + \frac{1}{(\lambda - (\lambda_n^\alpha \rho_2(n) + \epsilon))^2}\|q_2 y\|^2.$$

Then, if we put

$$N = \sup\left\{\frac{|\lambda|}{|\lambda - (\lambda_n^\alpha \rho_i(n) + \epsilon)|} : \lambda \in S_\theta, \quad n \ge 1; i = 1, 2.\right\},$$

we obtain

$$\| \left(\lambda - \overline{J}_{n\epsilon}\right)^{-1} y\|^2 \le \left(\frac{N}{|\lambda|}\right)^2 \left[\|q_1 y\|^2 + \|q_2 y\|^2\right].$$

Hence,

$$\| \left(\lambda - \overline{J}_{n\epsilon}\right)^{-1} \| \le \frac{N}{|\lambda|}, \quad \lambda \in S_\theta.$$

Now, if $\lambda \in S_\theta$, then

$$\begin{aligned} \mathcal{R}(\lambda, \mathcal{A}_\epsilon) z &= \sum_{n=1}^{\infty} (\lambda + A_n - \epsilon)^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n (\lambda + J_n - \epsilon)^{-1} K_n^{-1} P_n z \\ &= \sum_{n=1}^{\infty} K_n (\lambda - J_{n\epsilon})^{-1} K_n^{-1} P_n z. \end{aligned}$$

Then,

$$\begin{aligned} \|\mathcal{R}(\lambda, \mathcal{A}_\epsilon) z\|^2 &\le \sum_{n=1}^{\infty} \|K_n\|^2 \|K_n^{-1}\|^2 \| (\lambda - J_{n\epsilon})^{-1} \|^2 \|P_n z\|^2 \\ &\le \left(\frac{\Gamma_1(\eta, \gamma)}{\Gamma_2(\eta, \gamma)}\right)^2 \left(\frac{N}{|\lambda|}\right)^2 \|z\|^2 \end{aligned}$$

Therefore,

$$\|\mathcal{R}(\lambda, \mathcal{A}_\epsilon)\| \leq \frac{R}{|\lambda|}, \quad \lambda \in S_\theta.$$

Finally, if we define the following sector

$$S_{\theta,\epsilon}\{\lambda \in \boldsymbol{C} : \theta \leq |arg(\lambda + \epsilon)| \leq \pi, \quad \lambda \neq -\epsilon\},$$

then,

$$\|\mathcal{R}(\lambda, -\mathcal{A})\| \leq \frac{R}{|\lambda + \epsilon|}, \quad \lambda \in S_{\theta,\epsilon}.$$

□

3 The Controlled System

Now, we shall give the definition of controllability in terms of systems (2.16)-(2.18). To this end, for all $z_0 \in Z_r$ ($r = \alpha$ or $r = \frac{\beta}{2}$) the equation (2.16) or (2.18) has a unique mild solution given by

$$z(t) = T(t)z_0 + \int_0^t T(t-s)Bu(s)ds, \quad 0 \leq t \leq t_1. \tag{3.36}$$

Definition 3.1 *(***Exact Controllability***) We shall say that the system (2.16) (or (2.18)) is exactly controllable on* $[0, t_1]$, $t_1 > 0$, *if for all* $z_0, z_1 \in Z_r$ *there exists a control* $u \in L^2(0, t_1; I\!R^m)$ *(or* $u \in L^2(0, t_1; X)$ *) such that the solution* $z(t)$ *of (3.36) corresponding to* u*, verifies:* $z(t_1) = z_1$.

Consider the following bounded linear operator

$$G : L^2(0, t_1; U) \to Z_r, \quad Gu = \int_0^{t_1} T(t-s)B(s)u(s)ds, \quad U = I\!R^m \text{ or } U = X. \tag{3.37}$$

Then, the following proposition is a characterization of the exact controllability of the sytem (2.16).

Proposition 3.1 *The system (2.16) (or (2.18)) is exactly controllable on* $[0, t_1]$ *if and only if, the operator* G *is surjective, that is to say*

$$GL^2(0, t_1; U) = Range(G) = Z_r.$$

Definition 3.2 *(***Approximate Controllability***) We say that (2.16) is approximately controllable in* $[0, t_1]$ *if for all* $z_0, z_1 \in Z_r$ *and* $\epsilon > 0$*, there exists a control* $u \in L^2(0, t_1; I\!R^m)$ *such that the solution* $z(t)$ *given by (3.36) satisfies*

$$\|z(t_1) - z_0\| \leq \epsilon.$$

The following theorem can be found in [5] and [6].

Theorem 3.1 *(2.16) is approximately controllable on* $[0, t_1]$ *iff*

$$B^*T^*(t)z = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow z = 0. \tag{3.38}$$

3.1 Results on Approximate Controllability

$$2\alpha \geq \beta > \alpha \ \text{ or } \ 0 < 2\alpha \leq \beta \tag{3.39}$$

we will prove the following Theorem.

Theorem 3.2 *(2.16) is approximately controllable on* $[0, t_1]$ *iff the finite dimensional systems are controllable on* $[0, t_1]$

$$y' = A_j P_j y + P_j Bu, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \ldots, \infty. \tag{3.40}$$

Proposition 3.2 *The following statements are equivalent:*

(a) system (3.40) is controllable on $[0, t_1]$,

(b) $B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow y = 0,$

(c) Rank $\left[P_j B \vdots A_j P_j B \vdots A_j^2 P_j B \vdots \cdots A_j^{2\gamma_j - 1} P_j B \right] = 2\gamma_j$

(d) the operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ *given by:*

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} B B^* e^{-A_j^* s} ds, \tag{3.41}$$

is invertible.

Lemma 3.1 *Let* $\{\alpha_j\}_{j \geq 1}$ *and* $\{\beta_{i,j} : i = 1, 2, \ldots, m\}_{j \geq 1}$ *be two sequences of complex numbers such that:* $\alpha_1 > \alpha_2 > \alpha_3 \cdots$.

