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- Unitary operators in Banach spaces, Pac. Journal of Mathematics, 22 (1967), 467 - 475.
- Semigroups of scalar type operators in Banach spaces, Pac. Journal of Mathematics, 30 (1969), 489 - 517.

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THE BARTLE-DUNFORD-SCHWARTZ INTEGRAL VI. COMPLEMENTS TO THE THOMAS THEORY OF RADON INTEGRATION

T.V. PANCHAPAGESAN

The enumeration of sections will be continued from [P19]. We use the same notation and terminology given in [P15-P19].

Abstract

This chapter consists of Sections 25-30. In Section 25 we briefly indicate how the results in §1 of [T] can be extended to complex functions in $\mathcal{K}(T)$. Section 26 is devoted to integration with respect to a bounded weakly compact Radon operator, improving the complex versions of Theorems 2.2, 2.7, 2.12 and 2.7 bis and Proposition 2.5 of [T]. In Section 27, integration with respect to a prolongable Radon operator is studied, improving the complex versions of Theorems 3.3, 3.4, 3.11 and 3.20 of [T]. Section 28 is devoted to the complex Baire versions of Proposition 4.8 and Theorem 4.9 of [T]. The results of [P4] are generalized to vector measures in Section 29, while in Section 30, it is shown that $\mathcal{L}_p(u)$ is the same as $\mathcal{L}_p(\mathbf{m}_u)$ for $1 \le p < \infty$ when u is a bounded weakly compact Radon operator on $\mathcal{K}(T)$ and $\mathcal{L}_1(u)$ is the same as $\mathcal{L}_1(\mathbf{m}_u)$ when u is a prolongable Radon operator on $\mathcal{K}(T)$.

CONTENTS

25. INTEGRATION OF COMPLEX FUNCTIONS WITH RESPECT TO A RADON OPERATOR

Thomas developed in §1 of [T] a theory of vectorial Radon integration of real functions with respect to a Radon operator u on $\mathcal{K}(T,\mathbb{R})$ (see Notation 19.1 and Definition 19.3 of [P18]). In this section we indicate briefly how the results in §1 of [T] can be extended to complex functions in $\mathcal{K}(T)$.

In this section we extend Definition 1.1 of Thomas [T] to Radon operators on $\mathcal{K}(T)$ with values in a normed space X over $\mathbb{C}$.

Definition 25.1. Let $u : \mathcal{K}(T) \to X$ be a Radon operator in the sense of Definition 19.3 of P18], where X is a normed space over $\mathbb{C}$. We define

$$u^\bullet(f) = \sup_{|\varphi|\le f, \varphi\in\mathcal{K}(T)} |u(\varphi)|$$

for $f \in \mathcal{I}^+$, where $\mathcal{I}^+$ is the set of all non negative lower semicontinuous functions on T. When $f : T \to [0,\infty]$ has compact support we define

$$u^\bullet(f) = \inf_{f\le g, g\in\mathcal{I}^+} u^\bullet(g)$$

and when $f : T \to [0,\infty]$ is arbitrary, we define

$$u^\bullet(f) = \sup_{h\le f} u^\bullet(h)$$

where $h : T \to [0,\infty]$ has compact support. This Definition is similar to that in §1 in Chapter V of [B].

$u^\bullet(f)$ is called the semivariation of f with respect to u. For $A \subset T$, we define $u^\bullet(A) = u^\bullet(\chi_A)$. If $u^\bullet(A) = 0$, we say that A is u-null and use the expression u-almost everywhere (briefly, u-a.e.) correspondingly.

It is easy to verify that the definition is consistent.

Thomas [T] uses the terminology of Radon measure u instead of our terminology of Radon operator.

Unless otherwise stated, X will denote a normed space over $\mathbb{C}$ and $u : \mathcal{K}(T) \to X$ will denote a Radon operator.

Proposition 25.2. For $f \in \mathcal{I}^+$,

$$u^\bullet(f) = \sup_{0 \le \varphi \le f, \varphi \in \mathcal{K}(T)} u^\bullet(\varphi).$$

Proof. By Definition 25.1 we have

$$\begin{aligned} u^\bullet(f) &= \sup_{|\varphi| \le f, \varphi \in \mathcal{K}(T)} |u(\varphi)| \\ &= \sup_{\Psi \in \mathcal{K}(T), |\Psi| \le f} \; \sup_{|\varphi| \le |\Psi|, \varphi \in \mathcal{K}(T)} |u(\varphi)| \\ &= \sup_{\Psi \in \mathcal{K}(T), |\Psi| \le f} u^\bullet(|\Psi|) = \sup_{0 \le \varphi \le f, \varphi \in \mathcal{K}(T)} u^\bullet(\varphi). \end{aligned}$$

Hence the proposition holds.

We recall the following definition from [B].

Definition 25.3. Each element $u \in \mathcal{K}(T)^*$ is called a complex Radon measure and is sometimes identified with the complex measure μ_u induced by u in the sense of Definition 4.3 of [P3]. $|u|$ is the positive linear functional in $\mathcal{K}(T)^*$ given by (12) on p.55 of [B] and $|u|^*(f)$ for $f \in \mathcal{I}^+$ is given by Definition 1 on p.107 of Ch. IV of [B].

Proposition 25.4. If $u \in \mathcal{K}(T)^*$ and $f \in \mathcal{I}^+$, then

$$u^\bullet(f) = \sup_{0 \le \Psi \le f, \Psi \in \mathcal{K}(T)} |u|(\Psi) = |u|^\bullet(f) = |u|^*(f).$$

Consequently,

$$u^\bullet(f) = |u|^\bullet(f)$$

for $f : T \to [0, \infty]$.

Proof. For $f \in \mathcal{I}^+$, by Definition 25.1 we have

$$\begin{aligned} u^\bullet(f) &= \sup_{|\varphi| \le f, \varphi \in \mathcal{K}(T)} |u(\varphi)| \\ &= \sup_{0 \le \Psi \le f, \Psi \in \mathcal{K}(T)} \; \sup_{|\varphi| \le \Psi, \varphi \in \mathcal{K}(T)} |u(\varphi)| \\ &= \sup_{0 \le \Psi \le f, \Psi \in \mathcal{K}(T)} |u|(\Psi) \qquad (25.4.1) \end{aligned}$$

by (12) on p. 55 of Ch. III of [B].

Therefore, by (25.4.1)

$$\begin{aligned} |u|^\bullet(f) &= \sup_{|\varphi|\le f, \varphi\in\mathcal{K}(T)} ||u|(\varphi)| \\ &= \sup_{0\le\Psi\le f, \Psi\in\mathcal{K}(T)} |u|(\Psi) = |u|^\bullet(f) \end{aligned}$$

and hence again by (25.4.1) we have

$$|u|^\bullet(f) = \sup_{0\le\Psi\le f, \Psi\in\mathcal{K}(T)} |u|(\Psi) = u^\bullet(f).$$

Moreover, by the definition on p.107 of Ch. IV of [B]

$$|u|^\bullet(f) = \sup_{0\le\Psi\le f, \Psi\in\mathcal{K}(T)} |u|(\Psi) = |u|^*(f).$$

Now the last part is evident from Definition 25.1.

Definition 25.5. $\mathcal{F}^0(u) = \{f : T \to \boldsymbol{K}, u^\bullet(|f|) < \infty\}$ and we define $u^\bullet(f) = u^\bullet(|f|)$ if $f : T \to \boldsymbol{K}$

Clearly, $u^\bullet$ is a seminorm on $\mathcal{F}^0(u)$ and hence $\mathcal{F}^0(u)$ is a seminormed space with respect to $u^\bullet(\cdot)$.

The complex versions of Proposition 1.3 and of Lemmas 1.4 and 1.5 of [T] hold and consequently, we have the following definition.

Definition 25.6. The space $\mathcal{L}_1(u)$ of u-integrable functions is the closure of $\mathcal{K}(T)$ in the space $\mathcal{F}^0(u)$. Thus, a complex function f belongs to $\mathcal{L}_1(u)$, if given $\epsilon > 0$, there exists $\varphi \in \mathcal{K}(T)$ such that $u^\bullet(|\varphi - f|) < \epsilon$.

Remark 25.7. A complex function f on T belongs to $\mathcal{L}_1(u)$ if and only if, given $\epsilon > 0$, there exists $g \in \mathcal{L}_1(u)$ such that $u^\bullet(|f - g|) < \epsilon$. Also note that if the complex function $f = g$ u-a.e. in T and if $g \in \mathcal{L}_1(u)$, then $f \in \mathcal{L}_1(u)$.

Convention 25.8. Let $f : T \to [0, \infty]$. Then f is said to be u-integrable if there exists a complex function $g \in \mathcal{L}_1(u)$ such that $f = g$ u-a.e. in T.

The complex analogues of Proposition 1.7, Theorem 1.8 and Remark following it in [T] hold.

Definition 25.9. As $\mathcal{K}(T)$ is dense in $\mathcal{L}_1(u)$, the continuous linear extension of u to $\mathcal{L}_1(u)$ with values in $\widetilde{X}$, the completion of X, is denoted by $\int du$. Thus, if $f \in \mathcal{L}_1(u)$, then $\int f du \in \tilde{X}$.

Then the complex analogues of **1.10**, Theorem 1.11 and **1.12** of [T] hold.

The following result is the complex analogue of Proposition on p. 70 of [T].

Proposition 25.10. Let X be a normed space and $u : \mathcal{K}(T) \to X$ be a Radon operator. Let H be a norm determining set in X^* so that $|x| = \sup_{x^* \in H} | < x, x^* > |$ for $x \in X$. Then

$$u^\bullet(|f|) = \sup_{x^* \in H} |u_{x^*}|^\bullet(|f|) \tag{25.10.1}$$

for $f \in \mathcal{L}_1(u)$ or for $f \in \mathcal{I}^+$, where $u_{x^*} = x^* \circ u$.

Proof. By Lemma 18.13 of [P18], $H \subset \{x^* \in X^* : |x^*| \leq 1\}$. In view of the complex version of Lemma 1.4 of [T], the proof of **1.13** of [T] as given in [T] holds for complex functions too and hence the proposition holds.

Lemma 25.11. For $\mu \in \mathcal{K}(T)^*$,

$$\mu^\bullet(|f|) = |\mu|(|f|)) = \sup_{|\varphi| \leq 1, \varphi \in \mathcal{K}(T)} |\int_T \varphi f d\mu|$$

for $f \in \mathcal{L}_1(\mu)$ where $|\mu|$ is given by (12) on p. 55 of Ch. III of [B].

Proof. Let $\nu_f(\varphi) = \int_T \varphi f d\mu$ for $f \in \mathcal{L}_1(\mu)$ and $\varphi \in \mathcal{K}(T)$. Then

$$|\nu_f(\varphi)| \leq ||\varphi||_T \int_T |f| d|\mu|$$

so that $\nu_f \in \mathcal{K}(T)_b^*$. Hence by Theorem 3.3 of [P4] we have

$$\begin{aligned} |\nu_f|(T) = v(\mu_{\nu_f}|_{\mathcal{B}(T)}, \mathcal{B}(T)) &= ||\nu_f|| = \sup_{|\varphi|\le 1, \varphi\in\mathcal{K}(T)} |\nu_f(\varphi)| \\ &= \sup_{|\varphi|\le 1, \varphi\in\mathcal{K}(T)} |\int_T \varphi f d\mu|. \qquad (25.11.1) \end{aligned}$$

As noted in the beginning of the proof of Theorem 23.6 of [P19],

$$|\nu_f|(T) = \int_T |f| d|\mu|.$$

Then by (25.11.1) and by Proposition 25.4 we have

$$|\nu_f|(T) = \int_T |f|d|\mu| = \sup_{|\varphi|\le 1, \varphi\in\mathcal{K}(T)} |\int_T \varphi f d\mu| = |\mu|(|f|) = \mu^\bullet(|f|)$$

for $f \in \mathcal{L}_1(\mu)$.

Theorem 25.12. For $f \in \mathcal{L}_1(u)$,

$$u^\bullet(|f|) = \sup_{|\varphi|\le 1, \varphi\in\mathcal{K}(T)} |\int_T \varphi f du|.$$

Proof. Let H be a norm determining set in X^*. For $x^* \in H$, by Lemma 25.11 we have

$$u^\bullet_{x^*}(|f|) = \sup_{\varphi\in\mathcal{K}(T), |\varphi|\le 1} |\int_T \varphi f du_{x^*}|$$

for $f \in \mathcal{L}_1(u)$. Therefore, by Proposition 25.10 and by Proposition 24.4 of [P19] we have

$$\begin{aligned} u^\bullet(|f|) = \sup_{x^*\in H} u^\bullet_{x^*}(|f|) &= \sup_{x^*\in H} \sup_{\varphi\in\mathcal{K}(T), |\varphi|\le 1} |\int_T \varphi f du_{x^*}| \\ &= \sup_{\varphi\in\mathcal{K}(T), |\varphi|\le 1} \sup_{x^*\in H} |\int_T \varphi f du_{x^*}| \\ &= \sup_{\varphi\in\mathcal{K}(T), |\varphi|\le 1} |\int_T \varphi f du|. \end{aligned}$$

Definition 25.13. Let $u : \mathcal{K}(T) \to X$ be a Radon operator where X is a normed space. If Y is a topological space and $f : T \to Y$, then f is said to be u-measurable if, for every compact $K \subset T$ and $\epsilon > 0$, there exists a compact $K_1 \subset K$ such that $u^\bullet(K\backslash K_1) < \epsilon$ and $f|_{K_1}$ is continuous.

Replacing $||\mathbf{m}||$ by $u^\bullet$ and arguing as in the proof of Theorem 21.4 of [P19], we obtain the following theorem.

Theorem 25.14. Let u, X, f and Y be as in Definition 25.13. Then f is u-measurable if and only if, given $K \in \mathcal{C}$, there exist a u-null set $N \subset K$ and a countable disjoint family $(K_i)_1^\infty \subset \mathcal{C}$ such that $K \backslash N = \bigcup_1^\infty K_i$ and $f|_{K_i}$ is continuous for each $i \in \mathbb{N}$.

The set of u-measurable complex functions is evidently stable under the usual algebraic operations and under composition with a continuous function. Every continuous function on T is u-measurable.

The proof given in Appendix III of [T] holds good for complex normed spaces too and hence Lemma 1.19 of [T] holds for complex spaces too. The proofs of Propositions 1.20 and 1.21, Remark on p.74, Theorem 1.22, Lemmas 1.23, 1.24, 1.25 and 1.25 bis of [T] hold for complex functions too.

Proposition 25.15. Let X be a normed space and let H be a norm determining set in X^* so that $|x| = \sup_{x^* \in H} | < x, x^* > |$ for $x \in X$. Then for every u-measurable bounded positive function f,

$$u^\bullet(f) = \sup_{x^* \in H} |u_{x^*}|^\bullet(f).$$

Proof. By the complex version of Lemma 1.23 of [T], $f\chi_K \in \mathcal{L}_1(u)$ for $K \in \mathcal{C}$. Then by Proposition 25.10 we have

$$\begin{aligned} u^\bullet(f) = \sup_{K \in \mathcal{C}} u^\bullet(f\chi_K) &= \sup_{K \in \mathcal{C}} \sup_{x^* \in H} |u_{x^*}|^\bullet(f\chi_K) \\ &= \sup_{x^* \in H} \sup_{K \in \mathcal{C}} |u_{x^*}|^\bullet(f\chi_K) = \sup_{x^* \in H} |u_{x^*}|^\bullet(f). \end{aligned}$$

Corollary 25.16. Under the hypothesis of Proposition 25.15, for a u-measurable set A

$$u^\bullet(A) = \sup_{x^* \in H} |u_{x^*}|^\bullet(A).$$

Corollary 25.17. If f is locally u-integrable (i.e., if φf is u-integrable for each $\varphi \in \mathcal{K}(T)$), then

$$u^\bullet(f) = \sup_{x^* \in H} |u_{x^*}|^\bullet(f)$$

where H is a norm determining set in X^*.

Proof. By Urysohn's lemma, for each $K \in \mathcal{C}$, there exists $\varphi_K \in \mathcal{K}(T)$ such that $\chi_K \leq \varphi_K \leq 1$. Then

$$u^\bullet(f) = \sup_{K\in\mathcal{C}} u^\bullet(f\chi_K) \leq \sup_{K\in\mathcal{C}} u^\bullet(f\varphi_K) \leq \sup_{0\leq\varphi\leq 1, \varphi\in\mathcal{K}(T)} u^\bullet(f\varphi) \leq u^\bullet(f) \qquad (25.17.1)$$

and hence by Proposition 25.15, by Definition 25.5 and by (25.17.1) we have

$$\begin{aligned} u^\bullet(f) &= \sup_{0\leq\varphi\leq 1, \varphi\in\mathcal{K}(T)} u^\bullet(f\varphi) \\ &= \sup_{x^*\in H} \sup_{0\leq\varphi\leq 1, \varphi\in\mathcal{K}(T)} |u_{x^*}|^\bullet(\varphi f) \\ &= \sup_{x^*\in H} |u_{x^*}|^\bullet(f). \end{aligned}$$

Thus the corollary holds.

Definition 25.18. Let u ba a Radon operator with values in an lcHs X over $\mathbb{C}$. Let q ba a continuous seminorm on X. We denote $q(x)$ by $|x|_q$. The semivariation of u with respect to q for $f \in \mathcal{I}^+$ is defined by

$$u_q^\bullet(f) = \sup_{|\varphi|\leq f, \varphi\in\mathcal{K}(T)} |u(\varphi)|_q$$

and one completes the definition for $f : T \to [0, \infty]$ as in Definition 25.1 given in the case of a normed space.

For $q \in \Gamma$, let $X_q = X/q^{-1}(0)$ and let $\widetilde{X_q}$ be the Banach space completion of X_q with respect to $|\cdot|_q$. Let $\Pi_q : X \to \widetilde{X_q}$ be the canonical quotient map. (See the beginning of §10 of [P17].) Let $u_q = \Pi_q \circ u$. Then $|u_q(\varphi)| = |u(\varphi)|_q$ so that $u_q^\bullet$ is the semivariation of u_q.

Definition 25.19. For the Radon operator u on $\mathcal{K}(T)$ with values in the lcHs X and for $q \in \Gamma$, the family of continuous seminorms on X,

$$\mathcal{F}^0(u) = \{f : \mathcal{K}(T) \to \boldsymbol{K}, \ u_q^\bullet(|f|) < \infty \text{ for each } \ q \in \Gamma\}.$$

Then by the complex version of Lemma 1.5 of [T], $\mathcal{K}(T) \subset \mathcal{F}^0(u)$ and this permits the following definition.

Definition 25.20. Let $\mathcal{F}^0 = (u)$ be provided with the seminorms $u_q^\bullet(\cdot)$ for $q \in \Gamma$. The space $\mathcal{L}_1(u)$ of u-integrable functions is the closure of $\mathcal{K}(T)$ in the space $\mathcal{F}^0$. Hence a function $f : \mathcal{K}(T) \to \boldsymbol{K}$is u-integrable if, for each $q \in \Gamma$, there exists $\varphi_q \in \mathcal{K}(T)$ such that $u_q^\bullet(|\varphi_q - f|) < \epsilon$.

Thus $\mathcal{L}_1(u)$ is the intersection of the spaces $\mathcal{L}_1(u_q)$ provided with the smallest topology permitting the injections $\mathcal{L}_1(u) \subset \mathcal{L}_1(u_q)$ continuous for each $q \in \Gamma$.

Definition 25.21. We say that a function $f : T \to Y$ is u-measurable where Y is a topological space if f is u_q-measurable for each $q \in \Gamma$ and is u-null if it is u_q-null for each $q \in \Gamma$.

With these definitions the complex versions of Propositions 1.7, 1.20 and 1.21 and Theorem 1.22 of [T] hold without any modifications for complex lcHs valued Radon operators u. Thus a function f is u-integrable if and only if it is u-measurable and is dominated in modulus by a u-integrable function. The Hausdorff space $L_1(u)$ associated with $\mathcal{L}_1(u)$ consists of classes of functions in which two functions equal u-a.e in T are identified.

Definition 25.22. For a function $f \in \mathcal{L}_1(u)$, we denote by $u(f)$ or by $\int f du$ the value in f of the continuous linear extension of u to $\mathcal{L}_1(u)$. Thus this is an element in the completion $\widetilde{X}$ of X. Then the mapping $f \to \int f du$ is a continuous linear mapping of $\mathcal{L}_1(u)$ in $\widetilde{X}$.

Hereafter, by lcHs we mean a complex lcHs. i.e., an lcHs over $\boldsymbol{C}$ Then Propositions 1.28 and 1.30 and results **1.31**, **1.32, 1.33** and **1.34** of [T] hold for complex lcHs-valued u on $\mathcal{K}(T)$.

If X is a projective limit of Banach spaces X_i, then $\int f du$ is identified with the element $(\int f du_i)_i$(i.e., with the projective limit of $(\int f du_i)_i$).

Lemma 25.23. Suppose X is a quasicomplete lcHs. If $A \subset X$ is bounded and if x_0 belongs to the closure of A in $\widetilde{X}$, then $x_0 \in X$.

Proof. Let τ be the lcHs topology of X. By hypothesis there exists a net $(x_\alpha) \subset A$ such that $x_\alpha \to x_0$ in $\tilde{\tau}$, the topology of the completion $\widetilde{X}$. Thus (x_α) is Cauchy in $\tilde{\tau}$. As A is τ-bounded and as X is quasicomplete, the τ-closure of A is τ-complete. Since $\tilde{\tau}|_A = \tau$, it follows that (x_α)

is τ-Cauchy. Hence there exists x_1 in the τ-closure of A (so that $x_1 \in X$) such that $x_\alpha \to x_1$ in τ and hence in $\tilde{\tau}$. Since $\tilde{\tau}$ is Hausdorff, $x_0 = x_1 \in X$.

Theorem 25.24. Let u be a Radon operator on $\mathcal{K}(T)$ with values in a quasicomplete lcHs X. Then for each $f \in \mathcal{L}_1(u)$, $\int f du$ belongs to X. In other words, if X is an lcHs, then $\int f du$ belongs to the quasicompletion of X for each $f \in \mathcal{L}_1(u)$.

Proof. By Lemma 25.23 above, it suffices to show that $\int f du$ belongs to the closure in $\widetilde{X}$ of a τ-bounded set $A \subset X$ whenever $f \in \mathcal{L}_1(u)$.

For the sake of completeness, we give the proof of this result and we follow the proof of Theorem 1.35 of [T].

Case 1. Suppose f is bounded with compact support. Let ω be a relatively compact open set such that f is null in $T \backslash \omega$ and let $|f| \leq 1$.

If $| < \int \varphi du, x^* > | \leq 1$ for $\varphi \in \mathcal{K}(T)$ with $|\varphi| \leq \chi_\omega$, then

$$\begin{aligned} 1 \geq \sup_{|\varphi| \leq \chi_\omega, \varphi \in \mathcal{K}(T)} | < \int \varphi du, x^* > | &= \sup_{|\varphi| \leq \chi_\omega, \varphi \in \mathcal{K}(T)} |u_{x^*}(\varphi)| \\ &= u^\bullet_{x^*}(\chi_\omega) = |u_{x^*}|(\omega) \end{aligned}$$

by Proposition 25.4. As $|f| \leq \chi_\omega$, f is u-integrable and

$$| < \int f du, x^* > | = | \int f du_{x^*} | \leq \int |f| d |u_{x^*}| \leq |u_{x^*}|(\omega) \leq 1$$

and hence $\int f du \in A^{00}$ where $A = \{\int \varphi du : \varphi \in \mathcal{K}(T), |\varphi| \leq \chi_\omega\} \subset X \subset \widetilde{X}$.

Since A is absolutely convex, by the bipolar Theorem 8.1.5 of [E] A^{00} is the $\sigma(\widetilde{X}, X^*)$- closure of A, and hence by Theorem 3.12 of [Ru2] is also the $\tilde{\tau}$-closure of A. As A is weakly bounded, it is τ-bounded by Theorem by Theorem 3.18 of [Ru2]. Hence by Lemma 25.23, $\int f du \in X$.