Then

$$\sum_{j=1}^{\infty} e^{\alpha_j t} \beta_{i,j} = 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \cdots, m$$

iff

$$\beta_{i,j} = 0, \quad i = 1, 2, \cdots, m; j = 1, 2, \cdots, \infty.$$

Proof of Theorem 3.2- case $2\alpha \geq \beta > \alpha$**.** Suppose that each system (3.40) is controllable in $[0, t_1]$. It is easy to see that

$$B^* : Z_\alpha \to I\!R^m, \quad B^* z = (< b_1, z >, \cdots, < b_m, z >),$$

and

$$T^*(t)z = \sum_{j=1}^{\infty} e^{A_j^* t} P_j^* z, \quad z \in Z_\alpha, \quad t \geq 0.$$

Therefore,

$$B^*T^*(t)z = (< b_1, T^*(t)z >, \cdots, < b_m, T^*(t)z >).$$

Hence, system (2.16) is approximately controllable on $[0, t_1]$ iff

$$< b_i, T^*(t)z >= 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \cdots, m, \quad \Rightarrow z = 0. \tag{3.42}$$

Now, we shall check condition (3.42):

$$< b_i, T^*(t)z >= \sum_{j=1}^{\infty} < b_i, e^{A_j^* t} P_j^* z >= 0, \quad i = 1, 2, \cdots, m; \quad t \in [0, t_1].$$

Without loss of generality, we can assume that $\eta^2 - 4\gamma\lambda_1^{\beta-2\alpha} > 0$, which implies that the eigenvalues $\sigma_1(j)$ and $\sigma_2(j)$ of the 2×2 matrix B_j given by

$$\sigma_1(j) = -\lambda_j^\alpha \left(\frac{\eta + \sqrt{\eta^2 - 4\gamma\lambda_j^{\beta-2\alpha}}}{2} \right), \quad \sigma_2(j) = -\lambda_j^\alpha \left(\frac{\eta - \sqrt{\eta^2 - 4\gamma\lambda_j^{\beta-2\alpha}}}{2} \right) n = 1, 2, \ldots,$$

are real and

$$\begin{aligned} \lim_{j\to\infty} \sigma_1(j) &= -\infty, \\ \lim_{j\to\infty} \sigma_2(j) &= \frac{-1}{\eta} \lim_{j\to\infty} \lambda_j^\alpha (4\gamma\lambda_j^{\beta-2\alpha}) = \frac{-\gamma}{\eta} \lim j \to \infty \lambda_j^{\beta-\alpha} = -\infty, \quad (\beta > \alpha). \end{aligned}$$

Hence,

$$\begin{aligned} \sigma_1(1) &> \sigma_1(2) > \cdots > \sigma_1(j) > \ldots \\ \sigma_2(1) &> \sigma_2(2) > \cdots > \sigma_2(j) > \ldots. \end{aligned}$$

Since the eigenvalues of the matrix B_j are simple, there exists a complete family of complementaries projections $\{q_1(j), q_2(j)\}$ on $I\!R^2$ such that

$$e^{B_j^* t} = e^{\sigma_1(j)t} q_1(j) + e^{\sigma_2(j)t} q_2(j).$$

Therefore,

$$e^{A_j^* t} = e^{\sigma_1(j)t} P_{1,j} + e^{\sigma_2(j)t} P_{2,j}.$$

where $P_{s,j} = q_s(j) P_j = P_j q_s(j)$.

Hence,

$$\begin{aligned} < b_i, T^*(t)z >_\alpha &= \sum_{j=1}^{\infty} < b_i, e^{A_j^* t} P_j^* z >_\alpha = \sum_{j=1}^{\infty} < b_i, \sum_{s=1}^{2} e^{\sigma_s(j)t} P_{s,j}^* z >_\alpha \\ &= \sum_{j=1}^{\infty} \sum_{s=1}^{2} e^{\sigma_s(j)t} < b_i, P_{s,j}^* z >_\alpha = 0 \quad i = 1, 2, \cdots, m; \quad t \in [0, t_1]. \end{aligned}$$

Applying Lemma 3.1, we conclude that

$$< b_i, P_{s,j}^* z >_\alpha = 0 \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, \infty, \quad t \in [0, t_1].$$

Then,

$$< b_i, e^{A_j^* t} P_j^* z > = 0 \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, \infty, \quad t \in [0, t_1],$$

iff

$$B^* e^{A_j^* t} P_j^* z = 0; \quad j = 1, 2, \cdots, \infty, \quad t \in [0, t_1].$$

Since $P_j^* A_j^* = A_j^* P_j^*$ and $(P_j^*)^2 = P_j^*$, we obtain

$$(P_j B)^* e^{A_j^* t} P_j^* z = 0; \quad j = 1, 2, \cdots, \infty, \quad t \in [0, t_1].$$

From the controllability of the system (3.40), we get that $P_j^* z = 0, \quad j = 1, 2, \cdots, \infty$. Since $\{P_j^*\}_{j \geq 1}$ is a complete family of orthogonal projections on Z_α, we conclude that $z = 0$.

Conversely, assume that system (2.16) is approximately controllable on $[0, t_1]$ and there exists J such that the system

$$y' = -\lambda_J P_J A_J y + P_J B u, \quad y \in \mathcal{R}(P_J),$$

is not controllable on $[0, t_1]$. Then, there exists $V_J \in \mathcal{R}(P_J)$ such that

$$(P_J B)^* e^{A_j^* t} V_J = 0, \quad t \in [0, t_1] \quad \text{and} \quad V_J \neq 0.$$

Letting $z = P_J^* V_J$, we obtain

$$\begin{aligned} B^* T^*(t) z &= (< b_1, T^*(t) z >, \cdots, < b_m, T^*(t) z >) \\ &= (< b_1, e^{A_j^* t} V_J >, \cdots, < b_m, e^{A_j^* t} V_J >) \\ &= B^* e^{A_J^* t} V_J = (P_J B)^* e^{A_J^* t} V_J = 0, \end{aligned}$$

which contradicts the assumption. □

Proof of Theorem 3.2- case $2\alpha \leq \beta > \alpha$. Suppose that each system (3.40) is controllable in $[0, t_1]$. It is easy to see that

$$B^* : Z_{\beta/2} \to I\!R^m, \quad B^* z = (< b_1, z >, \cdots, < b_m, z >),$$

and

$$T^*(t)z = \sum_{j=1}^{\infty} e^{A_j^* t} P_j^* z, \quad z \in Z_\alpha, \quad t \geq 0.$$

Therefore,

$$B^*T^*(t)z = (< b_1, T^*(t)z >, \cdots, < b_m, T^*(t)z >).$$

Hence, system (2.16) is approximately controllable on $[0, t_1]$ iff

$$< b_i, T^*(t)z >= 0, \quad \forall t \in [0, t_1], \quad i = 1, 2, \cdots, m, \quad \Rightarrow z = 0. \tag{3.43}$$

Now, we shall check condition (3.43):

$$< b_i, T^*(t)z >= \sum_{j=1}^{\infty} < b_i, e^{A_j^* t} P_j^* z >= 0, \quad i = 1, 2, \cdots, m; \quad t \in [0, t_1].$$

Without loss of generality, we can assume that $\eta^2 - 4\gamma\lambda_1^{\beta-2\alpha} < 0$, which implies that the eigenvalues $\sigma_1(j)$ and $\sigma_2(j)$ of the 2×2 matrix B_j given by

$$\sigma_1(j) = r_j + il_j \ \text{ and } \ \sigma_2(j) = r_j - il_j$$

where

$$r_j = -\lambda_j^\alpha \frac{\eta}{2} \ \text{ and } \ l_j = \lambda_j^\alpha \frac{\sqrt{\eta^2 - 4\gamma\lambda_j^{\beta-2\alpha}}}{2}, j = 1, 2, \ldots.$$

Hence,

$$r_1 > r_2 > r_3 > \cdots - \infty.$$

Since the eigenvalues of the matrix B_j are simple, there exists a complete family of complementaries projections $\{q_1(j), q_2(j)\}$ on $I\!R^2$ such that

$$e^{B_j^* t} = e^{\sigma_1(j)t} q_1(j) + e^{\sigma_2(j)t} q_2(j).$$

Therefore,

$$e^{A_j^* t} = e^{r_j t - il_j t} P_{1,j} + e^{r_j t + il_j t} P_{2,j}.$$

where $P_{s,j} = q_s(j)P_j = P_j q_s(j)$.

Hence,

$$\begin{aligned} < b_i, T^*(t)z >_\alpha &= \sum_{j=1}^{\infty} < b_i, e^{A_j^* t} P_j^* z >_\alpha = \sum_{j=1}^{\infty} < b_i, e^{r_j t - il_j t} P_{1,j}^* z + e^{r_j t + il_j t} P_{2,j}^* z >_\alpha \\ &= \sum_{j=1}^{\infty} e^{r_j t} \left\{ e^{-il_j t} < b_i, P_{1,j}^* z > + e^{il_j t} < b_i, P_{2,j}^* z > \right\} = 0, \\ &= i = 1, 2, \cdots, m; \quad t \in [0, t_1]. \end{aligned}$$

Applying Lemma 3.1, we conclude that

$$e^{-il_jt} < b_i, P_{1,j}^* z > + e^{il_jt} < b_i, P_{2,j}^* z >= 0 \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, \infty, \quad t \in [0, t_1].$$

Since the two funcitions e^{il_jt}, e^{-il_jt} are linearly independent we obtain that

$$< b_i, P_{1,j}^* z >=< b_i, P_{2,j}^* z >= 0$$

Then,

$$< b_i, e^{A_j^* t} P_j^* z >= 0 \quad i = 1, 2, \cdots, m; \quad j = 1, 2, \cdots, \infty, \quad t \in [0, t_1].$$

From here, the proof follows in the same way as the foregoing case.

□

Theorem 3.3 *If* $< d_i, \phi_{jk} >\neq 0, \quad j = 1, 2, \ldots, \infty, \quad i = 1, 2, \ldots, m, \quad k = 1, 2, \ldots, \gamma_j$, *then system (2.16) is approximately controllable on* $[0, t_1]$.

Proof From the foregoing Theorem, it is enough to prove the controllability of the family of finite dimensional system (3.40). In order to check the algebraic condition (1.4) we have to find the matrix representation of the operators:

$$A_j P_j : \mathcal{R}(P_j) \to \mathcal{R}(P_j), \quad P_j B : I\!R^m \to \mathcal{R}(P_j).$$

To this end, we shall consider the canonical base $\mathcal{B} = \{e_1, e_2, \ldots, e_m, \}$ in $I\!R^m$ and the following base in $\mathcal{R}(P_j)$

$$\mathcal{B}_j = \{\phi_{jl}^1, \phi_{jl}^2 : l = 1, 2, \ldots, \gamma_j\},$$

where

$$\phi_{jl}^1 = \begin{bmatrix} \phi_{jl} \\ 0 \end{bmatrix}, \quad \phi_{jl}^1 = \begin{bmatrix} 0 \\ \phi_{jl} \end{bmatrix},$$

and for all $x \in X$ we have that

$$E_j x = \sum_{k=1}^{\gamma_j} < x, \phi_{j,k} > \phi_{j,k}.$$

Therefore,

$$A_j P_j \phi_{jl}^1 = -\gamma \lambda_j^\beta \phi_{jl}^2, \quad A_j P_j \phi_{jl}^2 = \phi_{jl}^1 - \eta \lambda_j^\alpha \phi_{jl}^2, \quad l = 1, 2, \ldots, \gamma_j,$$

and

$$P_j B e_l = \sum_{k=1}^{\gamma_j} < d_l, \phi_{j,k} > \phi_{j,k}^2.$$

Therefore,

$$A_jP_j = \begin{bmatrix} 0 & 0 & \dots & \dots & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & \dots & 0 & 0 & 1 & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & 0 & 0 & 0 & \dots & \dots & 1 \\ -\gamma\lambda_j^{\beta} & 0 & \dots & \dots & 0 & -\eta\lambda_j^{\alpha} & 0 & \dots & \dots & 0 \\ 0 & -\gamma\lambda_j^{\beta} & \dots & \dots & 0 & & -\eta\lambda_j^{\alpha} & \dots & \dots & 0 \\ \vdots & \vdots & & & \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & \dots & \dots & -\gamma\lambda_j^{\beta} & 0 & 0 & \dots & \dots & -\gamma\lambda_j^{\beta} \end{bmatrix} \tag{3.44}$$