Case 2. Let f be null outside a compact set.

First let us consider the case $f \geq 0$. Let $f_n = \min(f, n)$. Then by Case 1, $\int f_n du \in X$ for all n and by the lcHs analogue of Lemma 1.25 bis of [T], $\int f du = \lim_n \int f_n du$. Thus $\int f du$ belongs to the closure of $(\int f_n du)_{n=1}^\infty$ in $\widetilde{X}$ and as $(\int f_n)_{n=1}^\infty$ is convergent, it is bounded in X. Hence by Lemma 25.23, $\int f du \in X$. Consequently, by the complex analogue of Proposition 1.7 of [T], $\int f du$ belongs to X in this case too.

Case 3. f is an arbitrary element in $\mathcal{L}_1(u)$.

By the above cases, $\int f\varphi du \in X$ for each $\varphi \in \mathcal{K}(T)$. The set $B = \{\int f\varphi du\}_{|\varphi|\leq 1, \varphi\in\mathcal{K}(T)}$ is weakly bounded since

$$\sup_{|\varphi|\leq 1, \varphi\in\mathcal{K}(T)} |< \int f\varphi du, x^* > | \leq \int |f| d|u_{x^*}| < \infty$$

for each $x^* \in X^*$. Then by Theorem 3.18 of [Ru2], B is τ-bounded. If $| < \int f\varphi du, x^* > | \leq 1$ for $\varphi \in \mathcal{K}(T)$ with $|\varphi| \leq 1$, then by Theorem 25.12, $u^{\bullet}_{x^*}(|f|) \leq 1$ and hence $| < \int f du, x^* > | \leq 1$. Therefore, $\int f du \in B^{00}$. Then arguing as in Case 1 and appealing to Lemma 25.23, we conclude that $\int f du \in X$. This completes the proof of the theorem.

The proof of Proposition on p. 84 of [T] holds here for metrizable lcHs and hence we have:

Theorem 25.25. If X id a metrizable lcHs and if $u : \mathcal{K}(T) \to X$ is a Radon operator, then the space $\mathcal{L}_1(u)$ is seudo-metrizable and complete.

26. INTEGRATION WITH RESPECT TO A BOUNDED WEAKLY COMPACT RADON OPERATOR

The aim of this section is to improve the results in Section 2 of Thomas [T]. Remark 2 on p.161 of [G] and Theorem 6 of [G] when T is compact, play a key role in [T] to develop the theory of vectorial Radon integration with respect to a bounded weakly compact (respectively, a prolongable) Radon operator. Grothendieck comments in the said remark that his techniques developed in earlier sections of [G] are textually valid for $C_0(T)$, where T is a locally compact Hausdorff space. But, as shown in [P10], his techniques can be used to prove the said remark if and only if T is further σ-compact. However, by different methods, we established in [P9] and [P11] the validity of Theorem 6 of [G] for $C_0(T)$ where T is an arbitrary locally compact Hausdorff space, thereby restoring the validity of the Thomas theory in [T]. The proposition given in Complements of Section 2 of [T] improves Theorem 2.7 and Theorem 2.7 bis of [T]. But, we obtain here results which further improve the said proposition of [T].

Definition 26.1. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a Radon operator (see Definition 19.3 of [P17]). Then u is said to be bounded if $u : (C_c(T), ||\cdot||_T) \to X$ is continuous.

Notation 26.2. Whenever X ia a quasicomplete lcHs, Γ denotes the family of continuous seminorms on X.

Proposition 26.3. Let X be a quasicomplete lcHs. The Radon operator $u : \mathcal{K}(T) \to X$ is bounded if and only if $u_q^\bullet(T) < \infty$ for each $q \in \Gamma$.

Proof. Let u be bounded. Then, for each $q \in \Gamma$, by Definition 25.18 there exists a constant M_q such that $|u_q(\varphi)| = |u(\varphi)|_q = q(u(\varphi)) \le M_q||\varphi||_T$ for each $\varphi \in \mathcal{K}(T)$. Then

$$u_q^\bullet(T) = u_q^\bullet(\chi_T) = \sup_{|\varphi|\le 1, \varphi\in\mathcal{K}(T)} q(u(\varphi)) \le M_q < \infty$$

and hence $u_q^\bullet(T) < \infty$ for each $q \in \Gamma$.

Conversely, if $u_q^\bullet(T) = M_q < \infty$ for each $q \in \Gamma$, then for $\varphi \in \mathcal{K}(T)$, we have $|\varphi| \le ||\varphi||_T\chi_T$ and hence

$$q(u(\varphi)) = |u_q(\varphi)|_q \le ||\varphi||_T u_q^\bullet(\chi_T) = M_q||\varphi||_T$$

for $q \in \Gamma$. Hence u is bounded.

Convention 26.4. Let X be a quaicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. Then u has a continuous linear extension to the whole of $C_0(T)$ with values in X and hence we shall always assume that $u : (C_0(T), ||\cdot||_T) \to X$ is continuous whenever $u : \mathcal{K}(T) \to X$ is a bounded Radon operator.

Proposition 26.5. Let X be a quasicomplete lcHs. If $u : \mathcal{K}(T) \to X$ is a bounded Radon operator, then $C_0(T) \subset \mathcal{L}_1(u)$. Moreover,for $f \in C_0(T)$, $uf = \int f du$.

Proof. Let $f \in C_0(T)$ and let $q \in \Gamma$. Then there exists $(\varphi_n)_1^\infty \subset \mathcal{K}(T)$ such that $||\varphi_n - f||_T \to 0$. By hypothesis and by Proposition 26.3, $u_q^\bullet(T) < \infty$ and hence

$$u_q^\bullet(|\varphi_n - f|) \leq u_q^\bullet(||f - \varphi_n||_T \chi_T) \leq ||f - \varphi_n||_T u_q^\bullet(T) \to 0$$

as $n \to \infty$. Hence $f \in \mathcal{L}_1(u_q)$. As q is arbitrary in Γ, it follows that $f \in \mathcal{L}_1(u)$. Thus $C_0(T) \subset \mathcal{L}_1(u)$.

Moreover, $uf = \lim_n u\varphi_n$ and $\int f du = \lim_n u\varphi_n$. Hence $uf = \int f du$.

For a bounded Radon operator u, it is possible that $\mathcal{L}_1(u) = C_0(T)$ as shown below.

Example 26.6. Let $u : \mathcal{K}(T) \to C_0(T)$ be the identity operator. Then $u : (C_c(T), ||\cdot||_T) \to (C_0(T), ||\cdot||_T)$ is continuous and has a unique continuous extension to $C_0(T)$. Then $\mathcal{L}_1(u) = C_0(T)$.

If $f \in \mathcal{I}^+$, then

$$u^\bullet(f) = \sup_{|\varphi| \leq f, \varphi \in \mathcal{K}(T)} |u(\varphi)| = \sup_{|\varphi| \leq 1, \varphi \in \mathcal{K}(T)} ||\varphi||_T = ||f||_T \qquad (26.6.1)$$

since $f = \sup_{0 \leq \varphi \leq f, \varphi \in \mathcal{K}(T)} \varphi$.

If $f : T \to [0, \infty]$ has compact support, then by (26.6.1) and by Definition 25.1 we have

$$||f||_T \leq \inf_{f \leq g \in \mathcal{I}^+} ||g||_T = u^\bullet(f).$$

Let K be the support of f. By Urysohn's lemma there exists $\Psi \in C_c(T)$ with $0 \leq \Psi \leq 1$ and $\Psi|_K = 1$. Then $0 \leq f \leq ||f||_T \Psi \in \mathcal{I}^+$ and

$$u^\bullet(||f||_T \Psi) = ||f||_T u^\bullet(\Psi) = ||f||_T ||\Psi||_T = ||f||_T$$

by (26.6.1) as $\Psi \in \mathcal{I}^+$. Hence $u^\bullet(f) = ||f||_T$.

When $f : T \to [0, \infty]$ is arbitrary, then

$$u^\bullet(f) = \sup_{K \in \mathcal{C}} u^\bullet(f \chi_K) = \sup_{K \in \mathcal{C}} ||f \chi_K||_T = ||f||_T. \qquad (26.6.2)$$

If $f \in \mathcal{L}_1(u)$, then given $\epsilon > 0$, there exists $\varphi \in \mathcal{K}(T)$ such that $u^\bullet(|f-\varphi|) < \epsilon$. Then by (26.6.2), $||f-\varphi||_T < \epsilon$ and hence $f \in C_0(T)$. Therefore, $\mathcal{L}_1(u) = C_0(T)$.

Definition 26.7. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. (See convention 26.4.) Then u is called a bounded weakly compact Radon operator if $\{u\varphi : \varphi \in C_0(T), ||\varphi||_T \leq 1\}$ is relatively weakly compact in X.

Remark 26.8. Bounded Radon operators and bounded weakly compact Radon operators are respectively called bounded Radon measures and weakly compact bounded Radon measures in [T].

Lemma 26.9. Let X be a Banach space and let $u : \mathcal{K}(T) \to X$ be a continuous linear map. Then for each open set ω in T,

$$u^\bullet(\omega) = \sup_{|x^*|\leq 1} |x^*u|(\omega) \qquad (26.9.1)$$

where $|x^*u|(\omega) = \mu_{|x^*u|}(\omega)$ and $\mu_{|x^*u|}$ is the (complex) Radon measure induced by $|x^*u|$ in the sense of Definition 4.3 of [P3].

Proof. By Definition 25.1 we have

$$\begin{aligned}
u^\bullet(\omega) &= \sup_{|\varphi|\leq\chi_\omega, \varphi\in\mathcal{K}(T)} |u(\varphi)| \\
&= \sup_{|\varphi|\leq\chi_\omega, \varphi\in\mathcal{K}(T)} \sup_{|x^*|\leq 1} |x^*u(\varphi)| \\
&= \sup_{|x^*|\leq 1} \sup_{|\varphi|\leq\chi_\omega, \varphi\in\mathcal{K}(T)} |x^*u(\varphi)| \\
&= \sup_{|x^*|\leq 1} (x^*u)^\bullet(\omega) \\
&= \sup_{|x^*|\leq 1} |x^*u|^*(\omega)
\end{aligned}$$

by Proposition 25.4. Since ω is $|x^*u|^*$-measurable, we have

$$u^\bullet(\omega) = \sup_{|x^*|\leq 1} |x^*u|(\omega)$$

and hence (26.9.1) holds.

Using the above lemma and [P9], we give in the following theorem an improved version of Theorem 2.2 of [T] for complex functions.

Theorem 26.10. Let X be a Banach space and let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. Then the following statements are equivalent:

(i) Every bounded Borel (complex) function belongs to $\mathcal{L}_1(u)$.

(ii) Every bounded σ-Borel (complex) function belongs to $\mathcal{L}_1(u)$.

(iii) Every bounded (complex) Baire function belongs to $\mathcal{L}_1(u)$.

(iv) For every open set ω in T, the weak integral $\int_\omega du$ belongs to X; i.e., there exists a vector x_ω in X such that

$$\int_\omega d(x^*u) = x^*(x_\omega)$$

for each $x^* \in X^*$ and we say that the weak integral $\int_\omega du = x_\omega$.

(v) For every σ-Borel open set ω in T, the weak integral $\int_\omega$ belongs to X.

(vi) For every open Baire set ω in T, the weak integral $\int_\omega du$ belongs to X.

(vii) u is a bounded weakly compact Radon operator (so that by Convention 26.4, $u : C_0(T) \to X$ is weakly compact).

Proof. Clearly, (i)⇒(ii)⇒(iii).

(iii)⇒(vi) By (iii), for each open Baire set ω in T, there exists $x_\omega \in X$ such that $\int_\omega du = x_\omega$ and hence

$$x^*(x_\omega) = \int_\omega d(x^*u)$$

for $x^* \in X^*$. Hence (vi) holds.

(vi)⇒(vii) By (vi), for each open Baire set ω in T there exists $x_\omega \in X$ such that

$$\int_\omega d(x^*u) = x^*(x_\omega) \qquad (26.10.1)$$

for $x^* \in X^*$. As $x^*u \in C_0(T)^* = M(T)$, the complex Radon measure μ_{x^*u} induced by x^*u in the sense of Definition 4.3 of [P3] is a $\mathcal{B}(T)$-regular complex measure on $\mathcal{B}(T)$. Let K be a compact G_δ in T. Then by Theorem 55.A of [H] there exists $(\varphi_n)_1^\infty \subset C_0(T)$ such that $0 \leq \varphi_n \searrow \chi_K$. Then by LDCT and by 18.10 of [P18] we have

$$\mu_{x^*u}(K) = \lim_n \int_T \varphi_n d\mu_{x^*u} = \lim_n (x^*u)(\varphi_n) = \lim_n \int_T \varphi_n d(x^* \circ \mathbf{m}) = (x^* \circ \mathbf{m})(K)$$

where $\mathbf{m}$ is the representing measure of u. Then by the Baire regularity of $\mu_{x^*u}|_{\mathcal{B}_0(T)}$ and of $(x^* \circ \mathbf{m})_|\mathcal{B}_0(T)$ we have $\mu_{x^*u}|_{\mathcal{B}_0(T)} = (x^* \circ \mathbf{m})|_{\mathcal{B}_0(T)}$ and consequently, by Theorem 2.4 of [P4] and by the Borel regularity of μ_{x^*u} and of $x^* \circ \mathbf{m}$ on $\mathcal{B}(T)$, we conclude that

$$\mu_{x^*u} = x^* \circ \mathbf{m} \, \text{on} \, \mathcal{B}(T). \qquad (26.10.2)$$

Then by (26.10.1) and (26.10.2) we have

$$x^*(x_\omega) = \int_\omega d(x^*u) = \mu_{x^*u}(\omega) = (x^* \circ \mathbf{m})(\omega) = (x^* \circ u^{**})(\chi_\omega)$$

for $x^* \in X^*$. Since $u^{**}(\chi_\omega) \in X^{**}$, we conclude that $\mathbf{m}(\omega) = u^{**}(\chi_\omega) = x_\omega \in X$. Consequently, by Theorem 3(vii) of [P9], u is a weakly compact operator on $C_0(T)$ and hence (vii) holds.

(vii)⇒(iv) (resp. (vii)⇒(v), (vii)⇒(vi)) By (vii) and by Theorem 2(ii) of [P9], $u^{**}(\chi_A) \in X$ for each $A \in \mathcal{B}(T)$ and hence $u^{**}(\chi_\omega) \in X$ for each open set (resp. σ-Borel open set, open Baire set) ω in T. Let $u^{**}(\chi_\omega) = x_\omega \in X$. Then by (26.10.2) we have

$$x^*(x_\omega) = x^*u^{**}(\chi_\omega) = (x^* \circ \mathbf{m})(\omega) = \mu_{x^*u}(\omega) = \int_\omega d(x^*u)$$

and hence (iv) (resp. (v), (vi)) holds.

(vii)⇒(i) Since $u : C_0(T) \to X$ is weakly compact, u^* is also weakly compact and hence $\{\mu_{u^*x^*} : |x^*| \leq 1\} = \{\mu_{x^*u} : |x^*| \leq 1\}$ is relatively weakly compact in $M(T)$. Then by Theorem 1 of [P8], given a Borel set A in T and $\epsilon > 0$, there exist a compact set K and an open set U in T such that $K \subset A \subset U$ and $\sup_{|x^*| \leq 1} |\mu_{x^*u}|(U \backslash K) < \epsilon$. Then by Lemma 26.9 we have $u^\bullet(U \backslash K) < \epsilon$. Now choose $\varphi \in \mathcal{K}(T)$ such that $\chi_K \leq \varphi \leq \chi_U$ so that $u^\bullet(\chi_U - \varphi) \leq u^\bullet(\chi_U - \chi_A) = u^\bullet(U \backslash K) < \epsilon$. Then

$$u^\bullet(|\chi_A - \varphi|) \leq u^\bullet(\chi_U - \chi_A) \leq u^\bullet(\chi_U - \chi_K) = u^\bullet(U \backslash K) < \epsilon.$$

Therefore, by Definition 25.6, $\chi_A \in \mathcal{L}_1(u)$.

Consequently, every Borel simple function $s \in \mathcal{L}_1(u)$. If f is a bounded Borel (complex) function, then there exists a sequence (s_n) of Borel simple functions such that $||s_n - f||_T \to 0$. Then

$$u^\bullet(|f - s_n|) \leq ||f - s_n||_T u^\bullet(T) \to 0$$

as $n \to \infty$, since $u^\bullet(T)$ is finite by Proposition 26.3. Hence $f \in \mathcal{L}_1(u)$ and thus (i) holds.

Hence the statements (i)-(vii) are equivalent.

The following theorem is an improved complex version of Proposition 2.5 of Thomas [T].

Theorem 26.11. Let X be a Banach space and let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. Then the following statements are equivalent:

(i) u is weakly compact (see Convention 26.4).

(ii) Given $\epsilon > 0$, for each open set ω in T, there exists a compact $K \subset \omega$ such that $u^\bullet(\omega \backslash K) < \epsilon$.

(iii) Given $\epsilon > 0$, for each $A \in \mathcal{B}(T)$, there exist a compact K and an open set ω in T such that $K \subset A \subset \omega$ and $u^\bullet(\omega \backslash K) < \epsilon$.

(iv) Given $\epsilon > 0$, for each compact K in T there exists an open set U in T such that $K \subset U$ with $u^\bullet(U \backslash K) < \epsilon$ and there exists a compact C in T such that $u^\bullet(T \backslash C) < \epsilon$.

(v) Given $\epsilon > 0$, for each σ-Borel open set $\omega \subset T$ there exists a compact $K \subset \omega$ such that $u^\bullet(\omega \backslash K) < \epsilon$ and there exists a compact C in T such that $u^\bullet(T \backslash C) < \epsilon$.

(vi) Given $\epsilon > 0$, for each $A \in \mathcal{B}_c(T)$ there exist a compact K and a σ-Borel open set ω in T such that $K \subset A \subset \omega$ and $u^\bullet(\omega \backslash K) < \epsilon$.

(vii) Given $\epsilon > 0$, for each compact K in T there exists a σ-Borel open sets U in T such that $K \subset U$ with $u^\bullet(U \backslash K) < \epsilon$ and there exists a compact G_δ such that $u^\bullet(T \backslash C) < \epsilon$.

(viii) Given $\epsilon > 0$, for each open Baire set ω in T there exists a compact G_δ $K \subset \omega$ such that $u^\bullet(\omega \backslash K) < \epsilon$ and there exists a compact C such that $u^\bullet(T \backslash C) < \epsilon$.

(ix) Given $\epsilon > 0$, for each Baire set $A \in T$ there exist a compact G_δ K and an open Baire set ω in T such that $K \subset A \subset \omega$ and $u^\bullet(\omega \backslash K) < \epsilon$.

(x) Given $\epsilon > 0$, for each compact G_δ K in T there exists an open Baire set U in T such that $K \subset U$ with $u^\bullet(U \backslash K) < \epsilon$ and there exists a compact G_δ C such that $u^\bullet(T \backslash C) < \epsilon$.

Proof. Since u is a bounded operator on $C_0(T)$ by Convention 26.4, the set $F = \{\mu_{x^*u} : |x^*| \leq 1\} = \{\mu_{u^*x^*} : |x^*| \leq 1\}$ is bounded in $M(T)$. Let $|F| = \{\mu_{|x^*u|} : |x^*| \leq 1\}$. Then by Theorem 1 of [P8], F is relatively weakly compact in $M(T)$ if and only if $|F|$ is so (resp. if and only if u is weakly compact (as u is weakly compact if and only if u^* is weakly compact)). Moreover, by Theorem 4.11 of [P3] and by Theorem 3.3 of [P4], $\mu_{|x^*u|}(A) = v(\mu_{x^*u}, \mathcal{B}(T))(A) = |\mu_{x^*u}|(A)$ for $A \in \mathcal{B}(T)$. Then by Lemma 26.9 and by (vi) (resp. (vi)', (vi)") of Proposition 1 of [P9], (ix) (resp. (iii), (vi)) holds if and only if u is weakly compact. Hence (i)⇔(iii)⇔(vi) ⇔(ix).

(iii)⇒(ii) obviously.

(ii)⇒(i) Let ω be an open set in T and let $\epsilon > 0$. Then by hypothesis, there exists a compact K such that $K \subset \omega$ and $u^\bullet(\omega \backslash K) < \epsilon$. By Lemma 26.9, this means means $\sup_{|x^*| \leq 1} \mu_{|x^*u|}(\omega \backslash K) < \epsilon$. Then by Theorem 4.11 of [P3] and by Theorem 3.3 of [P4], $\sup_{|x^*| \leq 1} v(\mu_{x^*u}, \mathcal{B}(T))(\omega \backslash K) < \epsilon$ and hence by Theorem 1 of [P8], u is weakly compact. Thus (i) holds.

(i)⇔(iv) By Theorem 4.11 of [P3], $|F| = \{|\mu_{x^*u}| : |x^*| \leq 1\}$ and hence by Theorem 4.22.1 of [E], $|F|$ is relatively weakly compact in $M(T)$ if and only if , given $\epsilon > 0$, for each compact K in T there exists an open set U in T such that $K \subset U$ and $\sup_{|x^*| \leq 1} |\mu_{x^*u}|(U \backslash K) < \epsilon$ and there exists a compact C in T such that $\sup_{|x^*| \leq 1} |\mu_{x^*u}|(T \backslash C) < \epsilon$. Consequently, by Lemma 26.9, $|F|$ is relatively weakly compact in $M(T)$ if and only if (iv) holds and hence if and only if u is weakly compact.

Similarly, using Theorem 4.22.1 of [E], Theorem 4.11 of [P3], Theorem 50.D of [H] and Lemma 26.9 one can show that (i)⇔(vii) and (i)⇔(x).

By Proposition 1(iii) of [P9], by Theorem 4.11 of [P3] and by Lemma 26.9, (i)⇔(v) and (i)⇔(viii).

Hence the statements (i)-(x) are equivalent.

Definition and Notation 26.12. Let X be a Banach space and let $u : \mathcal{K}(T) \to X$ be a Radon operator. Let H be a subset of X^* separating the points of X. A function $f : T \to \boldsymbol{K}$ is said to be u-integrable with respect to the topology $\sigma(X, H)$ if f is x^*u-integrable for each $x^* \in H$. Then the integral of f with respect to $\sigma(X, H)$ is an element in the completion of $(X, \sigma(X, H))$ which is identified with $< H >^{alg}$, the set of all linear functionals on the liner span $< H >$ of H and the integral is denoted by $\int f d\tilde{u}$. This is identified with the function $x^* \to \int f d(x^*u)$ for $x^* \in H$. Thus

$$< x^*, \int f d\tilde{u} >= \int f du_{x^*} \qquad \text{for } x^* \in H.$$

The Orlicz property of a set H in X^* (see Definition 18.9 of [P18]) plays a key role in the sequel.

The following result improves the complex version of Theorem 2.7 of Thomas [T].

Theorem 26.13. Let X be a Banach space and let H be a norm determining subset of X^*. Suppose H possesses the Orlicz property. Let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. Then $u : C_0(T) \to (X, \sigma(X, H))$ is continuous (here we use Convention 26.4) and u is weakly compact on $C_0(T)$ if and only if $\int_\omega d\tilde{u} \in X$ for each open Baire set ω in T, where $\tilde{u}$ is the Radon operator obtained from u on providing X with the topology $\sigma(X, H)$.

Proof. Arguing as in the proof of Theorem 2.7 of [T] and using Theorem 1 of [P8] instead of Appendix I: C_2 of [T], we observe that the condition is sufficient.

Conversely, if u is weakly compact with its representing measure $\mathbf{m}$ in the sense of 18.10 of [P18], then by Theorem 2(ii) of [P9], $u^{**}(\chi_\omega) = \mathbf{m}(\omega) = x_\omega$ (say) $\in X$ for each open Baire set ω in T and hence

$$x^*(x_\omega) = \int_\omega d(x^*u)$$

for $x^* \in X^*$ and hence for $x^* \in H$. Then

$$\int_\omega d\tilde{u} = x_\omega \in X.$$

Therefore the condition is also necessary.

To improve the complex version of Theorem 2.12 of [T] we need the following lemma.