i.e.,

$$A_jP_j = \begin{bmatrix} O_{\gamma_j\times\gamma_j} & \vdots & I_{\gamma_j\times\gamma_j} \\ \dots\dots & \dots\dots & \dots\dots \\ -\gamma\lambda_j^{\beta}I_{\gamma_j\times\gamma_j} & \vdots & -\eta\lambda_j^{\alpha}I_{\gamma_j\times\gamma_j} \end{bmatrix}_{2\gamma_j\times 2\gamma_j}, \tag{3.45}$$

and

$$P_jB = \begin{bmatrix} 0 & 0 & \dots\dots & 0 \\ 0 & 0 & \dots\dots & 0 \\ \vdots & \vdots & \vdots\vdots & \vdots \\ 0 & 0 & \dots\dots & 0 \\ < d_1, \phi_{j1} > & < d_2, \phi_{j1} > & \dots\dots & < d_m, \phi_{j1} > \\ < d_1, \phi_{j2} > & < d_2, \phi_{j2} > & \dots\dots & < d_m, \phi_{j2} > \\ \vdots & \vdots & \vdots\vdots & \vdots \\ < d_1, \phi_{j\gamma_j} > & < d_2, \phi_{j\gamma_j} > & \dots\dots & < d_m, \phi_{j\gamma_j} > \end{bmatrix}_{2\gamma_j\times m}. \tag{3.46}$$

From here we can check the algebraic condition given by proposition 3.1 part (c). □.

As a special case we can consider the scalar strongly damped wave equation with a single control

$$\begin{cases} w_{tt} + \eta(-\Delta)^{1/2}w_t + \gamma(-\Delta)w = b(x)u, & t \geq 0, \quad 0 \leq x \leq 1, \\ w(t,1) = w(t,0) = 0, \quad t \geq 0, \quad 0 \leq x \leq 1, \end{cases} \tag{3.47}$$

In this case $\lambda_j = -j^2\pi^2$ and $\phi_j(x) = \sin j\pi x$. Therefore, the equation (3.47) is approximately controllable iff

$$\text{Rank}[P_jB \vdots A_jP_jB] = \text{Rank}\begin{bmatrix} 0 & < b, \phi_j > \\ < b, \phi_j > & -\eta\lambda_j^{1/2} < b, \phi_j > \end{bmatrix} = 2, \quad j = 1, 2, \dots, \infty.$$

Which is equivalent to:

$$< b, \phi_j > = \int_0^1 b(x)\sin j\pi x dx \neq 0, \quad j = 1, 2, \dots, \infty.$$

3.2 Results on Exact Controllability

Now, we are ready to formulate the main result about exact controllability of the system (2.18).

Theorem 3.4 *The system (2.18) is exactly controllable on* $[0, t_1]$.

Moreover, the control $u \in L^2(0, t_1; X)$ *steering an initial state* z_0 *to a final state* z_1 *in time* $t_1 > 0$ *is given by the following formula:*

$$u(t) = B^* T^*(-t) \sum_{j=1}^{\infty} W_j^{-1}(t_1) P_j (T(-t_1) z_1 - z_0). \tag{3.48}$$

Proof .

$$G : L^2(0, t_1; X) \to Z_{\beta/2}, \quad Gu = \int_0^{t_1} T(-s) B(s) u(s) ds. \tag{3.49}$$

$$GL^2(0, t_1; X) = \text{Range}(G) = Z_{\beta/2}?.$$

First, we shall prove that each of the following finite dimensional systems is controllable on $[0, t_1]$

$$y' = A_j P_j y + P_j B u, \quad y \in \mathcal{R}(P_j); \quad j = 1, 2, \ldots, \infty. \tag{3.50}$$

$$B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1], \quad \Rightarrow y = 0?.$$

In this case the operators $A_j = B_j P_j$ and $\mathcal{A}$ are given by

$$B_j = \begin{bmatrix} 0 & 1 \\ -\gamma \lambda_j^\beta & -\eta \end{bmatrix}, \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -\gamma A^\beta & -\eta I \end{bmatrix},$$

and the eigenvalues $\sigma_1(j)$, $\sigma_2(j)$ of the matrix B_j are given by

$$\sigma_1(j) = -c + i l_j, \quad \sigma_2(j) = -c - i l_j,$$

where,

$$c = \frac{\eta}{2} \quad \text{and} \quad l_j = \frac{1}{2} \sqrt{4 \gamma \lambda_j^\beta - \eta^2}.$$

Therefore, $A_j^* = B_j^* P_j$ with

$$B_j^* = \begin{bmatrix} 0 & -1 \\ \gamma \lambda_j^\beta & -\eta \end{bmatrix},$$

and

$$\begin{aligned} e^{B_j t} &= e^{-ct} \left\{ \cos l_j t I + \frac{1}{l_j} (B_j + cI) \right\} \\ &= e^{-ct} \begin{bmatrix} \cos l_j t + \frac{\eta}{2 l_j} \sin l_j t & \frac{\sin l_j t}{l_j} \\ -\gamma S(j) \lambda_j^{\beta/2} \sin l_j t & \cos l_j t - \frac{\eta}{2 l_j} \sin l_j t \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} e^{B_j^* t} &= e^{-ct}\left\{\cos l_j t I + \frac{1}{l_j}\left(B_j^* + cI\right)\right\} \\ &= e^{-ct}\left[\begin{array}{cc} \cos l_j t + \frac{\eta}{2l_j}\sin l_j t & -\frac{\sin l_j t}{l_j} \\ \gamma S(j)\lambda_j^{\beta/2}\sin l_j t & \cos l_j t - \frac{\eta}{2l_j}\sin l_j t \end{array}\right], \end{aligned}$$

$$B = \left[\begin{array}{c} 0 \\ I_X \end{array}\right], \quad B^* = [0, I_X] \text{ and } BB^* = \left[\begin{array}{cc} 0 & 0 \\ 0 & I_X \end{array}\right].$$

Now, let $y = (y_1, y_2)^T \in \mathcal{R}(P_j)$ such that

$$B^* P_j^* e^{A_j^* t} y = 0, \quad \forall t \in [0, t_1].$$

Then,

$$e^{-ct}\left[\begin{array}{c} \gamma S(j)\lambda_j^{\beta/2}\sin l_j t y_1 \\ \left(\cos l_j t - \frac{\eta}{2l_j}\sin l_j t\right) y_2 \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right], \quad \forall t \in [0, t_1],$$

which implies that $y = 0$.