Lemma 26.14. Let X be a Banach space and let $u : C_0(T) \to X$ be a continuous linear mapping with the representing measure $\mathbf{m}$ (in the sense of 18.10 of [P18]). Then

$$\int_A d(x^*u) = (x^* \circ \mathbf{m})(A) \qquad (26.14.1)$$

for $x^* \in X^*$ and for $A \in \mathcal{B}(T)$. If $\chi_A \in \mathcal{L}_1(u)$, then

$$\int_A du = \mathbf{m}(A) \qquad (26.14.2)$$

and consequently (26.14.2) holds for $A \in \mathcal{B}(T)$ if u is weakly compact.

Proof. By the proof of (vi)⇒(vii) of Theorem 26.10 (without using (26.10.1)), $\mu_{x^*u} = x^* \circ \mathbf{m}$ on $\mathcal{B}(T)$ for $x^* \in X^*$. Hence

$$\int_A d(x^*u) = \int_A d\mu_{x^*u} = \mu_{x^*u}(A) = (x^* \circ \mathbf{m})(A)$$

for $A \in \mathcal{B}(T)$ and for $x^* \in X^*$.

If $\chi_A \in \mathcal{L}_1(u)$, there exists $x_A \in X$ such that $\int_A du = x_A \in X$. Consequently, by (26.14.1) we have

$$x^*(x_A) = \int_A d(x^*u) = (x^* \circ \mathbf{m})(A)$$

for $x^* \in X^*$. As $\mathbf{m}$ has range in X^{**}, we conclude that $\mathbf{m}(A) = x_A = \int_A du$ for $A \in \mathcal{L}_1(u)$. If u is weakly compact, $\mathcal{B}(T) \subset \mathcal{L}_1(u)$ by Theorem 26.10(i) and hence the last part holds.

Using the above lemma we obtain the following improvement of the complex version of Theorem 2.12 of [T].

Theorem 26.15. Let X be a Banach space and let $u_n : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator for $n \in \mathbb{N}$. If for every open Baire set ω in T, the sequence $(\int_\omega du_n)_1^\infty$ is convergent in X, then there exists a bounded weakly compact Radon operator u on $\mathcal{K}(T)$ with values in X such that

$$\lim_n \int f du_n = \int f du$$

for each bounded (complex) Borel function f on T.

Proof. Let $\mathbf{m}_n$ be the representing measure of u_n in the sense of 18.10 of [P18]. Then by hypothesis and by Lemma 26.14,

$$\lim_n \mathbf{m}_n(\omega) \text{ exists in } X \qquad (26.15.1)$$

for each open Baire set ω in T. Moreover, by Theorems 6(xix) and 2 of [P9], $\mathbf{m}_n$ is Borel regular and σ-additive in the topology τ of X for each $n \in \mathbb{N}$. By Lemma 18.19 of [P18], φ is $\mathbf{m}_n$-integrable in T and by 18.10 of [P18] we have

$$x^* u_n(\varphi) = \int_T \varphi d(x^* \circ \mathbf{m}_n) = x^*(\int_T \varphi d\mathbf{m}_n)$$

for $x^* \in X^*$ and for $\varphi \in C_0(T)$. Then by the Hahn-Banach theorem

$$u_n(\varphi) = \int_T \varphi d\mathbf{m}_n \text{ for } \varphi \in C_0(T).$$

Consequently, by Lemma 18.20 of [P18] there exists an X-valued continuous linear mapping u on $C_0(T)$ such that

$$\lim_n u_n(\varphi) = u(\varphi)$$

for $\varphi \in C_0(T)$. Moreover, by (26.15.1) and by Lemma 18.18 of [P18], $\lim_n \mathbf{m}_n(U) \in X$ for each open set U in T. Consequently, by the complex version of Proposition 2.11 of [T], $\lim_n \int f du_n = \int f du \in X$ for each bounded complex Borel function f on T. Then particularly,

$$\lim_n \int \chi_A du_n = \int \chi_A du \in X \qquad (26.15.2)$$

for each $A \in \mathcal{B}(T)$. Hence $\mathcal{B}(T) \subset \mathcal{L}_1(u)$. If $\mathbf{m}$ is the representing measure of u as in 18.10 of [P18], then by Lemma 26.14 and by (26.15.2), $\mathbf{m}(A) = \int \chi_A du \in X$ for each $A \in \mathcal{B}(T)$ and hence by Theorem 2 of [P9], u is weakly compact.

This completes the proof of the theorem.

Remark 26.16. The proofs of Propositions 2.13, 2.14, 2.17 and 2.20, of Corollary 2.20 and of Lemma 2.21 of [T] hold for complex spaces too.

We need the following lemma to generalize Theorem 26.10 to quasicomplete lcHs.

Lemma 26.17. Let u be a Radon operator on $\mathcal{K}(T)$ with values in an lcHs X. For $q \in \Gamma$, let $u_q = \Pi_q \circ u$ where Γ and Π_q are as in the beginning of §10 of [P17]. Then for each open set ω in T,

$$u_q^\bullet(\omega) = \sup_{x^* \in U_q^0} |x^* u|^\bullet(\omega) = \sup_{x^* \in U_q^0} |x^* u|(\omega)$$

where U_q^0 is as in Notation 10.13 of [P17] and $u_q^\bullet$ is as in Definition 25.18.

Proof. Let $q \in \Gamma$. Then $u_q : \mathcal{K}(T) \to \widetilde{X_q}$ is a continuous linear map and hence by Lemma 26.9, by Proposition 10.14 of [P17] and by Proposition 25.4 we have

$$\begin{aligned} u_q^\bullet(\omega) = \sup_{x^* \in U_q^0} |\Psi_{x^*} u_q|^\bullet(\omega) &= \sup_{x^* \in U_q^0} |\Psi_{x^*} \circ (\Pi_q \circ u)|^\bullet(\omega) \\ &= \sup_{x^* \in U_q^0} |x^* u|^\bullet(\omega) = \sup_{x^* \in U_q^0} |x^* u|^*(\omega) \\ &= \sup_{x^* \in U_q^0} |x^* u|(\omega) \end{aligned}$$

since $\Psi_{x^*}(\Pi_q \circ u)(x) = \Psi_{x^*}(ux + q^{-1}(0)) = x^* ux$ for $x \in X$ and for $x^* \in U_q^0$ and since the open set ω is $|x^* u|^*$-measurable.

The following theorem generalizes Theorem 26.10 to quasicomplete lcHs.

Theorem 26.18. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. Then the following statements are equivalent:

(i) Every bounded (complex) Borel function belongs to $\mathcal{L}_1(u)$.

(ii) Every bounded (complex) σ-Borel function belongs to $\mathcal{L}_1(u)$.

(iii) Every bounded (complex) Baire function belongs to $\mathcal{L}_1(u)$.

(iv) For every open set ω in T the weak integral $\int_\omega du$ belongs to X; i.e. there exists a vector x_ω in X such that

$$\int_\omega d(x^* u) = x^*(x_\omega)$$

for each $x^* \in X^*$. Then we say that the weak integral $\int_\omega du = x_\omega$.

(v) For every σ-Borel open set ω in T, the weak integral $\int_\omega du$ belongs to X.

(vi) For every open Baire set ω in T, the weak integral $\int_\omega$ belongs to X.

(vii) u is weakly compact (see Convention 26.4).

Proof. Since the results mentioned in 18.10 of [P18] hold for lcHs valued continuous linear transformations on $C_0(T)$ and since the theorems in [P9] used in the proof of Theorem 26.10 are valid not only for Banach spaces but also for quasicomplete lcHs, the proof of the latter theorem excepting that of (vii)$\Rightarrow$(i) continues to be valid when X is a quasicomplete lcHs.

Now we shall show that (vii)$\Rightarrow$(i). Let $q \in \Gamma$ and let $U_q = \{x \in X : q(x) \leq 1\}$. Then U_q^0, the polar of U_q is equicontinuous and hence by Corollary 9.3.2 of [E] or by Proposition 4 of [P9], $u^*(U_q^0)$ is relatively weakly compact in $M(T)$. Then by Theorem 1 of [P8], given $A \in \mathcal{B}(T)$ and $\epsilon > 0$, there exist a compact set K and an open set U in T such that $K \subset A \subset U$ and $\sup_{x^* \in U_q^0} |x^*u|(U \backslash K) < \epsilon$. Consequently, by Lemma 26.17, $u_q^\bullet(U \backslash K) < \epsilon$. Then arguing as in the last part of the proof of (vii)$\Rightarrow$(i) of Theorem 26.10 we have a $\varphi_q \in \mathcal{K}(T)$ such that $\chi_K \leq \varphi_q \leq \chi_U$ so that

$$u_q^\bullet(|\chi_A - \varphi_q|) \leq u_q^\bullet(\chi_U - \chi_A) \leq u_q^\bullet(\chi_U - \chi_K) = u_q^\bullet(U \backslash K) < \epsilon.$$

Since q is arbitrary in Γ, by Definition 25.20 $\chi_A \in \mathcal{L}_1(u)$.

Then every Borel simple function s belongs to $\mathcal{L}_1(u)$. If f is a bounded Borel (complex) function, then there exists a sequence (s_n) of Borel simple functions such that $||s_n - f||_T \to 0$. Then, for each $q \in \Gamma$, we have

$$u_q^\bullet(|f - s_n|) \leq ||f - s_n||_T u_q^\bullet(T) \to 0$$

as $n \to \infty$, since $u_q^\bullet(T)$ is finite by Proposition 26.3 and by the hypothesis that $u_q : (C_c(T), ||\cdot||_T) \to \widetilde{X_q}$ is continuous.

This completes the proof of the theorem.

The following theorem gives an improvement of the complex version of Theorem 2.7 bis of [T].

Theorem 26.19. Let X be an lcHs and let H be a subset of X^* such that the topology τ of X is the same as the topology of uniform convergence in the equicontinuous subsets of H. Suppose H has the Orlicz property. Let u be a bounded Radon operator on $\mathcal{K}(T)$ with values in X. Then u is weakly compact if and only if, for every open Baire set ω in T, the ultra weak integral $\int_\omega d\tilde{u}$ (relative to the topology $\sigma(X,H)$) belongs to $\widetilde{X}$, the lcHs completion of X.

Proof. By hypothesis, τ is generated by the seminorms $\{q_E : E \in H_{\mathcal{E}}\}$ where $H_{\mathcal{E}} = \{E \subset H : E \text{ is eqicontinuous}\}$ and $q_E(x) = \sup_{x^* \in E} |x^*(x)|$. As observed in the proof of Theorem 18.16 of [P18], $\sigma(X,H)$ is Hausdorff.

By hypothesis, for each open Baire set ω in T, there exists a vector $x_\omega \in \widetilde{X}$ such that

$$\int_\omega d(x^*u) = \mu_{x^*u}(\omega) = x^*(x_\omega) \qquad (26.19.1)$$

for each $x^* \in H$ and hence for $x^* \in < H >$, the linear span of H. Then, given a disjoint sequence $(U_n)_1^\infty$ of open Baire sets in T, for each subsequence P of $\mathbb{N}$, by (26.19.1) we have

$$\sum_{n \in P} x^*(x_{U_n}) = \sum_{n \in P} \mu_{x^*u}(U_n) = \mu_{x^*u}(\bigcup_{n \in P} U_n) \in \mathbb{K}$$

and hence $\sum_1^\infty x_{U_n}$ is subseries convergent for the topology $\sigma(X,H)$. Since $(X, \sigma(X, < H >)^* =< H >$ by Theorem 5.3.9 of [DS], $\sum_1^\infty x_{U_n}$ is subseries convergent in $\sigma(X,H)$. As H has the Orlicz property by hypothesis, $\sum_1^\infty x_{U_n}$ is unconditionally convergent in τ. Hence

$$\lim_n q_E(x_{U_n}) = 0. \qquad (26.19.2)$$

Then by (26.19.1) and by (26.19.2) we have

$$\lim_n q_E(x_{U_n}) = \lim_n \sup_{x^* \in E} |x^*(x_{U_n})| = \lim_n \sup_{x^* \in E} |\mu_{x^*u}(U_n)| = 0. \qquad (26.19.3)$$

Since E is equicontinuous and since $u : C_0(T) \to X$ is continuous, by Lemma 2 of [P9] and by 18.10 of [P18], $u^*E = \{u^*x^* : x^* \in E\} = \{x^*u : x^* \in E\} = \{\mu_{x^*u} : x^* \in E\}$ is bounded in $M(T)$. Then by (26.19.3) and by Theorem 1 of [P8]

$$\{\mu_{x^*u} : x^* \in E\} \text{ is relatively weakly compact in } M(T). \qquad (26.19.4)$$

For $E \in H_{\mathcal{E}}$, $\Pi_{q_E} : \widetilde{X} \to \widetilde{X_{q_E}} \subset (\widetilde{\widetilde{X_{q_E}}})$. If Ψ_{x^*} is as in Proposition 10.12(i) of [P17] for $x^* \in E$, then $\{\Psi_{x^*} : x^* \in E\}$ is a norm determining subset of the closed unit ball of $(\widetilde{X_{q_E}})^*$ for $\widetilde{X_{q_E}}$ by Proposition 10.12(iii) of [P17] for $x^* \in E$. Then by Proposition 10.12(i) of [P17] and by 18.10 of [P18] we have

$$(\Psi_{x^*} \circ \Pi_{q_E} \circ u)(\varphi) = (x^* u)(\varphi) \qquad (26.19.5)$$

for $\varphi \in C_0(T)$ and hence $\Psi_{x^*} \circ \Pi_{q_E} \circ u \in \mathcal{K}(T)^*_b = (C_0(T), ||\cdot||_T)^* = M(T)$. Then by (26.19.4) and (26.19.5), $\{\mu_{(\Psi_{x^*} \circ \Pi_{q_E} \circ u)} : x^* \in E\}$ is relatively weakly compact in $M(T)$. Then by Corollary 18.15 of [P18], $\Pi_{q_E} \circ u$ is weakly compact for $E \in H_{\mathcal{E}}$. Consequently, by the complex analogue of Lemma 2.21 of [T], u is weakly compact.

Conversely, if u is weakly compact, then by Theorem 26.18(vi) the weak integral $\int_\omega f du$ belongs to X for each open Baire sets ω in T and hence there exists a vector $x_\omega \in X$ and hence in $\widetilde{X}$ such that

$$x^*(\int_\omega du) = \int_\omega d(x^* u) = x^*(x_\omega)$$

for each $x^* \in X^*$ and hence for each $x^* \in H$. Thus $\int_\omega d\tilde{u}$ (relative to the topology $\sigma(X, H)$) belongs to $\widetilde{X}$.

This completes the proof of the theorem.

Corollary 26.20. Under the hypothesis of Theorem 26.19 for X, H and the topology τ, a bounded Radon operator $u : \mathcal{K}(T) \to X$ is weakly compact (see Convention 26.4) if for each open set ω in T which is a countable union of closed sets, the ultra weak integral $\int_\omega d\tilde{u}$ (relative to the topology $\sigma(X, H)$) belongs to $\widetilde{X}$, the lcHs completion of X.

Proof. By Lemma 18.3 and by Theorem 26.19, the corollary holds.

Remark 26.21. Corollary 26.20 is obtained directly in Proposition on p. 98 of [T]. But Theorem 26.19 is much stronger than the said proposition of [T].

27. INTEGRATION WITH RESPECT TO A PROLONGABLE RADON OPERATOR

Following Thomas [T] we study the integration of complex functions with respect to a prolongable Radon operator u on $\mathcal{K}(T)$ (Thomas calls it a prolongable Radon measure) and improve most of the principal results such as the complex versions of Theorems 3.3, 3.4, 3.11, 3.13 and 3.20 of [T].

Definition 27.1. Let u be a Radon operator on $\mathcal{K}(T)$ with values in an lcHs. Then we say that u is prolongable if every bounded (complex) Borel function with compact support is u-integrable.

Notation 27.2. Let ω be an open set in T and let u be a Radon operator on $\mathcal{K}(T)$. We define $\hat{f}$ on T by $\hat{f}(t) = f(t)$ for $t \in \omega$ and 0 for $t \in T\backslash\omega$. For $\varphi \in \mathcal{K}(\omega)$, $\hat{\varphi} \in \mathcal{K}(T)$ and we identify $\mathcal{K}(\omega)$ with the set of functions in $\mathcal{K}(T)$ whose support is contained in ω. The Radon operator u_ω is defined as the restriction of u to $\mathcal{K}(\omega)$. i.e. $u_\omega(\varphi) = u(\hat{\varphi})$.

Lemma 27.3. Let u be a Radon operator on $\mathcal{K}(T)$ with values in a normed space, ω an open subset of T and u_ω the Radon operator induced by u on $\mathcal{K}(\omega)$. Then:

(i) For $f \in \mathcal{I}^+(\omega)$, $\hat{f} \in \mathcal{I}^+(T)$ and $(u_\omega)^\bullet(f) = u^\bullet(\hat{f})$.

(ii) For $f \geq 0$ with compact support in ω, $(u_\omega)^\bullet(f) = u^\bullet(\hat{f})$.

(iii) If f is a (complex) function with compact support in ω belonging to $\mathcal{L}_1(u_\omega)$, then$\hat{f}$ belongs to $\mathcal{L}_1(u)$ and $\int f du_\omega = \int \hat{f} du$ and the last conclusion also extends to Radon operators with values in an lcHs.

For the proof of the above lemma we refer to the proof of Lemma 3.2 of Thomas [T] given in Appendix III of [T] which holds for complex functions too.

Remark 27.4. If $u^\bullet(\omega) < \infty$ for ω in Lemma 27.3, then u_ω is a bounded operator on $\mathcal{K}(\omega)$ and hence particularly if ω is relatively compact in T, then u_ω is a bounded Radon operator.

Since $u^\bullet(\omega) = \sup_{|\varphi| \leq 1, \varphi \in \mathcal{K}(\omega)} |u_\omega(\varphi)|$, the above remark holds.

The following theorem improves Theorem 3.3 of Thomas [T].

Theorem 27.5. Let u be a Radon operator on $\mathcal{K}(T)$ with values in a quasicomplete lcHs X. Then the following statements are equivalent:

(i) u is prolongable in the sense of Definition 27.1.

(ii) Every bounded σ-Borel (complex) function with compact support belongs to $\mathcal{L}_1(u)$.

(iii) Every bounded complex Baire function with compact support belongs to $\mathcal{L}_1(u)$.

(iv) For each relatively compact open set ω in T, the weak integral $\int_\omega du$ belongs to X; i.e. there exists $x_\omega \in X$ such that $\int_\omega d(x^*u) = x^*(x_\omega)$ for $x^* \in X^*$.

(v) For each relatively compact open Baire set ω in T, the weak integral $\int_\omega du$ belongs to X.

(vi) If ω is a relatively compact open set in T, then $u|_{\mathcal{K}(\omega)}$ is a bounded weakly compact Radon operator.

(vii) For each relatively compact open Baire set ω in T, $u|_{\mathcal{K}(\omega)}$ is a bounded weakly compact Radon operator.

(viii) For each compact K in T, the weak integral $\int_K du$ belongs to X.

(ix) For each compact G_δ K in T, the weak integral $\int_K du$ belongs to X.

(x) A set $A \subset \mathcal{K}(T)$ is said to be bounded in $\mathcal{K}(T)$ if there exists $K \in \mathcal{C}$ such that supp $\varphi \subset K$ for each $\varphi \in A$ and $\sup_{\varphi \in A} ||\varphi||_T < \infty$. For each relatively compact open set ω in T, u transforms bounded subsets of $\mathcal{K}(\omega)$ into relatively weakly compact subsets of X.

(xi) For each compact K, $\lim_{\omega \searrow K} u^\bullet(\omega \backslash K) = 0$ where ω is open in T.

(xii) For every compact G_δ K, $\lim_{\omega \searrow K} u^\bullet(\omega \backslash K) = 0$ where ω is open in T.

Proof. Let $\mathcal{E}$ be the family of all equicontinuous subsets of X^*.

(i)$\Leftrightarrow$(ii) As shown in the proof of (1)$\Rightarrow$(18) of Theorem 19.12 of [P18], a Borel function with compact support is σ-Borel and a σ-Borel function is obviously Borel. Hence (i)$\Leftrightarrow$(ii).

(ii)$\Rightarrow$(iii) Obvious.

(iii)⇒(v) If ω is a relatively compact open Baire set in T, then χ_ω is a bounded Baire function with compact support and hence by (iii), $\chi_\omega \in \mathcal{L}_1(u)$. Then by Theorem 25.24 there exists a vector $x_\omega \in X$ such that $\int_\omega du = x_\omega$ and hence $\int_\omega d(x^*u) = x^*(x_\omega)$ for $x^* \in X^*$. Therefore, (v) holds.

(v)⇒(vi) Let ω be a relatively compact open set in T. Let $(\omega_n)_1^\infty$ be a disjoint sequence of open Baire sets in ω. Then $(\omega_n)_1^\infty \subset \mathcal{B}_0(T)$ as shown in the proof of Claim 1 in the proof of Theorem 19.12 of [P18]. Let $P \subset \mathbb{N}$ and let $\omega_P = \bigcup_{n\in P} \omega_n$. Then by (v) and by Theorem 25.24 there exists $x_P \in X$ and $(x_{\omega_n})_{n\in P} \subset X$ such that

$$x^*(x_{\omega_n}) = \int_{\omega_n} d(x^*u) \qquad (27.5.1)$$

and

$$x^*(x_P) = \int_{\omega_P} d(x^*u) = \sum_{n\in P} \int_{\omega_n} d(x^*u) = \sum_{n\in P} x^*(x_{\omega_n})$$

for $x^* \in X^*$. Thus $\sum_1^\infty x^*(x_{\omega_n})$ is subseries convergent for each $x^* \in X^*$ and hence by the Orlicz-Pettis theorem, $\sum_1^\infty x_{\omega_n}$ is unconditionally convergent in X. In other words, by (27.5.1) we have

$$\begin{aligned} \lim_n q_E(x_{\omega_n}) &= \lim_n \sup_{x^*\in E} |x^*(x_{\omega_n})| = \lim_n \sup_{x^*\in E} |(x^*u)(\omega_n)| \\ &= \lim_n \sup_{x^*\in E} |(u^*x^*)(\omega_n)| = \lim_n \sup_{\mu\in u^*E} |\mu(\omega_n)| = 0. \end{aligned}$$

Therefore the bounded set u^*E (see Lemma 2 of [P9]) is relatively weakly compact in $M(T)$ by Theorem 1 of [P8]. Since E is arbitrary in $\mathcal{E}$, by Proposition 4 of [P9] $u|_{\mathcal{K}(\omega)}$ is a weakly compact Radon operator (see Convention 26.4 with respect to $\mathcal{K}(\omega)$). Hence (vi) holds.

(vi)⇒(vii) Obvious.

(vii)⇒(i)
Claim. If A is a relatively compact Borel set, then $\chi_A \in \mathcal{L}_1(u)$ and consequently, each Borel simple function with compact support belongs to $\mathcal{L}_1(u)$.

In fact, $\bar{A} = K \in \mathcal{C}$ and hence by Theorem 50.D of [H], there exists a relatively compact open Baire set ω_0 such that $K \subset \omega_0$. By (vii), $u_{\omega_0} = u|_{C_0(\omega_0)}$ is weakly compact. Let $E \in \mathcal{E}$. Then $u_{q_E} = \Pi_{q_E} \circ u_{\omega_0} : C_0(\omega_0) \to \widetilde{X_{q_E}}$ is weakly compact and hence, given $\epsilon > 0$, by Theorem

26.11 there exist a compact C and an open set ω in ω_0 (hence open in T) such that $C \subset A \subset \omega$ and $(u_{\omega_0})^\bullet_q(\omega \backslash C) < \epsilon$. By Urysohn's lemma there exists $\varphi \in C_c(\omega)$ such that $0 \leq \varphi \leq 1$ and $\varphi|_C = 1$. Then $(u_{\omega_0})^\bullet_{q_E}(|\chi_A - \varphi|) \leq (u_{\omega_0})^\bullet_{q_E}(\omega \backslash C) < \epsilon$. Since E is arbitrary in $\mathcal{E}$, this shows that $\chi_A \in \mathcal{L}_1(u_{\omega_0})$. Since $\hat{\chi}_A = \chi_A$ by Lemma 27.3(iii), $\chi_A \in \mathcal{L}_1(u)$ and consequently, each Borel simple function with compact support belongs to $\mathcal{L}_1(u)$. Hence the claim holds.