From Proposition 3.2 the operator $W_j(t_1) : \mathcal{R}(P_j) \to \mathcal{R}(P_j)$ given by:

$$W_j(t_1) = \int_0^{t_1} e^{-A_j s} BB^* e^{-A_j^* s} ds = P_j \int_0^{t_1} e^{-B_j s} BB^* e^{-B_j^* s} ds P_j = P_j \overline{W}_j(t_1) P_j$$

is invertible.

Since

$$\|e^{-A_j t}\| \leq M(\eta, \gamma) e^{ct}, \quad \|e^{-A_j^* t}\| \leq M(\eta, \gamma) e^{ct},$$
$$\|e^{-A_j t} BB^* e^{-A_j^* t}\| \leq M^2(\eta, \gamma) \|BB^*\| e^{2ct},$$

then

$$\|W_j(t_1)\| \leq M^2(\eta, \gamma) \|BB^*\| e^{2ct_1} \leq L(\eta, \gamma), \quad j = 1, 2, \ldots.$$

Now, we shall prove that the following family of linear operators

$$W_j^{-1}(t_1) = \overline{W}_j^{-1}(t_1) P_j : Z_{\beta/2} \to Z_{\beta/2}$$

is bounded and $\|W_j^{-1}(t_1)\|$ is uniformly bounded. To this end we shall compute explicity the matrix $\overline{W}_j^{-1}(t_1)$. From the above formulas we obtain that

$$e^{B_j t} = e^{-ct}\left[\begin{array}{cc} a(j) & b(j) \\ -a(j) & c(j) \end{array}\right], \quad e^{B_j^* t} = e^{-ct}\left[\begin{array}{cc} a(j) & -b(j) \\ d(j) & c(j) \end{array}\right],$$

where

$$a(j) = \cos l_j t + \frac{\eta}{2l_j}\sin l_j t, \quad b(j) = \frac{\sin l_j t}{l_j},$$

$$c(j) = \gamma S(j)\lambda_j^{\beta/2}\sin l_j t, \quad d(j) = \cos l_j t - \frac{\eta}{2l_j}\sin l_j t,$$

and

$$S(j) = \sqrt{\frac{\lambda_j^\beta}{4\gamma\lambda_j^\beta - \eta^2}}.$$

Then

$$e^{-B_j s}BB^* e^{-B_j^* s} = \begin{bmatrix} b(j)c(j)\lambda_j^{\beta/2}I & -b(j)d(j)I \\ -d(j)c(j)\lambda_j^{\beta/2}I & d^2(j)I \end{bmatrix}.$$

Therefore,

$$\overline{W}_j(t_1) = \begin{bmatrix} \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}k_{11}(j) & \frac{1}{l_j}k_{12}(j) \\ -\gamma S(j)\lambda_j^{\beta/2}k_{21}(j) & k_{22}(j) \end{bmatrix},$$

where

$$\begin{aligned}
k_{11}(j) &= \int_0^{t_1} e^{2cs}\sin^2 l_j s ds \\
k_{12}(j) &= -\int_0^{t_1} e^{2cs}\left[\sin l_j s\cos l_j s - \frac{\eta\sin^2 l_j s}{2l_j}\right] ds \\
k_{21}(j) &= \int_0^{t_1} e^{2cs}\left[\sin l_j s\cos l_j s - \frac{\eta\sin^2 l_j s}{2l_j}\right] ds \\
k_{22}(j) &= \int_0^{t_1} e^{2cs}\left[\cos l_j s - \frac{\eta\sin l_j s}{2l_j}\right]^2 ds.
\end{aligned}$$

The determinant $\Delta(j)$ of the matrix $\overline{W}_j(t_1)$ is given by

$$\begin{aligned}
\Delta(j) &= \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}\left[k_{11}(j)k_{22}(j) - k_{12}(j)k_{21}(j)\right] \\
&= \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}\{\left(\int_0^{t_1} e^{2cs}\sin^2 l_j s ds\right)\left(\int_0^{t_1} e^{2cs}\left[\cos l_j s - \frac{\eta\sin l_j s}{2l_j}\right]^2 ds\right) \\
&- \left(\int_0^{t_1} e^{2cs}\left[\sin l_j s\cos l_j s - \frac{\eta\sin^2 l_j s}{2l_j}\right] ds\right)^2\}.
\end{aligned}$$

Passing to the limit when j goes to ∞ we obtain that

$$\lim_{j\to\infty}\Delta(j) = \frac{(e^{2ct_1}-1)(1-2e^{ct_1}+e^{2ct_1})}{2^4c^3}.$$

Therefore, there exist constants $R_1, R_2 > 0$ such that

$$0 < R_1 < |\Delta(j)| < R_2, \quad j = 1, 2, 3, \ldots$$

Hence,

$$\begin{aligned}\overline{W}^{-1}(j) &= \frac{1}{\Delta(j)}\begin{bmatrix} k_{22}(j) & -\frac{1}{l_j}k_{12}(j) \\ \gamma S(j)\lambda_j^{\beta/2}k_{21}(j) & \frac{\gamma S(j)\lambda_j^{\beta/2}}{l_j}k_{11}(j)\end{bmatrix} \\ &= \begin{bmatrix} b_{11}(j) & b_{12}(j) \\ b_{21}(j)\lambda_j^{\beta/2} & b_{22}(j)\end{bmatrix},\end{aligned}$$

where $b_{n,m}(j), \quad n = 1,2; m = 1,2; j = 1,2,\ldots$ are bounded. From here using the same computation as in Theorem 2.1 we can prove the existence of constant $L_2(\eta,\gamma)$ such that

$$\|W_j^{-1}(t_1)\|_{Z_{\beta/2}} \leq L_2(\eta,\gamma), \quad j = 1,2,\ldots.$$

Now, we define the following linear and bounded operators

$$W(t_1) : Z_{\beta/2} \to Z_{\beta/2}, \quad W^{-1}(t_1) : Z_{\beta/2} \to Z_{\beta/2},$$

by

$$W(t_1)z = \sum_{j=1}^{\infty} W_j(t_1)P_j z, \quad W^{-1}(t_1)z = \sum_{j=1}^{\infty} W_j^{-1}(t_1)P_j z.$$

Therefore, $W(t_1)W^{-1}(t_1)z = z$ and

$$W(t_1)z = \int_0^{t_1} T(-s)BB^*T^*(-s)zds.$$

Finally, we will show that given $z \in Z_{\beta/2}$ there exists a control $u \in L^2(0,t_1;X)$ such that $Gu = z$. In fact, let u be the following control

$$u(t) = B^*T^*(-t)W^{-1}(t_1)z, \quad t \in [0,t_1].$$

Then,

$$\begin{aligned} Gu &= \int_0^{t_1} T(-s)Bu(s)ds = \int_0^{t_1} T(-s)BB^*T^*(-s)W^{-1}(t_1)zds \\ &= \left(\int_0^{t_1} T(-s)BB^*T^*(-s)ds\right)W^{-1}(t_1)z = W(t_1)W^{-1}(t_1)z = z.\end{aligned}$$

Then, the control steering an initial state z_0 to a final state z_1 in time $t_1 > 0$ is given by

$$u(t) = B^*T^*(-t)W^{-1}(t_1)(T(-t_1)z_1 - z_0) = B^*T^*(-t)\sum_{j=1}^{\infty} W_j^{-1}(t_1)P_j(T(-t_1)z_1 - z_0).$$

□

4 Appendix: Some Results About C_0-Semigroups

In this section we prove a lemma that characterizes a very large class of C_0-semigroup appearing in many systems of partial differential equations, like reaction diffusion systems, second order systems wih dissipation, thermoelastic plate equations, beam equations, damped vibration of the string and others systems of partial differential equations. These first Lemma can be found in H. Leiva [14].

Definition 4.1 A family $\{T(t)\}_{t\geq 0}$ of bounded linear operators mapping the Banach space Z in to Z is called a C_0-semigroup if the following three conditions are satisfied:

(i) $T(t+s) = T(t)T(s), \quad t, s \geq 0;$

(ii) $T(0) = I \quad (I$ is the identity operator in $Z)$;

(iii) for each $z \in Z$, we have that

$$\lim_{h\to 0^+} \|T(h)z - z\| = 0.$$

Definition 4.2 (The infinitesimal generator) Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup in Z. Then the operator $\mathcal{A} : D(\mathcal{A}) \subset Z \to Z$ defined by the limit

$$\mathcal{A}z = \lim_{h\to 0^+} \frac{T(h)z - z}{h}, \quad z \in D(A), \tag{4.51}$$

where

$$D(\mathcal{A}) = \{z \in Z : \lim_{h\to 0^+} \frac{T(h)z - z}{h} \ \text{ exists}\} \tag{4.52}$$

is called the infinitesimal generator or simply the generator of the semigroup $\{T(t)\}_{t\geq 0}$.

The following theorem characterizes the fundamental properties of the infinitesimal generator of a C_0-semigroup.

Theorem 4.1 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup in Z and $\mathcal{A}$ its infinitesimal generator with domain $D(\mathcal{A})$. Then*

(a) $D(\mathcal{A})$ is a linear subspace in Z and $\mathcal{A}$ on $D(\mathcal{A})$ is a linear operator;

(b) if $z \in D(\mathcal{A})$, then $T(t)z \in D(\mathcal{A}), \quad t \geq 0$ is differentiable in t and

$$\frac{d}{dt}T(t)z = \mathcal{A}T(t)z = T(t)\mathcal{A}z, \quad t \geq 0; \tag{4.53}$$

(c) if $z \in D(\mathcal{A})$, then

$$T(t)z - T(s)z = \int_s^t T(u)\mathcal{A}zdu, \qquad t, s \geq 0; \tag{4.54}$$

(d) the linear subspace $D(\mathcal{A})$ is dense in Z, and A on $D(A)$ is a closed operator.

Theorem 4.2 *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup in Z and $\mathcal{A}$ its infinitesimal generator with domain $D(\mathcal{A})$. Then the Cauchy problem*

$$\begin{cases} z'(t) = & \mathcal{A}z(t), \quad t > 0 \\ z(0) = & z_0, \quad z_0 \in D(\mathcal{A}) \end{cases} \tag{4.55}$$

has the unique solution

$$z(t) = T(t)z_0 \tag{4.56}$$

Definition 4.3 (Analytic semigroup) A C_0-semigroup $\{T(t)\}_{t\geq 0}$ on Z is called analytic if for all $z \in Z$ the function $t \to T(t)z \in D(\mathcal{A})$ is real analytic on $0 < t < \infty$ and

$$\frac{d}{dt}T(t)z = \mathcal{A}T(t)z = T(t)\mathcal{A}z, \quad t \geq 0. \tag{4.57}$$

Therefore, $T(t)z \in D(\mathcal{A})$ for $t > 0$ and $z \in Z$.

Lemma 4.1 *Let Z be a separable Hilbert space and $\{A_n\}_{n\geq 1}$, $\{P_n\}_{n\geq 1}$ two families of bounded linear operators in Z with $\{P_n\}_{n\geq 1}$ being a complete family of orthogonal projections such that*

$$A_nP_n = P_nA_n, \quad n = 1, 2, 3, \ldots \tag{4.58}$$

Define the following family of linear operators

$$T(t)z = \sum_{n=1}^{\infty} e^{A_nt}P_nz, \qquad t \geq 0. \tag{4.59}$$

Then:

(a) $T(t)$ is a linear bounded operator if

$$\|e^{A_nt}\| \leq g(t), \quad n = 1, 2, 3, \ldots \tag{4.60}$$

for some continuous real-valued function $g(t)$.

(b) under the condition (4.60) $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup in the Hilbert space Z whose infinitesimal generator $\mathcal{A}$ is given by

$$\mathcal{A}z = \sum_{n=1}^{\infty} A_nP_nz, \qquad z \in D(\mathcal{A}) \tag{4.61}$$

with

$$D(\mathcal{A}) = \{z \in Z : \sum_{n=1}^{\infty} \|A_n P_n z\|^2 < \infty\} \tag{4.62}$$

(c) the spectrum $\sigma(\mathcal{A})$ *of* $\mathcal{A}$ *is given by*

$$\sigma(\mathcal{A}) = \overline{\bigcup_{n=1}^{\infty} \sigma(\bar{A}_n)}, \tag{4.63}$$

where $\bar{A}_n = A_n P_n$.