Let f be a bounded (complex) Borel function with compact support. Then there exists a sequence (s_n) of Borel simple functions such that $|s_n| \nearrow |f|$ and $s_n \to f$ uniformly in T. Then supp $s_n \subset$ supp$f = K$ (say) for all n. Let $\omega \in \mathcal{V}$ such that $K \subset \omega$. Then by the above claim, $(s_n)_1^\infty \subset \mathcal{L}_1(u)$. Then, given $\epsilon > 0$, choose n_0 such that

$$||s_n - f||_T (u_\omega)^\bullet_{q_E}(\chi_\omega) < \epsilon$$

for $n \geq n_0$. Then

$$(u_\omega)^\bullet_{q_E}(|s_n - f|) \leq ||s_n - f||_T (u_\omega)^\bullet_{q_E}(\chi_\omega) < \epsilon$$

for $n \geq n_0$. Thus $f \in \mathcal{L}_1(u_\omega)$ and consequently, $\hat{f} \in \mathcal{L}_1(u)$ by Lemma 27.3(iii). Since $\hat{f}(t) = f(t)$ for $t \in \omega$ and $\hat{f}(t) = 0$ for $t \in T \backslash \omega$ and $K \subset \omega$, $\hat{f} = f$ and hence $f \in \mathcal{L}_1(u)$. Thus (i) holds.

(i)⇒(iv) Let $\omega \in \mathcal{V}$. Then by Definition 27.1, $\chi_\omega \in \mathcal{L}_1(u)$ and hence $\int_\omega = x_\omega$ belongs to X. Then

$$\int_\omega d(x^*u) = x^*(\int_\omega du) = x^*(x_\omega)$$

and hence (iv) holds.

As shown above, (iii)⇒(v)⇒(vi)⇒(vii)⇒(i).

(i)⇒(viii) Obvious.

(viii)⇒(v) Let ω be a relatively compact open Baire set in T. Then $\bar{\omega}$ and the boundary of ω are compact. Hence there exist vectors $x_{\bar{\omega}}$ and y in X such that $x^*(x_{\bar{\omega}}) = \int_{\bar{\omega}} d(x^*u)$ and $x^*(y) = \int_A d(x^*u)$ for $x^* \in X^*$, where A is the boundary of ω. Then $x^*(x_{\bar{\omega}} - y) = \int_\omega d(x^*u)$ for $x^* \in X^*$. Hence (v) holds.

(viii)$\Rightarrow$(ix) Obvious.

(ix)$\Rightarrow$(v) Let V be a relatively compact open Baire set in T. Then arguing as in the proof of (7)$\Rightarrow$(5) of Theorem 19.12 of [P18], $V = K\backslash(K\backslash V)$ with $K \in \mathcal{C}_0$ and $K\backslash V \in \mathcal{C}_0$. Then by (ix) there exist x_K and $x_{K\backslash V}$ in X such that $\int_K d(x^*u) = x^*(x_K)$ and $\int_{K\backslash V} d(x^*u) = x^*(x_{K\backslash V})$ for $x^* \in X^*$. Then $x^*(x_K - x_{K\backslash V}) = \int_K d(x^*u)$ for $x^* \in X^*$. Hence (v) holds.

(i)$\Rightarrow$(x) Let $\omega \in \mathcal{V}$ and let $A \subset \mathcal{K}(\omega)$ be bounded. Then there exists $K \in \mathcal{C}$ such that supp $\varphi \subset K$ for $\varphi \in A$ and $\sup_{\varphi\in A} ||\varphi||_T = M < \infty$. Then by Theorem 50.D of [H] there exists a relatively compact open set ω in T such that $K \subset \omega$. Then $A \subset C_c(\omega)$. Then by (i), $u(A)$ is relatively weakly compact in X.

(x)$\Rightarrow$(vi) Let $\omega \in \mathcal{V}$ and let $A = \{\varphi \in C_0(\omega) : ||\varphi||_\omega \leq 1\}$. Then A is bounded in $\mathcal{K}(T)$ and hence by (x), $u(A)$ is relatively weakly compact in X. Hence $u_\omega = u|_{\mathcal{K}(\omega)}$ is a bounded weakly compact Radon operator. Hence (vi) holds.

(xi)$\Rightarrow$(viii) Suppose (xi) holds. Then given $\epsilon > 0$, there exists an open set $\omega \supset K$ such that $u^\bullet(\omega\backslash K) < \epsilon$. Then by Urysohn's lemma, there exists $\varphi \in C_c(T)$ such that $\chi_K \leq \varphi \leq \chi_\omega$ so that $u^\bullet(\varphi - \chi_K) < \epsilon$. Hence $\chi_K \in \mathcal{L}_1(u)$ so that $\int_K du \in X$. Then the weak integral $\int_K d\tilde{u} \in X$ and hence (viii) holds.

(xi)$\Rightarrow$(xii) Obvious.

(xii)$\Rightarrow$(ix) Let K be a compact G_δ in T. Then given $\epsilon > 0$, by (xii) there exists an open set ω in T such that $K \subset \omega$ and $u^\bullet(\omega\backslash K) < \epsilon$. By Urysohn's lemma there exists $\varphi \in c_c(T)$ such that $\chi_K \leq \varphi \leq \chi_\omega$ so that $u^\bullet(\varphi - \chi_K) < \epsilon$. Hence $\chi_K \in \mathcal{L}_1(u)$ so that $\int_K du \in X$. Particularly, the weak integral $\int_K du \in X$ and hence (ix) holds.

Thus (i)-(xii) are equivalent and this completes the proof of the theorem.

The following theorem is analogous to Theorems 26.13 and improves Theorem 3.4 of Thomas [T].

Theorem 27.6. (a) Let u be a Radon operator on $\mathcal{K}(T)$ with values in a Banach space (X, τ) and let $H \subset X^*$ be a norm determining set for X with the Orlicz property (respectively, (b) let u be a Radon operator on $\mathcal{K}(T)$ with values in a quasicomplete lcHs (X, τ) and let $H \subset X^*$ have the Orlicz property and let the topology τ of X be identical with the topology of uniform convergence in the equicontinuous subsets of H). Let $\tilde{u}$ be the operator obtained from u on providing X with the topology $\sigma(X, H)$ and let $\widetilde{X}$ be the lcHs completion of (X, τ). Then u is prolongable if and only if anyone of the following conditions holds:

(i) For each $\omega \in \mathcal{V}$ the ultra weak integral $\int_\omega d\tilde{u}$ belongs to $\widetilde{X}$; i.e. there exists $x_\omega \in \widetilde{X}$ such that

$$x^*(x_\omega) = \int_\omega d(x^*u)$$

for $x^* \in H$.

(ii) Similar to (i) with $\omega \in \mathcal{B}_0(T) \bigcap \mathcal{V}$.

(iii) For each $K \in \mathcal{C}$, the ultra weak integral $\int_K d\tilde{u}$ belongs to $\widetilde{X}$ (see (i)).

(iv) Similar to (iii) with $K \in \mathcal{C}_0$.

Proof. Since (b) subsumes (a), we shall prove (b).

Let $\mathcal{E}_H$ be the family of equicontinuous subsets of H. Let $\omega \in \mathcal{V}$. If u is prolongable, then $u|_{\mathcal{K}(\omega)}$ is a bounded weakly compact Radon operator and hence by Theorem 26.19, $\int_\omega d\tilde{u} \in \widetilde{X}$ so that (i) holds.

Clearly, (i)$\Rightarrow$(ii).

Let (ii) hold. Let $E \in \mathcal{E}_H$. For each $\omega \in \mathcal{B}_0(T) \bigcap \mathcal{V}$, there exists $x_\omega \in \widetilde{X}$ such that

$$x^*(x_\omega) = \int_\omega d(x^*u) \qquad (27.6.1)$$

for $x^* \in H$. Arguing as in the proof of Theorem 26.19, given a disjoint sequence $(\omega_n)_1^\infty$ of open Baire sets with $\bigcup_1^\infty \omega_n \subset \omega$ and using (27.6.1) in place of (26.19.1), we have

$$\lim_n q_E(x_{\omega_n}) = 0. \qquad (27.6.2)$$

Since E is equicontinuous and $u : C_0(\omega) \to X$ is continuous, by Lemma 2 of [P9] u^*E is bounded in $M(T)$. Then arguing as in the proof of Theorem 26.19, we conclude that $\Pi_{q_E} \circ u : \mathcal{K}(\omega) \to \widetilde{X_{q_E}}$ is a weakly compact Radon operator for $E \in \mathcal{E}_H$. Then by the complex version of Lemma 2.21 of [T], u is prolongable.

Conversely, let u be prolongable and let $K \in \mathcal{C}$. Then by Theorem 50.D of [H] there exists $U \in \mathcal{V}$ such that $K \subset U$. Then by (i), the ultra weak integrals $\int_U d\tilde{u} = x_U$ and $\int_{U\setminus K} d\tilde{u} = x_{U\setminus K}$ belong to $\widetilde{X}$. Then

$$\int_K d(x^*u) = \int_U d(x^*u) - \int_{U\setminus K} d(x^*u) = x^*(x_U - x_{U\setminus K})$$

for $x^* \in H$. Thus the ultra weak integral $\int_K d\tilde{u}$ belongs to $\widetilde{X}$ and hence (iii) holds.

(iii)$\Rightarrow$(iv) Obvious.

Let (iv) hold. Let ω be a relatively compact open Baire set. Then $\bar{\omega} \in \mathcal{C}$ and hence by Theorem 50.D of [H] there exists $K \in C_0$ such that $\bar{\omega} \subset K$. Then $\omega = K\setminus(K\setminus\omega)$ and $K\setminus\omega \in \mathcal{C}_0$ by Theorem 51.D of [H]. Then by hypothesis, there exist vectors x_K and $x_{K\setminus\omega}$ in $\widetilde{X}$ such that $\int_K d(x^*u) = x^*(x_K)$ and $\int_{K\setminus\omega} d(x^*u) = x^*(x_{K\setminus\omega})$ for $x^* \in H$. Consequently, $x^*(x_K - x_{K\setminus\omega}) = \int_\omega d(x^*u)$ for $x^* \in H$. Therefore, (iv)$\Rightarrow$(i).

Let (i) hold. Then, particularly, for each open Baire set $\omega \in \mathcal{V}$, $u|_{\mathcal{K}(\omega)}$ is a bounded weakly compact Radon operator by Theorem 26.19. If $V \in \mathcal{V}$, then $\bar{V} \in \mathcal{C}$ and hence by Theorem 50.D of [H] there exists an open Baire set $\omega \in \mathcal{V}$ such that $V \subset \bar{V} \subset \omega$. Then $u|_{\mathcal{K}(V)}$ is the restriction of u_ω in $\mathcal{K}(V)$ and hence $u|_{\mathcal{K}(V)}$ is weakly compact. Hence (i) implies that u is prolongable.

This completes the proof of the theorem.

Remark 27.7. The complex versions of Theorem 3.5, Corollary 3.6, Proposition 3.7 and Lemma 3.10 of [T] hold in virtue of Remark 26.16 above.

Proposition 27.8.Let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator where X is a quasicomplete lcHs. Then u is weakly compact if and only if u is prolongable and the function 1 belongs to $\mathcal{L}_1(u)$.

Proof. If u is weakly compact, then by Theorem 26.18 every bounded complex Borel function belongs to $\mathcal{L}_1(u)$ and hence the function 1 belongs to $L_1(u)$ and u is prolongable by Definition 27.1. Conversely, if u is prolongable and the function 1 is u-integrable, then by the complex version of Corollary 3.6 of [T] every bounded Borel function is u-measurable and hence every bounded complex Borel function is u-integrable by the complex version of Theorem 1.22 of [T]. Consequently, by Theorem 26.18, u is weakly compact.

Theorem 27.9. Let X be a Banach space (resp. a quasicomplete lcHs) and let $u : \mathcal{K}(T) \to X$ be a prolongable Radon operator. Then a scalar function f on T is u-integrable if and only if f is weakly u-integrable and for every open Baire set ω in T, the weak integral $\int_\omega f du$ belongs to X (resp. the weak integral $\int_\omega f d\tilde{u} \in X$ where $\tilde{u}$ is the Radon operator obtained from u on providing X with the topology $\sigma(X, X^*)$).

Proof. First we prove the theorem when X is a Banach space. By the complex versions of Theorem 1.22 and Corollary 3.6 of [T], the condition is necessary. Let $\tilde{u}$ be the operator obtained from u on providing X with the topology $\sigma(X, X^*)$ and let the hypothesis hold for each open Baire set. Then $\int_\omega f d\tilde{u} \in X$ for each open Baire set ω in T.

Let $\mathcal{F}$ be the complex vector space generated by the characteristic functions of open Baire sets in T, provided with the supremum norm. Then arguing as in the proof of Theorem 3.11 of [T] we have

$$\sup_{|g| \leq 1, g \in \mathcal{F}} \left| \int g f d\tilde{u} \right| < \infty.$$

Consequently, the mapping $\Phi : \mathcal{F} \to X$ given by

$$\Phi(g) = \int g f d\tilde{u}$$

is continuous. Then by Claim 4 in the proof of Theorem 22.3 of [P19], $C_0(T) \subset \bar{\mathcal{F}}$, where $\bar{\mathcal{F}}$ is the closure of $\mathcal{F}$ in the Banach space of all bounded complex functions on T.

If Ψ is the continuous extension of Φ to $C_0(T)$, then

$$< \Psi(\varphi), x^* >= \int \varphi f du_{x^*}$$

for $x^* \in X^*$, as there exists $(g_n)_1^\infty \subset \mathcal{F}$ such that $||g_n - \varphi||_T \to 0$ and $\Psi(g_n) = \Phi(g_n) = \int g_n f d\tilde{u}$. Thus

$$\Psi(\varphi) = \int \varphi f d\tilde{u},$$

$\int \varphi f d\tilde{u}$ belongs to X and Ψ is continuous on $C_0(T)$. Thus Ψ is a bounded Radon operator on $\mathcal{K}(T)$. By Theorem 55.A of [H], there exists $(\varphi_n)_1^\infty \subset C_0(T)$ such that $\varphi_n \searrow \chi_\omega$ since ω is an open Baire set. Then by LDCT we have

$$\int_\omega d(x^*\Psi) = \lim_n \int_T \varphi_n f d(x^* u) = \int_\omega f d(x^* u)$$

for $x^* \in X^*$. Thus $\int_\omega d\tilde{\Psi} = \int_\omega f d\tilde{u}$ which belongs to X by hypothesis. Then by the equivalence of (vi) and (vii) of Theorem 26.10, Ψ is weakly compact. Since $d\Psi_{x^*} = f du_{x^*}$ and since by hypothesis f is u_{x^*}-measurable for $x^* \in X^*$, the function f is Ψ_{x^*}-measurable for $x^* \in X^*$. Then by the complex version of Theorem 1.28 of [T], f is Ψ-measurable and consequently, by Theorem 26.11(vii), given $\epsilon > 0$, there exists a compact K such that $\Psi^\bullet(T \backslash K) < \frac{\epsilon}{2}$. As f is Ψ-measurable, there exists a compact $K_0 \subset K$ such that $f|_{K_0}$ is continuous and $\Psi^\bullet(K \backslash K_0) < \frac{\epsilon}{2}$. Then $\Psi^\bullet(T \backslash K_0) < \epsilon$. By Proposition 25.15, by Lemma 25.11 and by the fact that $d\Psi_{x^*} = f du_{x^*}$ we have

$$\begin{aligned} \epsilon > \Psi^\bullet(T\backslash K_0) = \sup_{|x^*|\le 1} |\Psi_{x^*}|^\bullet(T\backslash K_0) &= \sup_{|x^*|\le 1} |\Psi_{x^*}|(T\backslash K_0) \\ &= \sup_{|x^*|\le 1} \int_{T\backslash K_0} |f| d|u_{x^*}| \\ &= \sup_{|x^*|\le 1} \int_T |f - \chi_{K_0} f| d|u_{x^*}| \end{aligned}$$

since $\chi_{T\backslash K_0} \in \mathcal{L}_1(\Psi)$.

Since $\chi_{K_0} f$ is continuous in K_0, $\chi_{K_0} f$ is bounded (as K_0 is compact) and as u is prolongable, by Definition 27.1 $\chi_{K_0} f$ is u-integrable. Then by the complex version of Lemma 3.10 of [T], $f \in \mathcal{L}_1(u)$.

Now let X be a quasicomplete lcHs. For $q \in \Gamma$, let $\Pi_q : X \to X_q \subset \widetilde{X_q}$. Then Π_q is linear and continuous. If $y^* \in {X_q}^*$, then $y^* \circ \Pi_q \in X^*$. As f is weakly u-integrable, $f \in \mathcal{L}_1(x^* u)$ for each $x^* \in X^*$. Moreover, by hypothesis for each open Baire set $\omega \in T$ there exists a vector x_ω belonging to X such that

$$(y^* \circ \Pi_q)(x_\omega) = \int_\omega f d(y^* \circ \Pi_q) u$$

for $y^* \in X_q^*$. Hence $\Pi_q(\int_\omega f d\tilde{u}) = \int_\omega f d(\widetilde{\Pi_q \circ u}) = \Pi_q(x_\omega) \in \widetilde{X_q}$ for $q \in \Gamma$. Hence by the Banach space case, $f \in \mathcal{L}_1(\Pi_q \circ u)$ for each $q \in \Gamma$. Consequently, by the complex version of Proposition 1.28 of [T], f is u-integrable and therefore, by Theorem 25.24, $\int f du \in X$.

By the complex versions of Theorem 1.22 extended to lcHs (see p.77 of [T]) and of Corollary 3.6 of [T] and by **1.12** of [T] the conditions are also necessary.

This completes the proof of the theorem.

Lemma 27.10. Let $(\theta_n)_1^\infty \subset \mathcal{K}(T)^*$ be such that for each $\varphi \in \mathcal{K}(T)$,

$$\sum_1^\infty |\theta_n(\varphi)| < \infty.$$

Let $u(\varphi) = (\theta_n(\varphi))_1^\infty$ for $\varphi \in \mathcal{K}(T)$. Then u is a prolongable Radon operator with values in $\ell_1(\mathbb{N})$. Let f be a complex function which is θ_n-integrable for each $n \in \mathbb{N}$ such that

$$\sum_1^\infty |\int_\omega f d\mu_n| < \infty$$

for each open Baire set ω in T, where μ_n is the complex Radon measure induced by θ_n in the sense of Definition 4.3 of [P3]. Then $f \in \mathcal{L}_1(u)$. If $s(\varphi) = \sum_1^\infty \int \varphi d\mu_n$, then $s \in \mathcal{K}(T)^*$, f is s-integrable and

$$\int_\omega f ds = \sum_1^\infty \int_\omega f d\mu_n$$

for each open Baire set ω in T.

Proof. The proof of Lemma 3.14 of [T] holds here for the complex case too. Only change is that we have to use Corollary 18.5 of [P18] in place of Appendix I, T4 of [T]. The details are left to the reader.

The following theorem improves the complex version of Theorem 3.13 of [T].

Theorem 27.11. Let u be a prolongable Radon operator on $\mathcal{K}(T)$ with values in a Banach space X and let H be a norm determining set in X^*. Suppose H has the Orlicz property. Let $\tilde{u}$ be the operator obtained from u on providing X with the topology $\sigma(X, H)$. Then a complex function f on T is u-integrable if and only if f is $\tilde{u}$-integrable (i.e., f is u_{x^*}-integrable for each $x^* \in H$) and for each open Baire set ω in T, the integral $\int_\omega f d\tilde{u}$ belongs to X.

Proof. The proof of Theorem 3.13 of [T] holds here verbatim excepting that we have to apply Corollary 18.5 of [P18] instead of Appendix I T4 of [T]. The details are left to the reader.

The following theorem improves the complex version of Theorem 3.20 of [T].

Theorem 27.12. Let u be a Radon operator on $\mathcal{K}(T)$ with values in a complete lcHs X. Let H be a subset of X^* with the Orlicz property and let the topology τ of X be the same as the topology of uniform convergence in the equicontinuous subsets of H. Let $\tilde{u}$ be the operator obtained from u on providing X with the topology $\sigma(X, H)$. Then a complex function f on T is u-integrable if and only if f is u_{x^*}-integrable for each $x^* \in H$ and for each open Baire set ω in T, $\int_\omega d\tilde{u} \in X$ (i.e., there exists $x_\omega \in X$ such that $x^*(x_\omega) = \int_\omega f d(x^* u)$ for $x^* \in H$).

Proof. The proof of Theorem 3.20 holds here verbatim excepting that we have to use Theorem 26.19 in place of Theorem 2.7 bis of [T]. The details are left to the reader.

28. BAIRE VERSIONS OF PROPOSITION 4.8 AND THEOREM 4.9 OF [T]

Using the Baire version of the Diedonné-Grothendieck theorem we give a complex Baire version of Proposition 4.8 and Theorem 4.9 of [T] including that of the remark on p. 128 of [T]. For this we start with the following two lemmas.

Lemma 28.1. Let u be a prolongable operator on $\mathcal{K}(T)$ with values in a Banach space X and let $f \in \mathcal{L}_1(u)$. Then the operator $\Psi : C_0(T) \to X$ given by

$$\Psi(\varphi) = \int \varphi f du \text{ for } \varphi \in C_0(T)$$

is weakly compact.

Proof. By the complex version of Theorem 1.22 of [T], $\varphi f \in \mathcal{L}_1(u)$ for $\varphi \in C_0(T)$ and hence

Ψ is well defined. Clearly, Ψ is linear and

$$|\Psi(\varphi) \leq ||\varphi||_T u^\bullet(f)$$

for $\varphi \in C_0(T)$. Hence Ψ is continuous.

Since $f \in \mathcal{L}_1(u)$, by the complex versions of Theorem 1.22 and Corollary 3.6 of [T], $\chi_\omega f \in \mathcal{L}_1(u)$ for each open Baire set ω in T and thus

$$x_\omega = \int_\omega f du \in X. \qquad (28.1.1)$$

Then

$$x^*(x_\omega) = \int_\omega f d(x^* u) \qquad (28.1.2)$$

for each open Baire set ω in T.

Let $\mathcal{F}$ be the vector space spanned by the characteristic functions of open Baire sets in T and let it be provided with the supremum norm. Then by (28.1.1), for each $g \in \mathcal{F}$ there exists $x_g \in X$ such that

$$x_g = \int g f du \qquad (28.1.3)$$

so that

$$x^*(x_g) = \int g f d(x^* u) \qquad (28.1.4)$$

for $x^* \in X^*$.

Let $\Phi(g) = x_g$ for $g \in \mathcal{F}$. Then $\Phi : \mathcal{F} \to X$ is linear and

$$|\Phi(g)| = |\int g f du| \leq ||g|||_T u^\bullet(f).$$

Hence Φ is continuous. Therefore, Φ has a unique continuous linear extension $\hat{\Phi}$ on the closure $\bar{\mathcal{F}}$ in the Banach space of all bounded complex functions on T with the supremum norm. Then $C_0(T) \subset \bar{\mathcal{F}}$ by Claim 4 in the proof of Theorem 22.3 of [P19] and hence let $\Phi_0 = \hat{\Phi}|_{C_0(T)}$. Thus $\Phi_0 : C_0(T) \to X$ is continuous and linear and hence by Theorem 1 of [P9] its representing measure $\boldsymbol{\eta}$ is given by $\Phi_0^{**}|_{\mathcal{B}(T)}$. Moreover, by the same theorem, $(x^* \circ \boldsymbol{\eta}) \in M(T)$ for each $x^* \in X^*$ and

$$x^* \Phi_0(\varphi) = \int \varphi d(x^* \circ \boldsymbol{\eta}) \qquad (28.1.5)$$

for $\varphi \in C_0(T)$ and for $x^* \in X^*$.