Proof . (a) Since $A_n P_n = P_n A_n$, then for all $z \in Z$, $\{e^{A_n t} P_n z\}_{n=1}^{\infty}$ is an orthogonal family of vectors in Z. Therefore,

$$\|T(t)z\|^2 = \sum_{n=1}^{\infty} \|e^{A_n t} P_n z\|^2 \leq (g(t)\|z\|)^2.$$

So,

$$\|T(t)z\| \leq g(t)\|z\|.$$

(b) we first check condition (i) from definition (4.1)

$$\begin{aligned} T(t)T(s)z &= \sum_{n=1}^{\infty} e^{A_n t} P_n T(s) z \\ &= \sum_{n=1}^{\infty} e^{A_n t} P_n (\sum_{m=1}^{\infty} e^{A_m s} P_m z) \\ &= \sum_{n=1}^{\infty} e^{A_n (t+s)} P_n z = T(t+s)z. \end{aligned}$$

Condition (ii) from definition (4.1) follows from the completness of the family $\{P_n\}_{n \geq 1}$. That is: $z = \sum_{n=1}^{\infty} P_n z, \quad z \in Z$.

Let us check condition (iii) of definition (4.1).

$$\begin{aligned} \|T(t)z - z\|^2 &\leq \sum_{n=1}^{\infty} \|e^{A_n t} - I\|^2 \|P_n z\|^2 \\ &= \sum_{n=1}^{N} \|e^{A_n t} - I\|^2 \|P_n z\|^2 + \sum_{n=N+1}^{\infty} \|e^{A_n t} - I\|^2 \|P_n z\|^2 \end{aligned}$$

From (4.60) there exists a continuous function $k(t)$ such that

$$\|T(t)z - z\|^2 \leq \sup_{n=1,2,\ldots,N} \|e^{A_n t} - I\|^2 \sum_{n=1}^{N} \|P_n z\|^2 + k(t) \sum_{n=N+1}^{\infty} \|P_n z\|^2$$

Given $\epsilon > 0$ we can find N large enough such that

$$k(t)\sum_{n=N+1}^{\infty}\|P_n z\|^2 < \epsilon$$

for $t \in [0,\delta], \quad \delta > 0$. On the other hand, $\lim_{t\to 0^+}\sup_{n=1,2,\ldots,N}\|e^{A_n t} - I\| = 0$.

Hence, $\quad \lim_{t\to 0^+}\|T(t)z - z\| = 0$.

Let $\mathcal{A}$ be the infinitesimal generator of this semigroup. Then from definition 4.2, we have for all $z \in D(\mathcal{A})$

$$\mathcal{A}z = \lim_{t\to 0^+}\frac{T(t)z - z}{t} = \lim_{t\to 0^+}\sum_{n=1}^{\infty}\frac{(e^{A_n t} - I)}{t}P_n z.$$

Therefore,

$$\begin{aligned} P_m\mathcal{A}z &= P_m\left(\lim_{t\to 0^+}\sum_{n=1}^{\infty}\frac{\left(e^{A_n t} - I\right)}{t}P_n z\right) \\ &= \lim_{t\to 0^+}\frac{\left(e^{A_m t} - I\right)}{t}P_m z = A_m P_m z. \end{aligned}$$

Hence,

$$\mathcal{A}z = \sum_{n=1}^{\infty}P_n\mathcal{A}z = \sum_{n=1}^{\infty}A_n P_n z,$$

and

$$D(\mathcal{A}) \subset \{z \in Z : \sum_{n=1}^{\infty}\|A_n P_n z\|^2 < \infty\}.$$

Now, suppose $z \in \{z \in Z : \sum_{k=1}^{\infty}\|A_k P_k z\|^2 < \infty\}$. Then $\sum_{k=1}^{\infty}\|A_k P_k z\|^2 < \infty$ and $y = \sum_{k=1}^{\infty}A_k P_k z \in Z$.

Next, if we put $z_n = \sum_{k=1}^{n}P_k z$, then $z_n \in D(\mathcal{A})$ and $\mathcal{A}z_n = \sum_{k=1}^{n}A_k P_k z$.

Hence, $\lim_{n\to\infty} z_n = z$ and $\lim_{n\to\infty}\mathcal{A}z_n = y$, and since $\mathcal{A}$ is a closed linear operator we get that $z \in D(\mathcal{A})$ and $\mathcal{A}z = y$.

Proof of (c). It is equivalent to prove the following

$$\rho(\mathcal{A}) = \bigcap_{n=1}^{\infty}\rho(\bar{A}_n).$$

It is clear that $\bigcap_{n=1}^{\infty}\rho(\bar{A}_n) \subset \rho(\mathcal{A})$. We shall prove that $\rho(\mathcal{A}) \subset \bigcap_{n=1}^{\infty}\rho(\bar{A}_n)$. In fact, let λ be in $\rho(\mathcal{A})$. Then $(\lambda - \mathcal{A})^{-1} : Z \to D(\mathcal{A})$ is a bounded linear operator. We need to prove that

$$(\lambda - \bar{A}_m)^{-1} : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$$

exists and is bounded for $m \geq 1$. Suppose that $(\lambda - \bar{A}_m)^{-1}P_m z = 0$. Then

$$\begin{aligned}(\lambda - \mathcal{A})P_m z &= \sum_{n=1}^{\infty}(\lambda - A_n)P_n P_m z \\ &= (\lambda - A_m)P_m z = (\lambda - \bar{A}_m)P_m z = 0.\end{aligned}$$

Which implies that, $P_m z = 0$. So, $(\lambda - \bar{A}_m)$ is one to one.