Let $\varphi \in C_0(T)$. Then there exists $(g_n)_1^\infty \subset \mathcal{F}$ such that $g_n \to \varphi$ uniformly in T so that $\Phi_0(\varphi) = \lim_n \Phi(g_n)$. Then by (28.1.4) we have

$$x^*\Phi_0(\varphi) = \lim_n x^*\Phi(g_n) = \lim_n x^*(x_{g_n}) = \lim_n \int f g_n d(x^*u) \qquad (28.1.6)$$

for $x^* \in X^*$. Since $f \in \mathcal{L}_1(u)$, $f \in \mathcal{L}_1(x^*u)$ for $x^* \in X^*$ and hence $\int_T |f|d|x^*u| < \infty$ for each $x^* \in X^*$. Moreover, by the complex version of Theorem 1.22 of [T], $f\varphi \in \mathcal{L}_1(u)$ and hence $f\varphi \in \mathcal{L}_1(x^*u)$ for $x^* \in X^*$. Since $f \in \mathcal{L}_1(u)$, $f \in L_1(x^*u)$ for $x^* \in X^*$ and hence

$$\int |f|d|x^*u| < \infty. \qquad (28.1.7)$$

Consequently, by (28.1.7) we have

$$|\int f\varphi d(x^*u) - \int f g_n d(x^*u)| \leq |\varphi - g_n||_T \int |f|d|x^*u| \to 0$$

as $n \to \infty$ and therefore

$$\int f\varphi d(x^*u) = \lim_n \int f g_n d(x^*u) \qquad (28.1.8)$$

for $x^* \in X^*$. Then by (28.1.6) and (28.1.8) we have

$$x^*\Phi_0(\varphi) = \int f\varphi d(x^*u) \text{ for } x^* \in X^*. \qquad (28.1.9)$$

Let ω be an open Baire set in T. Then by Theorem 55.A of [H] there exists $(\varphi_n)_1^\infty \subset C_0(T)$ such that $\varphi_n \searrow \chi_\omega$. Consequently, by LDCT, by (28.1.2), by (28.1.5) and by (28.1.9) we have

$$\begin{aligned}(x^* \circ \boldsymbol{\eta})(\omega) = \int \chi_\omega d(x^* \circ \boldsymbol{\eta}) &= \lim_n \int \varphi_n d(x^* \circ \boldsymbol{\eta}) \\ &= \lim_n x^*\Phi_0(\varphi_n) = \lim_n \int f\varphi_n d(x^*u) \\ &= \int \chi_\omega f d(x^*u) = x^*(x_\omega).\end{aligned}$$

Thus

$$(x^* \circ \boldsymbol{\eta})(\omega) = x^*(x_\omega) \text{ for } x^* \in X^*$$

and for each open Baire set ω in T. Thus $\boldsymbol{\eta}(\omega) = x_\omega$ and hence by Theorem 3(vii) of [P9], Φ_0 is weakly compact.

On the other hand, by (28.1.9) and by the definition of Ψ (see the statement of Lemma 28.1)) we have

$$x^*\Psi(\varphi) = \int f\varphi d(x^*u) = x^*\Phi_0(\varphi)$$

for $\varphi \in C_0(T)$ and for $x^* \in X^*$. Then by the Hahn-Banach theorem, $\Psi = \Phi_0$ and hence Ψ is weakly compact.

Lemma 28.2. Let u, f, X and Ψ be as in Lemma 28.1 and let $M_\Psi = \{A \subset T : \chi_A \in \mathcal{L}_1(\Psi)\}$. Then:

(i) M_Ψ is a σ-algebra in T.

(ii) $\mathcal{B}(T) \subset M_\Psi$.

(iii) If $\boldsymbol{\mu}_{\boldsymbol{\Psi}}(A) = \int_A d\Psi$, then $\boldsymbol{\mu}_{\boldsymbol{\Psi}}(\omega) = \int_\omega f du$ for each open Baire set ω in T.

(iv) If $\boldsymbol{\lambda}(A) = \int_A f du$ for $A \in \mathcal{B}(T)$, then $\boldsymbol{\lambda}$ is σ-additive on $\mathcal{B}(T)$ and for each $x^* \in X^*$, $x^*\boldsymbol{\lambda}(\cdot)$ is Borel regular.

(v) $\boldsymbol{\mu}_{\boldsymbol{\Psi}}(A) = \boldsymbol{\lambda}(A)$ for $A \in \mathcal{B}(T)$ and consequently, $\boldsymbol{\lambda}$ is Borel regular.

(vi) For a bounded complex Borel function g on T

$$\int g d\Psi = \int g f du. \qquad (28.2.1)$$

Proof. Since Ψ is weakly compact by Lemma 28.1, (i) and (ii) hold by Theorem 29.4 (see the next section).

(iii) For $\varphi \in C_0(T)$, $\Psi(\varphi) = \int f\varphi du$ and hence

$$x^*\Psi(\varphi) = \int \varphi f d(x^*u) \qquad (28.2.1)$$

for $x^* \in X^*$. Then by Theorem 55.A of [H] there exists $(\varphi_n)_1^\infty \subset C_0(T)$ such that $\varphi_n \searrow \chi_\omega$. Then by Theorem 4.7 (LDCT) of [T] we have

$$\Psi(\omega) = \int \chi_\omega d\Psi = \lim_n \int \varphi_n d\Psi \qquad (28.2.2)$$

and hence by (28.2.1) and (28.2.2) and by LDCT for complex measures we have

$$x^*\Psi(\omega) = \lim_n \int \varphi_n d(x^*\Psi) = \lim_n (x^*\Psi)(\varphi_n) = \lim_n \int \varphi_n f d(x^*u) = \int \chi_\omega f d(x^*u)$$

for $x^* \in X^*$. Therefore

$$(x^*\Psi)(\omega) = \int_\omega f d(x^*u) \text{ for } x^* \in X^*.$$

Since $f \in \mathcal{L}_1(u)$, we have

$$\int_\omega f d(x^*u) = x^*(\int_\omega f du)$$

and hence

$$(x^*\Psi)(\omega) = x^*(\int_\omega f du) \text{ for } x^* \in X^*.$$

Then by the Hahn-Banach theorem and by the definition of $\boldsymbol{\mu}_\Psi$ we have

$$\boldsymbol{\mu}_\Psi(\omega) = \int_\omega f du$$

for open Baire sets ω in T. Hence (iii) holds.

(iv) As u is prolongable, by the complex version of Corollary 3.6 of [T] χ_A is u-measurable for each $A \in \mathcal{B}(T)$. As $|\chi_A f| \leq |f|$ for $A \in \mathcal{B}(T)$ and as $f \in \mathcal{L}_1(u)$, by the complex version of Theorem 1.22 of [T], $\chi_A f$ is u-integrable for each $A \in \mathcal{B}(T)$. Let $(A_n)_1^\infty$ be a disjoint sequence of Borel sets in T with $A = \bigcup_1^\infty A_n$. Then $\sum_{k=1}^n \chi_{A_k} \nearrow \chi_A$ and hence $(\sum_1^n \chi_{A_k})|f| \leq \chi_A|f| \in \mathcal{L}_1(u)$ and $(\sum_1^n \chi_{A_k})f \to \chi_A f$ in T. Hence by the complex version of Theorem 4.7 of [T]

$$\sum_1^\infty \int \chi_{Ak} f du = \sum_{k=1}^\infty \int_{A_k} f du = \int_A f du$$

and hence $\boldsymbol{\lambda}(\cdot)$ is σ-additive on $\mathcal{B}(T)$. Moreover,

$$x^*\boldsymbol{\lambda}(\cdot) = \int_{(\cdot)} f d(x^*u)$$

is Borel regular on $\mathcal{B}(T)$ by Theorem 23.6 of [P19] (see the beginning of the proof of Theorem 23.6 of [P19]) and hence (iv) holds.

(v) By (iii), $\boldsymbol{\lambda}(\omega) = \boldsymbol{\mu}_\Psi(\omega)$ for open Baire sets ω in T and hence by the Baire regularity of $\boldsymbol{\lambda}|_{\mathcal{B}_0(T)}$ and $\boldsymbol{\mu}_\Psi|_{\mathcal{B}_0(T)}$ we conclude that $\boldsymbol{\lambda}(A) = \boldsymbol{\mu}_\Psi(A)$ for $A \in \mathcal{B}_0(T)$. For $x^* \in X^*$, $x^*\boldsymbol{\lambda}(\cdot) = \int_{(\cdot)} f d(x^*u)$ is Borel regular and σ-additive on $\mathcal{B}(T)$ by Theorem 23.6 of [P19]. Since $\boldsymbol{\mu}_\Psi$ is Borel regular and σ-additive on $\mathcal{B}(T)$ by Theorem 29.4 (see Section 29), $x^*\boldsymbol{\mu}_\Psi$ is Borel regular and σ-additive on $\mathcal{B}(T)$, and hence by the uniqueness part of Proposition 1 of [DP1],

$$x^*\boldsymbol{\lambda}(A) = x^*\boldsymbol{\mu}_\Psi(A)$$

for $A \in \mathcal{B}(T)$ and for $x^* \in X^*$. Hence by the Hahn-Banach theorem, $\boldsymbol{\lambda}(A) = \boldsymbol{\mu}_\Psi(A)$ for $A \in \mathcal{B}(T)$. Consequently, $\boldsymbol{\lambda}$ is Borel regular on $\mathcal{B}(T)$.

(vi) By (iv) and (v)

$$\int s d\boldsymbol{\mu}_\Psi = \int sf du \qquad (28.2.3)$$

for a Borel simple function s. Given a bounded complex Borel function g, there exists $(s_n)_1^\infty$ of Borel simple functions such that $s_n \to g$ uniformly in T. Then by the complex version of **1.10** of [T] we have

$$|\int s_n f du - \int g f du| \leq ||s_n - g||_T u^\bullet(f) \to 0 \qquad (28.2.4)$$

as $n \to \infty$. Since $|\int g d\Psi - \int s_n d\Psi| = |\int (g - s_n) d\Psi| \leq ||g - s_n||_T \Psi^\bullet(T) \to 0$, by (iv) and (v) we have $\int g d\Psi = \lim_n \int s_n d\Psi = \lim_n \int s_n d\boldsymbol{\mu}_\Psi = \lim_n \int f s_n du = \int f g du$ by (28.2.3) and (28.2.4). Hence (vi) holds.

Theorem 28.3. Let u be a prolongable Radon operator on $\mathcal{K}(T)$ with values in a Banach space (resp. a quasicomplete lcHs) X. Let $(f_n)_1^\infty$ be a sequence of u-integrable complex functions converging u-a.e. to a function f in T. If the sequence $\int_\omega f_n du$ converges in τ (the topology of X) (respectively, converges weakly) in X for all open Baire sets ω in T, then the function f is u-integrable and $\int_\omega f_n du$ converges in τ (resp. weakly) to $\int_\omega f du \in X$ for each open Baire set ω in T. Moreover, for each bounded complex Borel function g on T,

$$\int f_n g du \to \int f g du \text{ in } \tau \text{ in } X$$

(resp.

$$\int f_n g du \to \int f g du \text{ weakly in } X.)$$

Proof. Let ω be an open Baire set in T. By hypothesis, there exists a vector $x_\omega \in X$ such that

$$\lim_n \int_\omega f_n du = x_\omega \text{ in } \tau \qquad (28.3.1)$$

(resp.

$$\lim_n \int_\omega f_n du = x_\omega \text{ weakly.} \qquad (28.3.1'))$$

In both the cases, by **1.34** of [T]

$$\lim_n \int_\omega f_n du_{x^*} = x^*(x_\omega) \qquad (28.3.2)$$

for $x^* \in X^*$. On the other hand, by hypothesis and by Theorem 23.6 of [P19] we have

$$\lim_n \int_\omega f_n du_{x^*} = \int_\omega f du_{x^*} \qquad (28.3.3)$$

for $x^* \in X^*$, since $u_{x^*} = x^*u \in \mathcal{K}(T)^*$. Then by (28.3.2) and (28.3.3) we have

$$x^*(x_\omega) = \int_\omega f du_{x^*} \qquad (28.3.4)$$

for each open Baire set ω in T and for each $x^* \in X^*$. Consequently, by the hypothesis that $f_n \to f$ u-a.e. in T so that $f_n \to f$ u_{x^*}-a.e. in T for $x^* \in X^*$ and by Theorem 23.6 of [P19] we have

$$f \in \mathcal{L}_1(x^*u) \qquad (28.3.5)$$

for $x^* \in X^*$. Then by (28.3.4) and (28.3.5) and by Theorem 27.9, f is u-integrable in both the cases of X.

Then for an open Baire set ω in T, by (28.3.4) we have

$$x^*(\int_\omega f du) = \int_\omega f d(x^*u) = \int_\omega f du_{x^*} = x^*(x_\omega)$$

for each $x^* \in X^*$. Since $\int_\omega f du \in X$, by the Hahn-Banach theorem we have

$$\int_\omega f du = x_\omega$$

and hence by (28.3.1) (resp. by (28.3.1'))

$$\int_\omega f_n du \to \int_\omega f du \text{ in } \tau \qquad (28.3.6)$$

(resp.

$$\int_\omega f_n du \to \int_\omega f du \text{ weakly.})$$

Let $x^* \in X^*$, X a quasicomplete lcHs and g be a bounded u-measurable complex function. Then g is an x^*u-measurable function. Clearly, $\theta = x^*u \in \mathcal{K}(T)^*$ and hence by Theorem 23.6 of [P19] we have

$$\lim_n \int g f_n d(x^*u) = \int g f d(x^*u). \qquad (28.3.7)$$

On the other hand, by the complex version of Theorem 1.22 of [T], gf_n and gf belong to $\mathcal{L}_1(u)$ and hence by (28.3.7) we have

$$\lim_n (x^* \int gf_n du) = \lim_n \int gf_n d(x^*u) = \int gfd(x^*u) = x^*(\int gfdu)$$

for each $x^* \in X^*$. Hence

$$\int gf_n du \to \int gfdu \text{ weakly.}$$

To prove the result for the convergence in τ, let $\Psi_n : C_0(T) \to X$ be given by $\Psi_n(\varphi) = \int \varphi f_n du$ for $\varphi \in C_0(T)$ and let $\Psi : C_0(T) \to X$ be given by $\Psi(\varphi) = \int \varphi f du$ for $\varphi \in C_0(T)$. Then by Lemma 28.1, Ψ_n and Ψ are weakly compact. By hypothesis, by Lemma 28.2(iii) and by (28.3.6)we have

$$\int_\omega d\Psi_n = \int_\omega f_n du \to \int_\omega fdu = \int_\omega d\Psi \text{ in } \tau. \qquad (28.3.8)$$

Case 1. X is a Banach space

By (28.3.8), by Theorem 26.15 and by Lemma 28.2

$$\lim_n \int gd\Psi_n = \lim_n \int gf_n du \to \int gd\Psi = \int fgdu$$

in τ for each bounded complex Borel function g on T.

Case 2. X is a quasicomplete lcHs

For each $q \in \Gamma$, by Lemma 28.1 and by the continuity of Π_q, $\Pi_q \circ \Psi$ and $\Pi_q \circ \Psi_n$ are weakly compact. By hypothesis and by the first part of the theorem, $\int_\omega f_n \to \int_\omega fdu$ in τ for each open Baire set in T. Then by Lemma 28.2,

$$\Pi_q(\int_\omega d\Psi_n) = \Pi_q(\int_\omega f_n du) = \int_\omega f_n d(\Pi_q \circ u) \to \int_\omega fd(\Pi_q \circ u) \text{ in } \widetilde{X_q}.$$

Hence by the case of Banach spaces, we have

$$\lim_n \int gd(\Pi_q \circ \Psi_n) = \lim_n \int gf_n d(\Pi_q \circ u) \to \int gfd(\Pi_q \circ u) = \int gd(\Pi_q \circ \Psi).$$

Hence

$$q(\int gf_n du - \int gfdu) = |\int gd(\Pi_q \circ \Psi_n) - \int gd(\Pi_q \circ \Psi)|_q = |\int gd\Psi_n - \int gd\Psi|_q \to 0$$

for each $q \in \Gamma$ and hence

$$\int g f_n du \to \int g f du \text{ in } \tau.$$

This completes the proof of the theorem.

Theorem 28.4. Let X be a Banach space with topology τ. Let u be a prolongable Radon operator on $\mathcal{K}(T)$ with values in X. Let $(f_n)_1^\infty$ be a sequence of u-integrable complex functions and suppose the sequence $(\int_\omega f_n du)_1^\infty$ converges in τ (resp. converges weakly) in X for each open Baire set ω in T. Then there exists a function $f \in \mathcal{L}_1(u)$, u-essentially unique, such that $\int_\omega f_n du \to \int_\omega f du$ in τ (resp. weakly)for each open Baire set ω in T. Moreover, when $(\int_\omega f_n du)$ converges in τ for each open Baire set ω in T, then for each bounded complex Borel function g,

$$\int f_n g du \to \int f g du \text{ in } \tau$$

as well as

$$\int f_n g du \to \int f g du \text{ weakly)}$$

in X.

Proof. Let $x^* \in X^*$. By hypothesis, in both the cases of convergence, $\int_\omega f_n d(x^* u)$ converges in $\boldsymbol{K}$for each open Baire set ω in T. Hence by the Baire version of the Dieudonné-Grothendieck theorem (i.e., Theorem 18.6 of [P18]),

$$\text{the sequence } (f_n) \text{ converges weakly in } \mathcal{L}_1(u_{x^*}) \tag{28.4.1}$$

for each $x^* \in X^*$.

Then by the complex version of Lemma 1 on p. 126 of [T] one can suppose that the f_n are null in the complement of $\bigcup_1^\infty K_p$, where $(K_p)_1^\infty \subset \mathcal{C}$. By the complex version of Lemma 2 on pp. 126-127 of [T], with the sequence $(K_p)_1^\infty$ we can associate a sequence $(x_i^*)_1^\infty \subset X^*$ with the property mentioned in the lemma. Then by the complex version of Lemma 3 on p. 127 of [T], there exists a sequence of barycenters g_n of the f_n given by

$$g_n = \sum_{i=n}^{N(n)} \alpha_i^{(n)} f_i, \ \ \alpha_i^{(n)} \geq 0, \text{ and } \sum_{i=n}^{N(n)} \alpha_i^{(n)} = 1 \tag{28.4.2}$$

such that (g_n) converges in mean in $\mathcal{L}_1(x_i^*u)$ and converges (x_i^*u)-a.e. in T for each $i \in I\!N$. Thus, for each $i \in I\!N$, there exists $N_i \subset T$ such that $|x_i^*u|^\bullet(N_i) = 0$ and $(g_n(t))_1^\infty$ is convergent in $T\backslash N_i$. (Compare with the proof of Theorem 23.12 of [P19].) Thus, if $N = \bigcap_1^\infty N_i$, then $(g_n(t))_1^\infty$ is convergent in $T\backslash N = \bigcup_1^\infty(T\backslash N_i)$ and $|x_i^*u|^\bullet(N) = 0$ for $i \in I\!N$. Therefore, by the complex version of Lemma 2 on pp. 126-127 of [T], g_n converges u-a.e. in T. Let f be the u-a.e. limit of the sequence (g_n). As (f_n) converges weakly in $\mathcal{L}_1(u_{x^*})$ for each $x^* \in X^*$ by (28.4.1) and as (g_n) is given by (28.4.2), it follows that (g_n) also converges weakly in $\mathcal{L}_1(u_{x^*})$ for each $x^* \in X^*$. Then by Theorem 23.6 of [P19] (taking $\theta = x^*u \in \mathcal{K}(T)^*$), $f \in \mathcal{L}_1(u_{x^*})$ for each $x^* \in X^*$ and

$$\lim_n \int g f_n du_{x^*} = \int g f du_{x^*} \qquad (28.4.3)$$

for $x^* \in X^*$ and for a bounded complex Borel function g on T. Thus f is weakly u-integrable and by (28.4.3) we have

$$\lim_n \int_\omega f_n d(x^*u) = \int_\omega f d(x^*u) \qquad (28.4.4)$$

for each open Baire set ω in T and for $x^* \in X^*$. But by hypothesis, in both the cases of convergence, there exists $x_\omega \in X$ such that

$$\lim_n \int_\omega f_n d(x^*u) = x^*(x_\omega) \qquad (28.4.5)$$

for each $x^* \in X^*$. Thus by (28.4.4) and (28.4.5) we have

$$\int_\omega f d(x^*u) = x^*(x_\omega) \qquad (28.4.6)$$

for $x^* \in X^*$. Then by Theorem 27.9, $f \in \mathcal{L}_1(u)$ and by (28.4.6) we have

$$x^*(\int_\omega f du) = \int_\omega f d(x^*u) = x^*(x_\omega) \qquad (28.4.7)$$

for $x^* \in X^*$. Then by (28.4.5) and (28.4.7), $\int_\omega f_n du \to \int_\omega f du$ weakly.

If $\int_\omega f_n du$ converges to x_ω in τ, then $\int_\omega f du = x_\omega$ by (28.4.7) and by the Hahn-Banach theorem and hence

$$\int_\omega f_n du \to \int_\omega f du \,\text{in}\, \tau.$$

Suppose there exists $h \in \mathcal{L}_1(u)$ such that

$$\int_\omega f_n du \to \int_\omega h du$$

in τ (resp. weakly). Then

$$\int_\omega f_n du_{x^*} \to \int_\omega h du_{x*}$$

and hence by Theorem 23.6 of [P19], $f = h$ u_{x^*}-a.e. in T for each $x^* \in X^*$. Then by the complex version of Proposition 3.7 of [T], $f = h$ u-a.e. in T and hence f is u-essentially unique.

Let $\Psi_n : C_0(T) \to X$ be given by

$$\Psi_n(\varphi) = \int \varphi f_n du, \;\; \varphi \in C_0(T).$$

Then by Lemma 28.1, Ψ_n is weakly compact for each n and by hypothesis and by Lemma 28.2(iii), $\lim_n \int_\omega d\Psi_n \in X$ in τ for each open Baire set ω in T. Then by Theorem 26.15 and by Lemma 28.2 there exists a bounded weakly compact operator Ψ on $\mathcal{K}(T)$ with values in X such that

$$\lim_n \int g d\Psi_n = \lim_n \int g f_n du = \int g d\Psi \tag{28.4.8}$$

in τ for each bounded complex Borel function g on T. Then by the fact that $f \in \mathcal{L}_1(u)$ and g is bounded, $gf \in \mathcal{L}_1(u)$. Moreover, by (28.4.3) and (28.4.8) we have

$$\lim_n x^* \int g f_n du = \lim_n \int g f_n du_{x^*} = \int g f du_{x^*} = x^*(\int g f du) = x^*(\int g d\Psi)$$

for $x^* \in X^*$. Consequently, by the Hahn-Banach theorem

$$\int g d\Psi = \int g f du$$

and hence by (28.4.8) we have

$$\lim_n \int g f_n du = \int g f du \text{ in } \tau$$

for each bounded complex Borel function g. Since gf_n and gf are u-integrable, (28.4.3) implies

$$\lim_n x^*(\int g f_n du) = \lim_n \int g f du_{x^*} = \int f g du_{x^*} = x^*(\int g f du)$$

and hence

$$\int f_n g du \to \int f g du \text{ weakly}.$$

This completes the proof of the theorem.

29. WEAKLY COMPACT AND PROLONGABLE RADON VECTOR MEASURES

If u is a bounded Radon operator with values in a quasicomplete lcHs, we define $M_u = \{A \subset T : \chi_A \in \mathcal{L}_1(u)\}$ and $\boldsymbol{\mu}_u(A) = \int_A du$ for $A \in M_u$. When u is a bounded weakly compact Radon operator, we show that M_u is a σ-algebra containing $\mathcal{B}(T)$ and $\boldsymbol{\mu}_u = \mathbf{m}_u$, where $\mathbf{m}_u$ is the representing measure of u in the sense of 18.10 of [P18], which is the Lebesgue completion of $\mathbf{m}_u|_{\mathcal{B}(T)}$. In Theorem 29.7 (resp. Theorem 29.8) we give several characterizations of a bounded weakly compact Radon (resp. a prolongable Radon) operator u. Theorems 29.9 and 29.11 study the regularity properties of $\boldsymbol{\mu}_u$ when u is a bounded weakly compact or a prolongable Radon operator, respectively. Then we study the outer measure $\boldsymbol{\mu}_u^*$ of $\boldsymbol{\mu}_u$ in the sense of [Si] and give the connection between M_u and $\boldsymbol{\mu}_u^*$-measurable sets in Theorem 29.20, where we also show that $M_u = M_{\boldsymbol{\mu}_u^*}$ and $\boldsymbol{\mu}_u^*(E) = \boldsymbol{\mu}_u(E)$ for $E \in M_u$. We introduce the concepts of Lebesgue-Radon completion and localized Lebesgue-Radon completion and in terms of them we generalize Theorems 4.4 and 4.6 of [P4].

Definition 29.1. Let X be a quasicomplete lcHs and $u : \mathcal{K}(T) \to X$ be a Radon operator. Let $M_u = \{A \subset T : \chi_A \in \mathcal{L}_1(u)\}$ and let $\boldsymbol{\mu}_u(A) = \int_A du$ for $A \in M_u$. Then $\boldsymbol{\mu}_u$ is called the Radon vector measure induced by u and M_u is called the domain of $\boldsymbol{\mu}_u$. $\boldsymbol{\mu}_u$ is called a weakly compact (resp. prolongable) Radon vector measure if u is a bounded weakly compact (resp. a prolongable) Radon operator on $\mathcal{K}(T)$.

Theorem 29.2. Let X, u, M_u and $\boldsymbol{\mu}_u$ be as in Definition 29.1. Then M_u is a ring of u-measurable sets and $\boldsymbol{\mu}_u$ is σ-additive on M_u.

Proof. Since $0 \in \mathcal{L}_1(u)$, $\emptyset \in M_u$. Let $q \in \Gamma$. For $A_1, A_2 \in M_u$ and $K \in \mathcal{C}$, by the complex version of Proposition 1.21 of [T], and by Theorem 25.14 there exist disjoint sequences $(K_{i,n})_{n=1}^{\infty} \subset \mathcal{C}$, $i = 1, 2$ and sets N_1 and N_2 with $u_q^\bullet(N_1 \cup N_2) = 0$ such that

$$K = \bigcup_{n=1}^{\infty} K_{i,n} \cup N_i$$

with $K_{i,n} \subset K \cap A_i$ or $K_{i,n} \subset K \backslash A_i$, $i = 1, 2$.

Let $J_i = \{n : K_{i,n} \subset K \cap A_i\}$ and $I_i = \{n : K_{i,n} \subset K\backslash A_i\}$, $i = 1, 2$. Then

$$K \cap (A_1\backslash A_2) = (\bigcup_{n\in J_1} K_{1,n} \cup F_1)\bigcap(\bigcup_{n\in I_2} K_{2,n} \cup F_2)$$

and

$$K\backslash(A_1\backslash A_2) = (\bigcup_{n\in I_1} K_{1,n} \cup F_3)\bigcup(\bigcup_{n\in J_2} K_{2,n} \cup F_4)$$

with $u_q^\bullet(F_i) = 0$, $i = 1, 2, 3, 4$. Consequently, $A_1\backslash A_2$ is u_q-measurable. As $\chi_{A_1\backslash A_2} \le \chi_{A_1} \in \mathcal{L}_1(u)$, by the complex version of Theorem 1.22 of [T] $A_1\backslash A_2 \in M_u$. Since $\chi_{A_1\cup A_2} = \chi_{A_2} + \chi_{A_1\backslash A_2}$, $A_1 \cup A_2 \in M_u$. Hence M_u is a ring of u-measurable sets.

For $x^* \in X^*$, $x^*u \in \mathcal{K}(T)^*$ and hence by the complex version of Proposition 1.30 of [T], $\mathcal{L}_1(u) \subset \mathcal{L}_1(x^*u)$. Then μ_{x^*u} is the complex Radon measure induced by x^*u in the sense of Definition 4.3 of [P3]. Thus

$$\mu_{x^*u}(A) = (x^*u)(\chi_A) = x^*\boldsymbol{\mu}_u(A) \qquad (29.2.1)$$

for $A \in M_u$. Let $(A_i)_1^\infty$ be a disjoint sequence in M_u with $A = \bigcup_1^\infty A_i \in M_u$. Then by (29.2.1) we have

$$x^*\boldsymbol{\mu}_u(A) = \mu_{x^*u}(A) = \sum_1^\infty \mu_{x^*u}(A_i) = \sum_1^\infty x^*\boldsymbol{\mu}_u(A_i)$$

for each $x^* \in X^*$, since μ_{x^*u} is σ-additive on M_{x^*u} and since $M_u \subset M_{x^*u}$. Now by the Orlicz-Pettis theorem which holds for lcHs by McArthur [McA] we conclude that

$$\boldsymbol{\mu}_u(A) = \sum_1^\infty \boldsymbol{\mu}_u(A_i)$$

and hence $\boldsymbol{\mu}_u$ is σ-additive on M_u.

Remark 29.3. It is possible that $M_u = \{\emptyset\}$. For example, let u be the identity operator on $C_0([0, 1))$. Then by Example 26.6, $C_0([0, 1)) = \mathcal{L}_1(u)$ and clearly, 0 is the only idempotent function in $C_0([0, 1))$.

Theorem 29.4. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Then the following statements hold:

(i) M_u is a σ-algebra in T.

(ii) $\mathcal{B}(T) \subset M_u$.

(iii) If $\mathbf{m}_u$ is the representing measure of the continuous extension of u on $C_0(T)$, then $\boldsymbol{\mu}_u|_{\mathcal{B}(T)} = \mathbf{m}_u$ and hence $\boldsymbol{\mu}_u|_{\mathcal{B}(T)}$ is $\mathcal{B}(T)$-regular.

(iv) For each $A \in M_u$,

$$\boldsymbol{\mu}_u(A) = \lim_{K \in \mathcal{C}} \boldsymbol{\mu}_u(A \cap K)$$

where $\mathcal{C}$ is directed by the relation $K_1 \leq K_2$ if $K_1 \subset K_2,\ K_1, K_2 \in \mathcal{C}$.

Proof. Let $q \in \Gamma$. Then $U_q^0 = \{x^* \in X^* : |x^*(x)| \leq 1 \,\text{for}\, x \in U_q\}$ is equicontinuous in X^*. As u is a weakly compact operator on $C_0(T)$ (see Convention 26.4), by Corollary 9.3.2 of Edwards [E] and by 18.10 of [P18], $u^*(U_q^0) = \{u^*x^* : x^* \in U_q^0\} = \{x^* \circ u : x^* \in U_q^0\}$ is relatively weakly compact in $M(T)$. Let $\epsilon > 0$. Then by Theorem 4.22.1 of [E] there exists a compact set K_q in T such that

$$\sup_{x^* \in U_q^0} |x^* \circ u|(T \backslash K_q) < \epsilon \qquad (29.4.1)$$

and by Proposition 1 of [P9], given an open set U in T there exists a compact $C_q \subset U$ such that

$$\sup_{x^* \in U_q^0} |x^* \circ u|(U \backslash C_q) < \epsilon. \qquad (29.4.2)$$

(i) By Theorem 29.2, M_u is a ring of u-measurable sets. By (29.4.1) and by Lemma 26.17, $u_q^\bullet(T \backslash K_q) < \epsilon$. By Theorem 50.D of [H] there exists $\varphi_q \in K(T)$ such that $\chi_{K_q} \leq \varphi_q \leq \chi_T$ so that $u_q^\bullet(1 - \varphi_q) \leq u_q^\bullet(T \backslash K_q) < \epsilon$. As q is arbitrary in Γ, this shows that $1 \in \mathcal{L}_1(u)$ and hence $T \in M_u$.

Let $(A_n)_1^\infty$ be a disjoint sequence in M_u with $A = \bigcup_1^\infty A_n$. Then $\sum_{k=1}^n \chi_{A_k} \nearrow \chi_A \leq \chi_T \in \mathcal{L}_1(u)$. Since $\sum_1^n \chi_{A_k} = \chi_{\bigcup_1^n A_k} \in \mathcal{L}_1(u)$, by the complex version of Theorem 4.7 of [T], $\chi_A \in \mathcal{L}_1(u)$ and hence $A \in M_u$. Therefore, (i) holds.

(ii) Let U be an open set in T. Then by Lemma 26.17 and by (29.4.2) there exists a compact C_q in T such that $C_q \subset U$ and $u_q^\bullet(U \backslash C_q) < \epsilon$. By Urysohn's lemma there exists $\varphi_q \in \mathcal{K}(T)$ such that $\chi_{C_q} \le \varphi_q \le \chi_U$ and hence $u_q^\bullet(\chi_U - \varphi_q) \le u_q^\bullet(U \backslash C_q) < \epsilon$. Therefore, $U \in M_u$. As M_u is a σ-algebra, we conclude that $\mathcal{B}(T) \subset M_u$.

(iii) By 18.10 of [P18], $x^*u = u^*x^* = x^* \circ \mathbf{m}_u$ for $x^* \in X^*$. Given an open set U in T and an $\epsilon > 0$, as in the beginning of the proof choose a compact $K_q \subset U$ for which (29.4.2) holds. Then by Urysohn's lemma there exists $\varphi_q \in \mathcal{K}(T)$ such that $\chi_{K_q} \le \varphi_q \le \chi_U$ so that by (29.4.2) we have

$$u_q^\bullet(\chi_U - \varphi_q) \le u_q^\bullet(U \backslash K_q) < \epsilon. \qquad (29.4.3)$$

Then

$$\begin{aligned}
|\mathbf{m}_u(U) - \int_T \varphi_q d\mathbf{m}_u|_q &= \sup_{x^* \in U_q^0} |(x^*(\mathbf{m}_u(U) - \int_T \varphi_q d\mathbf{m}_u)| \\
&= \sup_{x^* \in U_q^0} |\int_T \chi_U d(x^*u) - \int_T \varphi_q dx^*\mathbf{m}_u)| \\
&= \sup_{x^* \in U_q^0} |\int_T \chi_U d(x^*u) - \int_T \varphi_q d(x^*u)| \\
&= \sup_{x^* \in U_q^0} \int_T |\chi_U - \varphi_q| dv(x^*u) \qquad (29.4.4) \\
&\le \sup_{x^* \in U_q^0} |x^*u|(U \backslash K_q) \\
&= u_q^\bullet(U \backslash K_q) < \epsilon
\end{aligned}$$

by Lemma 26.17 and (29.4.3). Consequently, by 18.10 of [P18] we have

$$|\mathbf{m}_u(U) - u(\varphi_q)|_q = \sup_{x^* \in U_q^0} |x^*(\mathbf{m}_u(U) - \int_T \varphi_q d\mathbf{m}_u)| < \epsilon \qquad (29.4.5)$$

since φ_q is $\mathbf{m}_u$-integrable in T. On the other hand,

$$\begin{aligned}
|\boldsymbol{\mu}_u(U) - u(\varphi_q)|_q &= |\int \chi_U du - u(\varphi_q)|_q \\
&= \sup_{x^* \in U_q^0} |\int_T \chi_U d(x^*u) - x^*u(\varphi_q)| \\
&\le \sup_{x^* \in U_q^0} |\int_T |\chi_U - \varphi_q| dv(x^*u) \qquad (29.4.6) \\
&\le u_q^\bullet(U \backslash K_q) < \epsilon
\end{aligned}$$

by (29.4.4).

Therefore, by (29.4.5) and (29.4.6) we have

$$|\mathbf{m}_u(U) - \boldsymbol{\mu}_u(U)|_q < 2\epsilon.$$

Since ϵ is arbitrary,

$$|\mathbf{m}_u(U) - \boldsymbol{\mu}_u(U)|_q = 0.$$

Now, as q is arbitrary in Γ, $\mathbf{m}_u(U) = \boldsymbol{\mu}_u(U)$.

If U_1, U_2 are open sets with $U_1 \subset U_2$, then

$$\mathbf{m}_u(U_2 \backslash U_1) = \mathbf{m}_u(U_2) - \mathbf{m}_u(U_1) = \boldsymbol{\mu}_u(U_2) - \boldsymbol{\mu}_u(U_1) = \boldsymbol{\mu}_u(U_2 \backslash U_1)$$

and consequently, $\boldsymbol{\mu}_u(E) = \mathbf{m}_u(E)$ for E in the ring generated by $\mathcal{U}$, the family of open sets in T.

Let $\mathcal{M} = \{A \in \mathcal{B}(T) : \boldsymbol{\mu}_u(A) = \mathbf{m}_u(A)\}$. If $(E_n)_1^\infty$ is a monotone sequence in $\mathcal{M}$, by the σ-additivity of $\boldsymbol{\mu}_u$ on $\mathcal{B}(T)$ by Theorem 29.2 and by (ii) above and by the σ-additivity of $\mathbf{m}_u$ on $\mathcal{B}(T)$ by Theorem 2 of [P9] as u is weakly compact, we have $\boldsymbol{\mu}_u(\lim_n E_n) = \mathbf{m}_u(\lim_n E_n)$ and hence $\mathcal{M}$ is a monotone class. Then by Theorem 6.B of [H], $\boldsymbol{\mu} = \mathbf{m}_u$ on $\mathcal{B}(T)$. Consequently, by Theorem 6 of [P9], $\boldsymbol{\mu}_u|_{\mathcal{B}(T)}$ is Borel regular. Hence (iii) holds.

(iv) By (ii), $\mathcal{C} \subset M_u$ and hence $A \cap K \in M_u$ for $A \in M_u$ and $K \in \mathcal{C}$. Given $q \in \Gamma$, by the lcHs complex version of Lemma 1.24 of [T] we have

$$\lim_{K \in \mathcal{C}} u_q^\bullet(\chi_{A \backslash K}) = \lim_{K \in \mathcal{C}} u_q^\bullet(A \backslash (A \cap K)) = 0.$$

As q is arbitrary in Γ, this shows that $\chi_{A \cap K} \to \chi_A$ in the topology of $\mathcal{L}_1(u)$ and consequently,

$$\int \chi_A du = \lim_{K \in \mathcal{C}} \int \chi_{A \cap K} du.$$

$$i.e., \boldsymbol{\mu}_u(A) = \lim_{K \in \mathcal{C}} \boldsymbol{\mu}_u(A \cap K).$$

This complete the proof of the theorem.

Definition 29.5. Let X be a quasicomplete lcHs and let $u : C_0(T) \to X$ be a continuous linear mapping. As in 18.10 of [P18], let $\mathbf{m}_u = u^{**}|_{\mathcal{B}(T)}$, the representing measure of u and let $\widetilde{\mathbf{m_u}}$ be the Lebesgue-completion of $\mathbf{m}_u$ with respect to $\mathcal{B}(T)$ and let $\widetilde{\mathcal{B}(T)}$ be the Lebesgue completion of $\mathcal{B}(T)$ with respect to $\mathbf{m}_u$. In the light of Theorem III.10.17 of [DS], we define $\mathbf{m}_u(A \cup N) = \mathbf{m}_u(A)$ where $A \in \mathcal{B}(T)$ and $N \subset M \in \mathcal{B}(T)$ with $v(\mathbf{m}_u, \mathcal{B}(T))(M) = 0$.(See Definition III.10.18 of [DS]). Thus we use the symbol $\mathbf{m}_u$ to denote its Lebesgue completion on $\widetilde{\mathcal{B}(T)}$ also.

Theorem 29.6. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator and let $q \in \Gamma$. By Convention 26.4, $u : C_0(T) \to X$ is continuous and let $\mathbf{m}_u$ be the representing measure of u with $\widetilde{\mathbf{m_u}}$ whose Lebesgue completion with respect to $\mathcal{B}(T)$ is denoted by $\mathbf{m}_u$ also. By Theorem 6 of [P9], $\mathbf{m}_u|_{\mathcal{B}(T)}$ is Borel regular. Then:

(i) For $M \in \mathcal{B}(T)$, $u_q^\bullet(M) = ||\mathbf{m}_u||_q(M)$. (Recall $u_q = \Pi_q \circ u$ from Lemma 26.17.)

(ii) For $A \in \widetilde{B(T)}$, the Lebesgue completion of $\mathcal{B}(T)$ with respect to $\mathbf{m}_u$, suppose there exists $M \in \mathcal{B}(T)$ such that $A \subset M$ and $v(\Pi_q \circ \mathbf{m}_u, \mathcal{B}(T))(M) = 0$. Then $u_q^\bullet(A) = ||\mathbf{m}_u||_q(A) = 0$.

(iii) A set A in T is u-integrable if and only if it is $\mathbf{m}_u$-integrable and hence $\widetilde{\mathcal{B}(T)} = M_u$. Consequently, for $A \in M_u$, $\boldsymbol{\mu}_u(A) = \mathbf{m}_u(A)$. (See Definition 29.5.)

(iv) A function $f : T \to \boldsymbol{K}$is u-measurable if and only if it is $\mathbf{m}_u$-measurable.

Proof. (i) Since $\mathcal{B}(T) \subset M_u$ by Theorem 29.4(ii), the set M is u-measurable and u-integrable. Then by the complex version of Theorem 1.11 of [T], M is $\Pi_q \circ u$- integrable and hence $\chi_M \in \mathcal{L}_1(u_q)$. Let $\Psi_{x^*}(x + q^{-1}(0)) = x^*(x)$ for $x \in X$ and $x^* \in U_q^0$. Then Ψ_{x^*} is well defined, linear and continuous and $\{\Psi_{x^*} : x^* \in U_q^0\}$ is a norm determining subset of the closed unit ball of $(X_q)^*$ by Proposition 10.14 of [P17]. Consequently, by the complex version of **1.13** of [T], by 18.10 of [P18] and by Lemma 25.11 we have

$$\begin{aligned} u_q^\bullet(M) = u_q^\bullet(\chi_M) &= \sup_{x^* \in U_q^0} |x^* u|^\bullet(\chi_M) \\ &= \sup_{x^* \in U_q^0} |u^* x^*|(\chi_M) \\ &= \sup_{x^* \in U_q^0} |(x^* \circ \mathbf{m}_u)|(M) \\ &= ||\mathbf{m}_u||_q(M). \end{aligned}$$

Hence (i) holds.

(ii) By (i), $u_q^\bullet(M) = 0$ implies $||\mathbf{m}_u||_q(M) = 0$. As $A \subset M$, $u_q^\bullet(A) \leq u_q^\bullet(M) = 0$ and $||\mathbf{m}_u||_q(A) \leq ||\mathbf{m}_u||_q(M) = 0$. Hence (ii) holds.

(iii) Suppose $A \subset T$ is u-integrable. Then it is u-measurable and hence by the analogue of Theorem 21.9 of [P19], given a compact K in T and $q \in \Gamma$, there exists a disjoint sequence $(K_n^{(q)})_1^\infty$ of compacts and a set N contained in K such that $K \backslash N = \bigcup_1^\infty K_n^{(q)}$, $u_q^\bullet(N) = 0$ and $K_n^{(q)} \subset K \cap A$ or $K_n^{(q)} \subset K \backslash A$ for each n. Then $N = K \backslash (K \backslash N) = K \backslash \bigcup_1^\infty K_n^{(q)} \in \mathcal{B}(T)$. Consequently, by (i) we have $||\mathbf{m}_u||_q(N) = 0$. Then by Theorem 21.9 of [P19] and by the arbitrariness of $q \in \Gamma$ we conclude that A is Lusin $\mathbf{m}_u$-measurble. Consequently, by Theorem 21.5 of [P19], A is $\mathbf{m}_u$-measurable. Since $1 \in \mathcal{L}_1(\mathbf{m}_u)$ by Theorem 19.14(b) of [P18], by the domination principle χ_A is $\mathbf{m}_u$-integrable.

Conversely, let χ_A be $\mathbf{m}_u$-integrable. Then A is $\mathbf{m}_u$-measurable. Therefore, by Theorem 21.9 of [P19], given a compact K and $q \in \Gamma$, there exist a disjoint sequence $(K_n^{(q)})_1^\infty$ of compacts and a set $N \subset K$ such that $K \backslash N = \bigcup_1^\infty K_n^{(q)}$ with $K_i^{(q)} \subset A$ or $K_i^{(q)} \subset (K \backslash A)$ and with $||\mathbf{m}_u||_q(N) = 0$. Then $N \in \mathcal{B}(T)$ and by (i), $u_q^\bullet(N) = 0$. Since q is arbitrary in Γ, by the analogue of Theorem 21.9 of [P19], A is u-measurable. Since $1 \in \mathcal{L}_1(u)$ by Theorem 29.4(i), by the domination principle χ_A is u-integrable and hence $A \in M_u$. Consequently, $M_u = \widetilde{\mathcal{B}(T)}$.

For $A \in M_u = \widetilde{\mathcal{B}(T)}$, $\boldsymbol{\mu}_u(A) = \int \chi_A du$ and hence by 18.10 of [P18] we have

$$x^* \boldsymbol{\mu}_u(A) = \int_A d(x^* u) = \int_A d(u^* x^*) = \int_A d(x^* \circ \mathbf{m}_u) = x^* \mathbf{m}_u(A)$$

for $x^* \in X^*$. As $\mathbf{m}_u(A)$ and $\boldsymbol{\mu}_u(A)$ belong to X, by the Hahn-Banach theorem $\boldsymbol{\mu}_u(A) = \mathbf{m}_u(A)$ for $A \in M_u$. Thus (iii) holds.

(iv) If f is u-measurable, given a compact K and $q \in \Gamma$, there exist a disjoint sequence $(K_i^{(q)})_1^\infty \subset \mathcal{C}$ and a set N with $u_q^\bullet(N) = 0$ such that $K \backslash N = \bigcup_1^\infty K_i^{(q)}$ and with $f|_{K_i^{(q)}}$ continuous for each i. Since $N \in \mathcal{B}(T)$, by (i), $||\mathbf{m}_u||_q(N) = u_q^\bullet(N) = 0$ and hence by Theorem 21.4 of [P19], f is Lusin $\mathbf{m}_u$- measurable. Then by Theorem 21.5 of [P19], f is $\mathbf{m}_u$-measurable. Conversely, if f is $\mathbf{m}_u$ measurable, then by Theorem 21.5 of [P19], f is Lusin $\mathbf{m}_u$-measurable and hence reversing the argument and using (i), one can easily show that f is u-measurable.

This completes the proof of the theorem.

The following theorem generalizes Theorem 3.3 of [P4] to bounded weakly compact Radon operators on $\mathcal{K}(T)$ and improves the first part of Theorem 9.13 of [P13]. Moreover, Theorem 9.13 of [P13] was announced earlier in [P6].

Theorem 29.7. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded Radon operator. Let M_u and $\mathbf{m}_u$ be given as in Definitions 29.1 and 29.5 respectively. Then the following statements are equivalent:

(i) u is a bounded weakly compact Radon operator.

(ii) M_u is a σ-algebra in T and $\mathcal{C}_0 \subset M_u$.

(iii) $\mathcal{B}(T) \subset M_u$.

(iv) $\mathcal{B}_c(T) \subset M_u$.

(v) $\mathcal{B}_0(T) \subset M_u$.

(vi) Every bounded u-measurable complex function f belongs to $\mathcal{L}_1(\mathbf{m}_u)$.

(vii) Every bounded complex Borel function f belongs to $\mathcal{L}_1(\mathbf{m}_u)$.

(viii) Every bounded complex σ-Borel function f belongs to $\mathcal{L}_i(\mathbf{m}_u)$.

(ix) Every bounded complex Baire function f belongs to $\mathcal{L}_1(\mathbf{m}_u)$.

(x) Every bounded u-measurable complex function f belongs to $\mathcal{L}_1(u)$.

(xi) Every bounded complex Borel function f belongs to $\mathcal{L}_1(u)$.

(xii) Every bounded complex σ-Borel function f belongs to $\mathcal{L}_1(u)$.

(xiii) Every bounded complex Baire function f belongs to $\mathcal{L}_1(u)$.

(xiv) For every open set U in T there exists a vector $x_U \in X$ such that the weak integral $\int_U du = x_U$ in the sense that

$$x^*(x_U) = \int_U d(x^*u)$$

for each $x^* \in X^*$, where x^*u is treated as a complex Radon measure in T.