Now, given y in $\mathcal{R}(P_m)$ we want to solve the equation $(\lambda - \bar{A}_m)w = y$. In fact, since $\lambda \in \rho(\mathcal{A})$ there exists $z \in Z$ such that

$$(\lambda - \mathcal{A})z = \sum_{n=1}^{\infty}(\lambda - A_n)P_n z = y.$$

Then, applying P_m to the both side of this equation we obtain

$$P_m(\lambda - \mathcal{A})z = (\lambda - A_m)P_m z = (\lambda - \bar{A}_m)P_m z = P_m y = y.$$

Therefore, $(\lambda - \bar{A}_m) : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$ is a bijection, and since $\mathcal{R}(P_m)$ is a closed, it is a Banach space. So, we can invoke the Open Mapping Theorem to coclude that $(\lambda - \bar{A}_m) : \mathcal{R}(P_m) \to \mathcal{R}(P_m)$ exists and is a bounded linear operator. Hence, $\lambda \in \rho(\bar{A}_m)$ for all $m \geq 1$. We have proved that

$$\rho(\mathcal{A}) \subset \bigcap_{n=1}^{\infty} \rho(\bar{A}_n) \iff \bigcup_{n=1}^{\infty} \sigma(\bar{A}_n) \subset \sigma(\mathcal{A}).$$

□

Lemma 4.2 *Suppose the conditions of Lemma 4.1 holds and S is a bounded subset of $\boldsymbol{C}$ with $Re(S) > 0$ such that*

$$-\frac{1}{\lambda_n}\sigma(A_n) \subset S, \quad \lambda_n > 0 \quad for \quad n = 1, 2, \ldots,$$

Then, the operator $\mathcal{A}$ given by (4.61) generates an analytic C_0-semigroup.

Proof If we put $D_n = -\frac{1}{\lambda_n}A_n$, then $A_n = -\lambda_n D_n$, $\sigma(D_n) \subset S$ and the operator $\mathcal{A}$ can be written as follows

$$-\mathcal{A}z = \sum_{n=1}^{\infty} \lambda_n D_n P_n z, \quad z \in D(\mathcal{A})$$

From Theorem 3.2 it is enough to prove the operator $-\mathcal{A}$ is sectorial. In fact, let $\theta \in (0, \pi/2)$ such that for any $\lambda \in \sigma(S)$ we have that $|arg\lambda| < \theta$.

We shall prove the sector

$$S_\theta = \{\lambda \in \boldsymbol{C} : \theta \leq |arg\lambda| \leq \pi, \quad \lambda \neq 0\}$$

is in the resolvent set of $-\mathcal{A}$ and there exists a constant M such that

$$\|(\lambda + \mathcal{A})^{-1}\| \leq \frac{M}{|\lambda|}, \quad \lambda \in S_\theta. \tag{4.64}$$

Since $\lambda \in S_\theta$., then $\frac{\lambda}{\lambda_n}$ is not in $\sigma(D_n)$ for all $n \geq 1$ and the operator $\lambda - \lambda_n D_n$ is invertible. Moreover, we shall prove the existence of constant $M > 0$ such that

$$\|(\lambda - \lambda_n D_n)^{-1}\| \leq \frac{M}{|\lambda|}, \quad n = 1, 2, \ldots.$$

In fact, for such λ we have the following estimate

$$\begin{aligned}
\|\mathcal{R}(\lambda, D_n)\| &= \|(\lambda - D_n)^{-1}\| = \|(\lambda - I)^{-1}\left\{I - (D_n - I)(\lambda - I)^{-1}\right\}^{-1}\| \\
&\leq \frac{1}{|\lambda - 1|}\|\left\{I - (D_n - I)(\lambda - I)^{-1}\right\}^{-1}\| \\
&\leq \frac{1}{|\lambda - 1|}\left\{1 - \frac{\|D_n - I\|}{|\lambda - 1|}\right\}^{-1} \\
&\leq \frac{C(\|D_n\|)}{|\lambda|},
\end{aligned}$$

if $|\lambda|$ is sufficiently large.

On the other hand, we have that

$$\|D_n\| = \sqrt{r(D_n D_n^*)} = \sqrt{\sup\{\lambda : \lambda \in \sigma(D_n D_n^*)\}} \leq K, \quad n = 1, 2, \ldots.$$

where $r(D_n D_n^*)$ denotes the spectral radius of $D_n D_n^*$. From here, we obtain the existence of M such that

$$\|(\lambda - D_n)^{-1}\| \leq \frac{M}{|\lambda|}, \quad \lambda \in S_\theta, \quad n \geq 1.$$

Hence,

$$\begin{aligned}
\|(\lambda - \lambda_n D_n)^{-1}\| &= \frac{1}{|\lambda_n|}\|(\frac{\lambda}{\lambda_n} - D_n)^{-1}\| \\
&\leq \frac{1}{|\lambda_n|}\frac{M}{\frac{|\lambda|}{|\lambda_n|}} = \frac{M}{|\lambda|}, \quad \lambda \in S_\theta, \quad n \geq 1.
\end{aligned}$$

Now, we consider the equation

$$\lambda z + \mathcal{A} z = y, \quad z \in D(\mathcal{A}), \quad y \in Z$$

If $y = \sum_{n=1}^{\infty} P_n y$, then the foregoing equation is equivalent to

$$\sum_{n=1}^{\infty}(\lambda - \lambda_n D_n)P_n z = \sum_{n=1}^{\infty} P_n y$$

i.e.,

$$(\lambda - \lambda_n D_n) P_n z = P_n y \iff P_n z = (\lambda - \lambda_n D_n)^{-1} P_n y, \quad n = 1, 2, \ldots$$

Therefore, $z = \sum_{n=1}^{\infty} (\lambda - \lambda_n D_n)^{-1} P_n y$ is well defined and $(\lambda + \mathcal{A})^{-1}$ is a bounded linear operator. So, λ is in the resolvent set of $-\mathcal{A}$ for all λ in the sector S_θ, and (4.64) holds. □

Corollary 4.1 *Suppose the conditions of Lemma 4.1 holds and* $\sigma(A_n) = -\lambda_n \sigma(D_n)$, $D_n \in L(\mathcal{R}(P_n))$, $\sigma(D_n) \subset S$ *for* $n = 1, 2, \ldots$, *where* S *is a bounded subset of* $\boldsymbol{C}$ *with* $Re(S) > 0$ *and*

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n \to \infty.$$

Then, the operator $\mathcal{A}$ *given by (4.61) generates an analytic* C_0*-semigroup.*

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HUGO LEIVA
Departamento de Matemáticas,Facultad de Ciencias,
Universidad de Los Andes
Mérida 5101, Venezuela
e-mail: hleiva@ula.ve