(xv) Similar to (xiv) except that the open set U is σ-Borel.

(xvi) Similar to (xiv) except that the open set U is an open Baire set.

(xvii) Every bounded u-measurable complex function f belongs to $\mathcal{L}_1(\boldsymbol{\mu}_u)$.

(xviii) Every bounded complex Borel function f belongs to $\mathcal{L}_1(\boldsymbol{\mu}_u)$.

(xix) Every bounded complex σ-Borel function f belongs to $\mathcal{L}_1(\boldsymbol{\mu}_u)$.

(xx) Every bounded complex Baire function f belongs to $\mathcal{L}_1(\boldsymbol{\mu}_u)$.

Proof. By Theorem 29.4, (i)$\Rightarrow$(ii) and (i)$\Rightarrow$(iii)$\Rightarrow$(iv)$\Rightarrow$(v). Obviously, (ii)$\Rightarrow$(v).

(v)$\Rightarrow$(i) Since $\mathcal{B}_0(T) \subset M_u$, every open Baire set U is u-integrable and hence there exists a vector $x_U \in X$ such that $\int \chi_U du = x_U$. Consequently, the weak integral $\int_U du$ belongs to X and therefore, by (vi)$\Rightarrow$(vii) of Theorem 26.10, (i) holds.

Thus (i)$\Leftrightarrow$(ii)$\Leftrightarrow$(iii)$\Leftrightarrow$(iv)$\Leftrightarrow$(v).

(i)$\Rightarrow$(vi) By the hypothesis (i) and by Theorem 29.6, $A \in M_u$ if and only if A $\mathbf{m}_u$-integrable in T. If f is a bounded u-measurable complex function, then by Theorem 29.4(i) χ_T is u-integrable and hence by the complex lcHs version of Theorem 1.22 of [T], $f \in \mathcal{L}_1(u)$. Moreover, by Theorem 29.6(iv) f is $\mathbf{m}_u$-measurable and hence, given $q \in \Gamma$, there exists a set $N \in \mathcal{B}(T)$ with $||\mathbf{m}_u||_q(N) = 0$ and a sequence $(s_n)_1^\infty$ of $\mathcal{B}(T)$-simple functions such that $|s_n| \le |f|$ and $s_n \to f\chi_{T\setminus N}$ uniformly in T. Then by the complex lcHs versions of Proposition 1.3 and of **1.10** of [T]

$$|\int s_n du - \int f du|_q \le u_q^\bullet(|s_n - f|) \le ||f - s_n||_T u_q^\bullet(1) \to 0$$

as $n \to \infty$ since $u_q^\bullet(1) < \infty$ by Proposition 26.3. Hence

$$\int f du = \lim_n \int s_n du \qquad (29.7.1)$$

since q is arbitrary in Γ.

Since M_u is the set of all $\mathbf{m}_u$-measurable sets which are $\mathbf{m}_u$-integrable in T and as $\boldsymbol{\mu}_u(A) = \mathbf{m}_u(A)$ for $A \in M_u$ by Theorem 29.6(iii), each s_n is $\mathbf{m}_u$-integrable in T and as $\boldsymbol{\mu}(A) = \mathbf{m}_u(A)$ for $A \in M_u$ by the said theorem, we have

$$\int s_n du = \int_T s_n d\mathbf{m}_u. \qquad (29.7.2)$$

Consequently, by (29.7.1),

$$\int f du = \lim_n \int_T s_n d\mathbf{m}_u \qquad (29.7.3)$$

and hence by (29.7.1), (29.7.2) and (29.7.3) we have

$$\int_T f d\mathbf{m}_u = \lim_n \int_T s_n d\mathbf{m}_u = \lim_n \int s_n du = \int f du.$$

Thus every bounded u-measurable complex function f is u-integrable and $\mathbf{m}_u$-integrable in T and

$$\int f du = \int_T f d\mathbf{m}_u. \qquad (29.7.4)$$

Hence (i)⇒(vi).

As shown in the above, (vi)⇒(x). Obviously, by the complex version of Corollary 2.18 of [T], (x)⇒(xi)⇒(xii) ⇒(xiii). Clearly, (xi)⇒(xiv) (resp. (xii)⇒(xv), (xiii) ⇒(xvi)) and consequently, (xiv) (resp. (xv), (xvi)) implies by Theorem 26.10 that u is a weakly compact Radon operator. Clearly, (vi)⇒(vii)⇒(viii) ⇒(ix) and (ix) implies that $\int_U d\mathbf{m}_u = x_U \in X$ for each open Baire set U in T. Then by (29.7.4), $\int_U du = \int_U d\mathbf{m}_u = x_U$ for each open Baire set U in T and hence by Theorem 26.10, (ix)⇒(i). Thus (i)⇔(vi)⇔(vii) ⇔(viii)⇔(ix).

By (29.7.4), $\int_U du = \int_U d\mathbf{m}_u = x_U \in X$ for each open Baire set U in T and hence by Theorem 26.10, (xiii)⇒(i). Hence (i)⇔(x)⇔(xi)⇔(xii)⇔(xiii).

(iii)⇒(xiv)⇒(xv)(obvious)⇒(xvi) obvious and (xvi)⇒(i) by Theorem 26.10. Hence (i)⇔(iii)⇔(xiv)⇔(xv) ⇔(xvi).

Since $\mathbf{m}_u = \boldsymbol{\mu}_u$ on M_u, (xvii)⇔(vi); (xviii)⇔(vii); (xix)⇔(viii) and (xx)⇔(ix) and hence all the statements are equivalent.

This complete the proof of the theorem.

The following theorem has been given without proof in Theorem 9.14 of [P13]. It was also announced earlier in [P6].

Theorem 29.8. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a Radon operator. Then the following statements are equivalent:

(i) u is prolongable.

(ii) $\delta(\mathcal{C}) \subset M_u$.

(iii) $\delta(\mathcal{C}_0) \subset M_u$.

(iv) M_u is a δ-ring containing all relatively compact open sets in T.

(v) M_u is a δ-ring containing $\mathcal{C}$.

(vi) M_u is a δ-ring containing $\mathcal{C}_0$.

(vii) Every bounded complex Borel function with compact support belongs to $\mathcal{L}_1(u)$.

(viii) For every compact K in T, there exists $x_K \in X$ such that $\mu_{x^*u}(K) = x^*(x_K)$ for $x^* \in X^*$, where μ_{x^*u} is the complex Radon measure induced by x^*u in the sense of Definition 4.3 of [P3].

(ix) Similar to (viii) for each relatively compact open set U instead of K.

(x) Similar to (viii) with K compact G_δ.

Proof. (i)⇒(ii) Suppose u is prolongable. Let $V \in \mathcal{V}$. Then by (i), $u : \mathcal{K}(V) \to X$ is a bounded weakly compact Radon operator and hence by Theorem 29.4, $\mathcal{B}(V) \subset M_{u_V}$ where u_V is $u|_{\mathcal{K}(V)}$. Since V is arbitrary in $\mathcal{V}$ and since $\delta(\mathcal{C}) = \bigcup_{V \in \mathcal{V}} \mathcal{B}(V)$, by the complex analogue of Lemma 3.2 of [T] (ii) holds.

(ii)⇒(iii) Obvious.

(i)⇒(iv) Let (A_n) be a decreasing sequence in M_u with $A_n \searrow A$. Since $\chi_{A_n} \to \chi_A$ and $\chi_A \le \chi_{A_n}$ for all n, by the complex lcHs analogue of Theorem 4.7 of [T] $\chi_A \in \mathcal{L}_1(u)$ and hence $A \in M_u$. Since M_u is a ring of sets by Theorem 29.2, M_u is a δ-ring of sets. Since (i)⇒(ii), $\mathcal{V} \subset M_u$ and hence (i)⇒(iv).

(iv)⇒(v) If $C \in \mathcal{C}$, then by Theorem 50.D of [H] there exists $V \in \mathcal{V}$ such that $C \subset V$. Then $C = V \backslash (V \backslash C) \in M_u$ by (iv). Hence (iv)⇒(v).

(v)$\Rightarrow$(vi) Obvious.

(vi)$\Rightarrow$(iii) Obvious.

(i)$\Rightarrow$(vii) Let f be a bounded complex Borel function with supp $f = K \in \mathcal{C}$. Let $U \in \mathcal{V}$ with $K \subset U$. By (i), $u|_{\mathcal{K}(U)}$ is a bounded weakly compact Radon operator and hence by Theorem 29.7(xi), f is $u|_{\mathcal{K}(U)}$-integrable. Then given $\epsilon > 0$ and $q \in \Gamma$ there exists $\varphi \in \mathcal{K}(U)$ such that $(u|_{\mathcal{K}(U)})^{\bullet}_{q}(|f-\varphi|) < \epsilon$. Now by the complex analogue of Lemma 3.2 of [T], $\hat{\varphi} = \varphi$ and $\hat{f} = f$ and hence we have

$$u^{\bullet}_{q}(|f-\varphi|) = u^{\bullet}_{q}(|\hat{f}-\hat{\varphi}|) = (u_q|_{\mathcal{K}(U)})^{\bullet}(|f-\varphi|) < \epsilon.$$

Hence f is u-integrable and thus (vii) holds.

(vii)$\Rightarrow$(viii) Let $K \in \mathcal{C}$. Then by (vii), $\chi_K \in \mathcal{L}_1(u)$ and hence $\int_K du = x_K(\text{say}) \in X$. Then $x^*(x_K) = x^*(\int_K du) = \int_K d(x^*u) = \mu_{x^*u}(K)$.

(vii)$\Rightarrow$(ix) Let $U \in \mathcal{V}$. Then by (vii), $\chi_U \in \mathcal{L}_1(u)$ and hence $\int_U du = x_U(\text{say}) \in X$. Then $x^*(x_U) = x^*(\int_U du) = \int_U d(x^*u) = \mu_{x^*u}(U)$.

(vii)$\Rightarrow$(x) The proof is similar to that of (vii)$\Rightarrow$(viii).

(viii)$\Rightarrow$(i) as (iv)$\Leftrightarrow$(i) of Theorem 19.13 of [P18] and as $(x^* \circ \mathbf{m}_u) = x^*u$.

(ix)$\Rightarrow$(i) as (ii)$\Leftrightarrow$(i) of Theorem 19.13 of [P18] and as $x^* \circ \mathbf{m}_u = x^*u$.

(x)$\Rightarrow$(i) as (ix)$\Leftrightarrow$(i) of Theorem 19.13 of [P18] and as $x^* \circ \mathbf{m}_u = x^*u$.

(iii) $\Rightarrow$(i) By (iii), every $K \in \mathcal{C}_0$ belongs to M_u and hence is u-integrable. Hence $\int_L du \in X$. But $\int_K du = \boldsymbol{\mu}_u(K) = \mathbf{m}_u(K)$, and hence $\mathbf{m}_u(K) \in X$. Then by the equivalence (7) and (1) of Theorem 19.12 of [P18], u is prolongable and hence (i) holds.

Hence the statements (i)-(x) are equivalent and this completes the proof of the theorem.

Theorem 29.9. Let X be a quasicomplete lcHs and suppose $u : \mathcal{K}(T) \to X$ is a bounded weakly compact Radon operator. Then:

(i) $\boldsymbol{\mu}_u$ is M_u-regular. That is, given $A \in M_u$ and a neighborhood W of 0 in X, there exist a compact $C \subset A$ and an open set $U \supset A$ such that $\boldsymbol{\mu}_u(F) \in W$ for all $F \in M_u$ with $F \subset U \backslash C$.

(ii) For $A \in M_u$

$$\boldsymbol{\mu}_u(A) = \lim_{K \in \mathcal{C}(A)} \boldsymbol{\mu}_u(K) = \lim_{U \in \mathcal{U}(A)} \boldsymbol{\mu}_u(U) = \lim_{K \in \mathcal{C}(A)} \mathbf{m}_u(K) = \lim_{U \in \mathcal{U}(A)} \mathbf{m}_u(U)$$

where $\mathcal{C}(A) = \{K \in \mathcal{C} : K \subset A\}$ is directed by the relation $K_1 \leq K_2$ if $K_1 \subset K_2$ and $\mathcal{U}(A) = \{U \in \mathcal{U} : A \subset U\}$ is directed by the relation $U_1 \leq U_2$ if $U_2 \subset U_1$.

Proof. Let W be a neighborhood of 0 in X and letW_0 be a closed balanced neighborhood of 0 in X such that $W_0 + W_0 \subset W$. Then there exist an $\epsilon > 0$ and a finite family $(q_i)_1^n$ in Γ such that

$$\bigcap_{i=1}^{n} \{x : q_i(x) < 2\epsilon\} \subset W_0. \qquad (29.9.1)$$

Let $A \in M_u$. Then by the complex lcHs version of Lemma 1.24 of [T] there exists $K_0 \in \mathcal{C}$ such that

$$u_{q_i}^{\bullet}(A \backslash A \cap K) = u_{q_i}^{\bullet}(\chi_A \chi_{T \backslash K}) < \epsilon \qquad (29.9.2)$$

for all $K \in \mathcal{C}$ with $K \supset K_0$ and for $i = 1, 2, ..., n$. By Theorem 29.2,$\boldsymbol{\mu}_u$ is additive on M_u and by Theorem 29.4, $A \cap K \in M_u$ for $K \in \mathcal{C}$. Hence by (29.9.2) and by Theorem 29.2 and by the complex lcHs version of **1.10** of [T] we have

$$q_i(\boldsymbol{\mu}_u(A) - \boldsymbol{\mu}_u(A \cap K)) = q_i(\boldsymbol{\mu}_u(A \backslash (A \cap K)) \leq u_{q_i}^{\bullet}(A \backslash (A \cap K)) < \epsilon \qquad (29.9.3)$$

for $i = 1, 2, ..., n$ and for $K \in \mathcal{C}$ with $K \supset K_0$. On the other hand, as the members of M_u are u-measurable by Theorem 29.2, there exists a compact $K_i \subset K_0$ such that $\chi_A|_{K_i}$ is continuous and $u_{q_i}^{\bullet}(K_0 \backslash K_i) < \epsilon$ for $i = 1, 2, ..., n$. Let $J = \{k : K_k \subset A \cap K_0\}$ and $C = \bigcup_{k \in J} K_k$. If J is empty, then $K_i \subset K_0 \backslash A$ for all i and in that case, $C = \emptyset$. Thus $C \in \mathcal{C}$, $C \subset A \cap K_0$ and in both the cases we have

$$\begin{aligned} u_{q_i}^{\bullet}((A \cap K_0) \backslash C) &\leq u_{q_i}^{\bullet}(\{(K_0 \cap A) \cup (K_0 \backslash A)\} \backslash \bigcup_1^n K_j) \\ &= u_{q_i}^{\bullet}(K_0 \backslash \bigcup_1^n K_j) \leq u_{q_i}^{\bullet}(K_0 \backslash K_i) < \epsilon \end{aligned}$$

for $i = 1, 2, ..., n$.

Thus there exists $C \in \mathcal{C}$ with $C \subset A \cap K_0$ such that

$$u^{\bullet}_{q_i}((A \cap K_0)\backslash C) < \epsilon \tag{29.9.4}$$

for $i = 1, 2, ..., n$. Then by (29.9.3) and (29.9.4) and by the complex lcHs version of **1.10** of [T] we have

$$\begin{aligned} q_i(\boldsymbol{\mu}_u(A) - \boldsymbol{\mu}_u(C) &\leq q_i(\boldsymbol{\mu}_u(A) - \boldsymbol{\mu}_u(A \cap K_0)) + q_i(\boldsymbol{\mu}_u(A \cap K_0) - \boldsymbol{\mu}_u(C)) \\ &\leq \epsilon + q_i(\boldsymbol{\mu}_u(A \cap K_0)\backslash C) \\ &\leq \epsilon + u^{\bullet}_{q_i}((A \cap K_0)\backslash C) \\ &< 2\epsilon \end{aligned}$$

for $i = 1, 2, ..., n$. Hence $\boldsymbol{\mu}_u(A) - \boldsymbol{\mu}_u(C) \in W_0$.

Now let $F \subset A\backslash C$ with $F \in M_u$. Then by (29.9.2) and (29.9.4) we have

$$\begin{aligned} q_i(\boldsymbol{\mu}_u(F)) &\leq u^{\bullet}_{q_i}(F) \leq u^{\bullet}_{q_i}(A\backslash C) \\ &= u^{\bullet}_{q_i}(\{(A \cap K_0) \cup (A\backslash K_0)\}\backslash C) \\ &\leq u^{\bullet}_{q_i}((A \cap K_0)\backslash C) + u^{\bullet}_{q_i}(A\backslash K_0) \\ &< 2\epsilon \end{aligned}$$

for $i = 1, 2, ..., n$. Hence

$$\boldsymbol{\mu}_u(F) \in W_0. \tag{29.9.5}$$

Since M_u is a σ-algebra by Theorem 29.4(i), $A \in M_u$ if and only if $A' \in M_u$ and hence by the above argument applied to A' in place of A, there exists a compact $K \subset A'$ such that $\boldsymbol{\mu}_u(F) \in W_0$ for all $F \in M_u$ with $F \subset A'\backslash K$. Let $U = K' = T\backslash K$. Then U is open, $A \subset U$ and

$$\boldsymbol{\mu}_u(F) \in W_0 \tag{29.9.6}$$

for all $F \in M_u$ with $F \subset U\backslash A$. Thus $C \subset A \subset U$, $C \in \mathcal{C}$, $U \in \mathcal{U}$ and for $F \in M_u$ with $F \subset U\backslash C$ we have

$$\begin{aligned} \boldsymbol{\mu}_u(F) &= \boldsymbol{\mu}_u(F \cap (U\backslash A)) + \boldsymbol{\mu}_u(F \cap (A\backslash C)) \\ &\in W_0 + W_0 \subset W \end{aligned}$$

by (29.9.6) and (29.9.5), respectively.

Hence (i) holds.

(ii) Given a neighborhood W of 0 in X, by (i) there exist $K_0 \in \mathcal{C}$ and $U_0 \in \mathcal{U}$ such that $K_0 \subset A \subset U_0$ and $\boldsymbol{\mu}_u(F) \in W$ for all $F \in M_u$ with $F \subset U_0 \backslash K_0$. Let $K \in \mathcal{C}$ with $K_0 \subset K \subset A$ and $U \in \mathcal{U}$ with $A \subset U \subset U_0$. Such K and U exist by the regularity of $\boldsymbol{\mu}_u$ in M_u. Then

$$\boldsymbol{\mu}_u(A) - \boldsymbol{\mu}_u(K) = \boldsymbol{\mu}_u(A \backslash K) \in W \tag{29.9.7}$$

as $A \backslash K \subset A \backslash K_0 \subset U_0 \backslash K_0$ and

$$\boldsymbol{\mu}_u(U) - \boldsymbol{\mu}_u(A) = \boldsymbol{\mu}_u(U \backslash A) \in W \tag{29.9.8}$$

as $U \backslash A \subset U_0 \backslash A \subset U_0 \backslash K_0$.

Since $\boldsymbol{\mu}_u = \mathbf{m}_u$ by Theorem 29.6(iii), (ii) holds by (29.9.7) and (29.9.8).

Lemma 29.10. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a prolongable Radon operator. Let U be a relatively compact open set in T. Let $v = u|_{\mathcal{K}(U)}$. Then for each compact set $K \subset U$ and $E \in M_u$, $E \cap K \in M_v$. Moreover, given $q \in \Gamma$ and $\epsilon > 0$, there exist open sets O and V in T such that $E \cap K \subset O \subset V \subset \bar{V} \subset U$, $\{F \subset O \backslash (E \cap K) : F \in M_u\} = \{F \subset O \backslash (E \cap K) : F \in M_v\}$, $\boldsymbol{\mu}_u(F) = \boldsymbol{\mu}_v(F)$ for such F and $u^\bullet_q(F) = v^\bullet_q(F) \leq v^\bullet_q(O \backslash (E \cap K)) = u^\bullet_q(O \backslash (E \cap K)) < \epsilon$.

Proof. Since $K \in \mathcal{C}$ and $K \subset U$, by Theorem 50.D of [H] there exists an open set V such that $K \subset V \subset \bar{V} \subset U$. Since $E \in M_u$ and since $\delta(\mathcal{C}) \subset M_u$ by Theorem 29.8(ii), $E \cap K \in M_u$.

Claim.For each $\varphi \in \mathcal{K}(T)$, $\varphi \chi_K \in \mathcal{L}_1(v)$. (29.10.1)

In fact, as v is weakly compact on $C_0(U)$, given $q \in \Gamma$ and $\epsilon > 0$, by Theorem 6 of [P9] there exists an open set G such that $K \subset G \subset V$ with

$$\sup_{x^* \in U_q^0} |x^* \circ v|(G \backslash K) < \frac{\epsilon}{||\varphi||_T}.$$

Then by Lemma 26.17 we have $v_q^\bullet(G\backslash K) < \frac{\epsilon}{||\varphi||_T}$. By Urysohn's lemma there exists $g \in \mathcal{K}(U)$ such that $\chi_K \leq g \leq \chi_G$. Then $\varphi g \in \mathcal{K}(U)$ and

$$v_q^\bullet(|\varphi\chi_K - \varphi g|) \leq ||\varphi||_T v_q^\bullet(|\chi_K - g|) \leq ||\varphi||_T v_q^\bullet(G\backslash K) < \epsilon.$$

This shows that $\varphi\chi_K \in \mathcal{L}_1(v)$.

As $E \in M_u$, there exists $\Psi \in \mathcal{K}(T)$ such that $u_q^\bullet(|\chi_E - \Psi|) < \epsilon$. By the above claim $\Psi\chi_K \in \mathcal{L}_1(v)$ and $v_q^\bullet(|\chi_{E\cap K} - \Psi\chi_K|) = v_q^\bullet(\chi_K(|\chi_E - \Psi|)) = u_q^\bullet(\chi_K|\chi_E - \Psi|) \leq u_q^\bullet(|\chi_E - \Psi|) < \epsilon$, since $\chi_K|\chi_E - \Psi|$ has compact support contained in U so that by the complex version of Lemma 3.2 of [T] applies. Thus $E \cap K \in M_v$. Moreover, $E \cap K$ is relatively compact in T and hence by the complex lcHs version of Lemma 1.19 of [T] and by Theorem 29.8(ii), given $q \in \Gamma$ and $\epsilon > 0$, there exists an open set O in T with $O \in M_v$ such that $E \cap K \subset O \subset V \subset \bar{V} \subset U$ such that $v_q^\bullet(O\backslash(E \cap K)) < \epsilon$. Then each $F \in M_v$ with $F \subset O\backslash(E \cap K)$ has compact support contained in U and hence by the complex version of Lemma 3.2 of [T] $F \in M_u$, $\boldsymbol{\mu}_u(F) = \boldsymbol{\mu}_v(F)$ and $u_q^\bullet(F) = v_q^\bullet(F) \leq v_q^\bullet(O\backslash(E \cap K)) < \epsilon$.

Conversely, let us suppose $F \subset O\backslash(E \cap K)$ and $F \in M_u$. Then given $q \in \Gamma$ and $\epsilon > 0$, let $\varphi \in K(T)$ such that $u_q^\bullet(|\varphi - \chi_F|) < \epsilon$. Clearly, as $F \subset O \subset V \subset \bar{V} \subset U$, we have

$$|\varphi\chi_{\bar{V}} - \chi_F| = |\varphi\chi_{\bar{V}} - \chi_{\bar{V}}\chi_F| \leq |\varphi - \chi_F|,$$

$\varphi\chi_{\bar{V}} \in \mathcal{L}_1(v)$ by the above claim and by the complex version of Lemma 3.2 of [T] we have

$$v_q^\bullet(|\chi_F - \varphi\chi_{\bar{V}}|) \leq u_q^\bullet(|\chi_F - \varphi|) < \epsilon$$

as $|\chi_F - \varphi\chi_{\bar{V}}| \leq |\chi_F - \varphi|$. Thus $\chi_F \in \mathcal{L}_1(v)$ and hence $F \in M_v$.

Theorem 29.11. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a prolongable Radon operator. Then:

(i) $\boldsymbol{\mu}_u$ is M_u-inner regular in the sense that given $E \in M_u$ and a neighborhood W of 0 in X, there exists $C \in \mathcal{C}$ such that $C \subset E$ and $\boldsymbol{\mu}_u(F) \in W$ for all $F \subset E\backslash C$ with $F \in M_u$.

(ii) $\boldsymbol{\mu}_u$ is restrictedly M_u-outer regular in the sense that given $E \in M_u$, $K \in \mathcal{C}$ and a neighborhood W of 0 in X, there exists a relatively compact open set O in T such that $E \cap K \subset O$ with $\boldsymbol{\mu}_u(F) \in W$ for all $F \subset O\backslash(E \cap K)$ with $F \in M_u$.

(iii) $\boldsymbol{\mu}_u|_{\delta(\mathcal{C})}$ is $\delta(\mathcal{C})$-regular.

(iv) For each $E \in M_u$,

$$\boldsymbol{\mu}_u(E) = \lim_{K \subset E, K \in \mathcal{C}} \boldsymbol{\mu}_u(K) = \lim_{K \in \mathcal{C}} \lim_{E \cap K \subset O, O \in \mathcal{U} \cap \delta(\mathcal{C})} \boldsymbol{\mu}_u(O)$$

where $\mathcal{C}$ is directed by the relation $K_1 \leq K_2$ if $K_1 \subset K_2$ and $\mathcal{U} \cap \delta(\mathcal{C})$ is directed by the relation $O_1 \leq O_2$ if $O_2 \subset O_1$.

Proof. Choose a neighborhood W_0 of 0 in X such that $W_0 + W_0 \subset W$. Let $(q_i)_1^n \subset \Gamma$ and $\epsilon > 0$ such that $\bigcap_{i=1}^n \{x : q_i(x) < 2\epsilon\} \subset W_0$. Let $E \in M_u$. Then E is u-integrable and hence by the complex lcHs version of Lemma 1.24 of T there exists $K_0 \in \mathcal{C}$ such that

$$u^\bullet_{q_i}(E \backslash (E \cap K)) < \epsilon \qquad (29.11.1)$$

for $K_0 \subset K \in \mathcal{C}$ and for $i = 1, 2, ..., n$. Since u is prolongable, by Theorem 29.8, M_u is a δ-ring containing $\delta(\mathcal{C})$ and $\boldsymbol{\mu}_u$ is σ-additive on M_u by Theorem 29.2. Consequently, $E \cap K \in M_u$ for $K \in \mathcal{C}$ and by (29.11.1) we have

$$|\boldsymbol{\mu}_u(E) - \boldsymbol{\mu}_u(E \cap K)|_{q_i} = |\boldsymbol{\mu}_u(E \backslash (E \cap K))|_{q_i} \leq u^\bullet_{q_i}(E \backslash (E \cap K)) < \epsilon$$

for $i = 1, 2, ..., n$ and for $K_0 \subset K \in \mathcal{C}$. Hence

$$\boldsymbol{\mu}_u(E) = \lim_{K \in \mathcal{C}} \boldsymbol{\mu}_u(E \cap K). \qquad (29.11.2)$$

(i) Since E is u-measurable, there exists a compact $K_i \subset K_0$ such that $\chi_E|_{K_i}$ is continuous so that $K_i \subset E \cap K_0$ or $K_i \subset E' \cap K_0$ and $u^\bullet_{q_i}(K_0 \backslash K_i) < \epsilon$ for $i = 1, 2, ..., n$. Let $J = \{i : K_i \subset E \cap K_0\}$ and let $C = \bigcup_{i \in J} K_i$. If $J = \emptyset$, take $C = \emptyset$. Then

$$C \in \mathcal{C}, C \subset K_0 \cap E \text{ and } u^\bullet_{q_i}((E \cap K_0) \backslash C) < \epsilon \qquad (29.11.3)$$

for $i = 1, 2, ..., n$, since

$$\begin{aligned} u^\bullet_{q_i}((E \cap K_0) \backslash E) &\leq u^\bullet_{q_i}(\{(E \cap K_0) \cup (E' \cap K_0)\} \backslash \bigcup_1^n K_i) \\ &= u^\bullet_{q_i}(K_0 \backslash \bigcup_1^n K_i) \\ &\leq u^\bullet_{q_i}(K_0 \backslash K_i) \\ &< \epsilon. \end{aligned}$$

Then $C \subset E$, $C \in \mathcal{C}$ and by (29.11.1) and (29.11.3) we have

$$u^\bullet_{q_i}(E\backslash C) \leq u^\bullet_{q_i}(E\backslash(E\cap K_0)) + u^\bullet_{q_i}((E\cap K_0)\backslash C) < 2\epsilon$$

for $i = 1, 2, ..., n$. Then for $F \in M_u$ with $F \subset E\backslash C$ we have

$$|\boldsymbol{\mu}_u(F)|_{q_i} \leq u^\bullet_{q_i}(F) \leq u^\bullet_{q_i}(E\backslash C) < 2\epsilon$$

for $i = 1, 2, ..., n$ and hence $\boldsymbol{\mu}_u(F) \in W_0 \subset W$. Thus (i) holds.

(ii) Let $K \in \mathcal{C}$ and let U be a relatively compact open set in T such that $K \subset U$. Then by Lemma 29.10 there exist open sets O and V such that $E \cap K \subset O \subset V \subset \bar{V} \subset U$, $\{F \subset O\backslash(E\cap K) : F \in M_u\} = \{F \subset O\backslash(E\cap K) : F \in M_v\}$, $\boldsymbol{\mu}_u(F) = \boldsymbol{\mu}_v(F)$ for such F and $v^\bullet_{q_i}(O\backslash(E\cap K)) = u^\bullet_{q_i}(O\backslash(E\cap K)) < \epsilon$ for $i = 1, 2, ..., n$ where $v = u|\mathcal{K}(U)$. This shows that for all $F \in M_u$ with $F \subset O\backslash(E\cap K)$,

$$|\boldsymbol{\mu}_u(F)|_{q_i} \leq u^\bullet_{q_i}(O\backslash(E\cap K)) < \epsilon$$

for $i = 1, 2, ..., n$ and hence $\boldsymbol{\mu}_u(F) \in W_0 \subset W$ for $F \in M_u$ with $F \subset O\backslash(E\cap K)$. Hence (ii) holds.

(iii) Let $\boldsymbol{\omega} = \boldsymbol{\mu}_u|_{\delta(\mathcal{C})}$ and let $E \in \delta(\mathcal{C})$. By (i) there exists $C \subset E$, $C \in \mathcal{C}$ such that

$$\boldsymbol{\omega}(F) = \boldsymbol{\mu}_u(F) \in W_0$$

for $F \subset E\backslash C$ with $F \in \delta(\mathcal{C})$.

As $E \in \delta(\mathcal{C})$, there exist a compact C and a relatively compact open set V in T such that $E \subset C \subset V$. Then $E = E\cap C$ and hence by (ii) there exists a relatively compact open set O such that $E \subset O$ and $\boldsymbol{\mu}_u(F) \in W_0$ for all $F \in M_u$ with $F \subset O\backslash E$. Thus particularly for all $F \in \delta(\mathcal{C})$ with $F \subset O\backslash E$, $\boldsymbol{\omega}(F) = \boldsymbol{\mu}_u(F) \in W_0$. Then $C \subset E \subset O$, $C \in \mathcal{C}$, $O \in \mathcal{U} \cap \delta(\mathcal{C})$, and for $F \in \delta(\mathcal{C})$ with $F \subset O\backslash C$ we have

$$\boldsymbol{\omega}(F) = \boldsymbol{\mu}_u(F) = \boldsymbol{\mu}_u(F\cap(O\backslash E)) + \boldsymbol{\mu}_u(F\cap(E\backslash C)) \in W_0 + W_0 \subset W.$$

Hence $\boldsymbol{\mu}_u|\delta(\mathcal{C})$ is $\delta(\mathcal{C})$-regular.

(iv) Let $q \in \Gamma$, $\epsilon > 0$ and $K \in \mathcal{C}$. By (ii) there exists $O_1 \in \mathcal{U}\cap\delta(\mathcal{C})$ such that $K\cap E \subset O_1$ and for all $O \in \mathcal{U}$ with $E\cap K \subset O \subset O_1$ we have $|\boldsymbol{\mu}_u(F)|_q < \epsilon$ for all $F \in M_u$ with $F \subset (O_1\backslash E\cap K)$.

Then we have

$$|\boldsymbol{\mu}_u(E \cap K) - \boldsymbol{\mu}_u(O)|_q = |\boldsymbol{\mu}_u(O \backslash (E \cap K))|_q < \epsilon$$

and hence

$$\lim_{E \cap K \subset O \in \mathcal{U} \cap \delta(\mathcal{C})} \boldsymbol{\mu}_u(O) = \boldsymbol{\mu}_u(E \cap K). \tag{29.11.3}$$

Then by (29.11.2) and (29.11.3) we have

$$\boldsymbol{\mu}_u(E) = \lim_{K \in \mathcal{C}} \boldsymbol{\mu}_u(E \cap K) = \lim_{K \in \mathcal{C}} \lim_{E \cap K \subset O \in \mathcal{U} \cap \delta(\mathcal{C})} \boldsymbol{\mu}_u(O). \tag{29.11.4}$$

Finally, as $\boldsymbol{\mu}_u$ is inner regular in M_u by (i), we have

$$\boldsymbol{\mu}_u(E) = \lim_{K \subset E, K \in \mathcal{C}} \boldsymbol{\mu}_u(K). \tag{29.11.5}$$

and hence by (29.11.4) and (29.11.5), (iv) holds.

This completes the proof of the theorem.

Definition 29.12. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator with $\boldsymbol{\mu}_u$ as in Definition 29.1. For each open set U in T, by Theorem 29.9(ii),

$$\boldsymbol{\mu}_u(U) = \lim_{K \in \mathcal{C}, K \subset U} \boldsymbol{\mu}_u(K). \tag{29.12.1}$$

Let $A \subset T$ and let

$$\boldsymbol{\mu}_u^*(A) = \lim_{A \subset U \in \mathcal{U}} \boldsymbol{\mu}_u(U)$$

whenever the limit exists, where $\mathcal{U}$ is directed by the relation $U_1 \leq U_2$ if $U_2 \subset U_1$.

Theorem 29.13. Let X, u and $\boldsymbol{\mu}_u^*$ be as in Definition 29.12. Then $\boldsymbol{\mu}_u^*(A)$ exists in X for each $A \subset T$.

Proof. $\boldsymbol{\mu}_u$ is σ-additive on M_u by Theorem 29.2 and M_u is a σ-algebra in T by Theorem 29.4(i). Therefore, the range of $\boldsymbol{\mu}_u$ is relatively weakly compact in X by Theorem on Extension of [K3] (or by Corollary 2 of [P7]) and hence is bounded in the lcHs topology τ of X. Since

$\mathcal{B}(T) \subset M_u$ by Theorem 29.4(ii) and since $\boldsymbol{\mu}_u$ is σ-additive on M_u, for each increasing sequence $(K_n)_1^\infty \subset \mathcal{C}$, $\lim_n \boldsymbol{\mu}_u(K_n) \in X$. Moreover, by Theorem 29.9(i), $\boldsymbol{\mu}_u$ is M_u-regular and hence, given $K \in \mathcal{C}$ and a neighborhood W of 0 in X, there exists $U \in \mathcal{U}$ such that $K \subset U$ and for each compact C with $K \subset C \subset U$, $\boldsymbol{\mu}_u(C) - \boldsymbol{\mu}_u(K) \in W$. Thus conditions 6.1 of Sion [Si] are satisfied by $T, \mathcal{C}, \mathcal{U}$ and $\boldsymbol{\mu}_u$, excepting that X is a quasicomplete lcHs so that every bounded closed set in X is complete. Since $\boldsymbol{\mu}_u$ is σ-additive on M_u and since $\mathcal{U} \subset M_u$, for every monotone sequence $(U_n)_1^\infty$ of open sets in T, $\lim_n \boldsymbol{\mu}_u(U_n) \in X$. Consequently, by Lemma 2.5 of Sion [Si], $\{\boldsymbol{\mu}_u(U)\}_{A \subset U \in \mathcal{U}}$ is a Cauchy net in X. Since X is quasicomplete and since the range of $\boldsymbol{\mu}_u$ is bounded, $\lim_{A \subset U \in \mathcal{U}} \boldsymbol{\mu}_u(U)$ exists in X and hence $\boldsymbol{\mu}_u^*(A) \in X$ for each $A \subset T$.

Theorem 29.14. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Then a subset A of T is u-integrable if and only if, for each $q \in \Gamma$ and $\epsilon > 0$, there exist a compact C and an open set U in T such that $C \subset A \subset U$ and $u_q^\bullet(U \backslash C) < \epsilon$.

Proof. Suppose A is u-integrable. Proceeding as in the proof of Theorem 29.9(i) with $q \in \Gamma$, by the complex lcHs analogue of Lemma 1.24 of [T] there exists $K_0 \in \mathcal{C}$ such that

$$u_q^\bullet(A \backslash (A \cap K)) = u_q^\bullet(\chi_A \chi_{T \backslash K}) < \frac{\epsilon}{4} \qquad (29.14.1)$$

for all $K \in \mathcal{C}$ with $K \supset K_0$. Since A is u-measurable, there exists a compact $C \subset K_0$ such that $\chi_A|_C$ is continuous and $u_q^\bullet(K_0 \backslash C) < \frac{\epsilon}{4}$. Then $C \subset A$, $C \in \mathcal{C}$ and

$$\begin{aligned} u_q^\bullet(A \backslash C) &= u_q^\bullet((A \cap K_0) \backslash C) \cup ((A \backslash K_0) \backslash C) \\ &\leq u_q^\bullet((A \cap K_0) \backslash C) + u_q^\bullet(A \backslash K_0) \\ &< \frac{\epsilon}{2}. \end{aligned}$$

By a similar argument applied to $A' = T \backslash A \in M_u$, there exists a compact $K \subset A'$ such that $u_q^\bullet(A' \backslash K) < \frac{\epsilon}{2}$. Let $U = T \backslash K$. Then U is open, $A \subset U$ and $u_q^\bullet(U \backslash A) = u_q^\bullet(A' \backslash K) < \frac{\epsilon}{2}$. Consequently, $C \subset A \subset U$, $C \in \mathcal{C}$, $U \in \mathcal{U}$ and $u_q^\bullet(U \backslash C) < \epsilon$.

Conversely, let us suppose that the conditions are satisfied for each $q \in \Gamma$ and for each $\epsilon > 0$. For $q \in \Gamma$ and $\epsilon > 0$, let $C \subset A \subset U$, $C \in \mathcal{C}$, $U \in \mathcal{U}$ with $u_q^\bullet(U \backslash C) < \epsilon$. Then by Urysohn's lemma, there exists $\varphi \in \mathcal{K}(T)$ such that $\chi_C \leq \varphi \leq \chi_U$. Then $|\chi_A - \varphi| \leq \chi_U - \chi_C$ and hence we have

$$u_q^\bullet(\chi_A - \varphi) \leq u_q^\bullet(\chi_{U \backslash C}) = u_q^\bullet(U \backslash C) < \epsilon.$$

Therefore, $\chi_A \in \mathcal{L}_1(u)$ and hence $A \in M_u$.

Definition 29.15. Let $M_{\boldsymbol{\mu}_u^*} = \{E \subset T : \boldsymbol{\mu}_u^*(A) = \boldsymbol{\mu}_u^*(A \cap E) + \boldsymbol{\mu}_u^*(A \backslash E) \text{ for all } A \subset T\}$. Members of $M_{\boldsymbol{\mu}_u^*}$ are called Carathéodory-Sion $\boldsymbol{\mu}_u^*$-measurable sets.

Theorem 29.16. $M_{\boldsymbol{\mu}_u^*} \subset M_u$ and $\boldsymbol{\mu}_u^*(E) = \boldsymbol{\mu}_u(E)$ for $E \in M_{\boldsymbol{\mu}_u^*}$. Consequently, each $E \in M_{\boldsymbol{\mu}_u^*}$ is u-measurable and u-integrable.

Proof. Since the range of $\boldsymbol{\mu}_u$ is bounded and since X is quasicomplete, Theorem 6.3 of Sion [Si] holds here and hence $\mathcal{B}(T) \subset M_{\boldsymbol{\mu}_u^*}$. Let $A \in M_{\boldsymbol{\mu}_u^*}$, $q \in \Gamma$ and $\epsilon > 0$. Then by (3) of Theorem 6.3 of Sion [Si] there exists $K \in \mathcal{C}$ and $U \in \mathcal{U}$ such that $K \subset A \subset U$ and $q(\boldsymbol{\mu}_u^*(F)) < \frac{\epsilon}{4}$ for all $F \in M_{\boldsymbol{\mu}_u^*}$ with $F \subset U \backslash K$. As $\mathcal{B}(T) \subset M_u$ by Theorem 29.4(ii), by Definition 29.12 and by Theorem 29.9(ii) we have $\boldsymbol{\mu}_u^*(A) = \lim_{A \subset U \in \mathcal{U}} \boldsymbol{\mu}_u(U) = \boldsymbol{\mu}_u(A)$ for $A \in \mathcal{B}(T)$ and hence $q(\boldsymbol{\mu}_u(F)) < \frac{\epsilon}{4}$ for all $F \in \mathcal{B}(T)$ with $F \subset U \backslash K$. Since $U \backslash K$ is open, by Lemma 26.17 we have

$$\begin{aligned}
u_q^{\bullet}(U \backslash K) &= \sup_{x^* \in U_q^0} |x^* \circ u|(U \backslash K) \\
&\leq 4 \sup_{x^* \in U_q^0, F \subset U \backslash K, F \in \mathcal{B}(T)} |\mu_{x^* u}(F)| \\
&= 4 \sup_{x^* \in U_q^0, F \subset U \backslash K, F \in \mathcal{B}(T)} |(x^* \boldsymbol{\mu}_u)(F)| \\
&= 4 \sup_{F \subset U \backslash K, F \in \mathcal{B}(T)} q(\boldsymbol{\mu}_u(F)) < 4 \frac{\epsilon}{4} = \epsilon
\end{aligned}$$

where we use the relation

$$x^* \boldsymbol{\mu}_u(E) = \mu_{x^* u}(E) \text{ for } E \in M_u \text{ and hence for } E \in \mathcal{M}_u \tag{29.16.1}$$

as $mathcalB(T) \subset M_u$. In fact, as $E \in M_u$, $E \in M_{x*u}$. If $f \in \mathcal{L}_1(u)$, then $f \in \mathcal{L}_1(x^* u)$ and hence $\mu_{x^* u}(f) = \int f d(x^* u) = x^*(\int f du)$ and hence $\mu_{x^* u}(E) = x^*(\int_E du) = x^* \boldsymbol{\mu}_u(E)$ so that $x^* \boldsymbol{\mu}_u(E) = \mu_{x^* u}(E)$ for $E \in \mathcal{M}_u$.

Consequently, by Theorem 29.14, $A \in M_u$ and hence $M_{\boldsymbol{\mu}_u^*} \subset M_u$. Then by Theorem 29.9(ii) it follows that $\boldsymbol{\mu}_u^*(A) = \boldsymbol{\mu}_u(A)$ for $A \in M_{\boldsymbol{\mu}_u^*}$. The last part is evident from Definition 29.1.

This completes the proof of the theorem.

To prove that $M_{\boldsymbol{\mu}_u^*} = M_u$, we proceed as below and prove the following lemmas.

Let θ be a positive linear functional on $\mathcal{K}(T)$. For $E \subset T$, let

$$\mu_\theta^*(E) = \inf_{\chi_E \leq g \in \mathcal{I}^+} \sup\{\theta(\Psi) : \Psi \leq g, \Psi \in C_c^+(T)\}.$$

Then by Rudin [Ru1], μ_θ^* is an outer measure on $\mathcal{P}(T)$, the family of all subsets of T. Let $M_{\mu_\theta^*} = \{E \subset T : E$ is μ_θ^* -measurable $-$ that is, $\mu_\theta^*(A) = \mu_\theta^*(A \cap E) + \mu_\theta^*(A \backslash E)$ for all $A \subset T\}$. Then $M_{\mu_\theta^*}$ is a σ-algebra in T and contains $\mathcal{B}(T)$. (See Theorem 2.2 of [P3]).

Lemma 29.17. Let $\theta \in \mathcal{K}(T)^*$. For $A \subset T$ with $\mu_{|\theta|}^*(A) < \infty$, let

$$\mu_\theta^*(A) = \{(\mu_{\theta_1^+}^* - \mu_{\theta_1^-}^*) + i(\mu_{\theta_2^+}^* - \mu_{\theta_2^-}^*)\}(A)$$

where $\theta = Re\theta + iIm\theta, Re\theta = \theta_1^+ - \theta_1^-,$ and $Im\theta = \theta_2^+ - \theta_2^-$. Let $E \in M_\theta = \{E \in M_{\mu_{\theta_1^+}^*} \cap M_{\mu_{\theta_1^-}^*} \cap M_{\mu_{\theta_2^+}^*} \cap M_{\mu_{\theta_2^-}^*}$ with $\mu_{|\theta|}(E) < \infty\}$. Then

$$\mu_\theta^*(A) = \mu_\theta^*(A \cap E) + \mu_\theta^*(A \backslash E). \qquad (29.17.1)$$

Proof. Let $E \in M_\theta$. Then for $A \subset T$, we have

$$\mu_{\theta_i^+}^*(A) = \mu_{\theta_i^+}^*(A \cap E) + \mu_{\theta_i^+}^*(A \backslash E) \qquad (29.27.2)$$

and

$$\mu_{\theta_i^-}^*(A) = \mu_{\theta_i^-}^*(A \cap E) + \mu_{\theta_i^-}^*(A \backslash E) \qquad (29.17.3)$$

for $i = 1, 2$. If $\mu_{|\theta|}^*(A) < \infty$, then $\mu_{\theta_i^+}^*(A) < \infty$ and $\mu_{\theta_i^-}^*(A) < \infty$ for $i = 1, 2$ and hence by (29.17.2) and (29.17.3) and by the definition of $\mu_\theta^*(A)$, (29.17.1) holds.

Lemma 29.18. If $\theta \in \mathcal{K}(T)^*$ and is bounded, then for $A \subset T$

$$\mu_\theta^*(A) = \lim_{A \subset U \in \mathcal{U}} \mu_\theta^*(U).$$

Proof. By Theorem 2.2 of [P3],

$$\begin{aligned} \mu_{\theta_i^+}^*(A) &= \inf\{\mu_{\theta_i^+}^*(U) : A \subset U \in \mathcal{U}\} \\ &= \lim_{A \subset U \in \mathcal{U}} \mu_{\theta_i^+}^*(U)\} \end{aligned}$$

for $i = 1, 2$ for $A \subset T$. A similar expression holds for $\mu^*_{\theta_i^-}(A)$ for $i = 1, 2$. Then by the definition of $\mu^*_\theta(A)$ as given in Lemma 29.17, the lemma holds.

Lemma 29.19. Let θ, M_θ, θ_1 and θ_2 be as in Lemma 29.17 and let $\mu_1 = \mu_{\theta_1^+}$, $\mu_2 = \mu_{\theta_1^-}$, $\mu_3 = \mu_{\theta_2^+}$ and $\mu_4 = \mu_{\theta_2^-}$. For $A \subset T$ and $E \in M_\theta$,

$$\begin{aligned} \mu_j^*(A \cap E) = \inf_{A \cap E \subset U \in \mathcal{U}} \mu_j^*(U) &= \lim_{A \cap E \subset U \in \mathcal{U}} \mu_j^*(U) \\ &= \inf_{A \subset U \in \mathcal{U}} \mu_j^*(U \cap E) = \lim_{A \subset U \in \mathcal{U}} \mu_j^*(U \cap E) \end{aligned} \tag{29.19.1}$$

and

$$\begin{aligned} \mu_j^*(A \backslash E) = \inf_{A \backslash E \subset U \in \mathcal{U}} \mu_j^*(U) &= \lim_{A \backslash E \subset U \in \mathcal{U}} \mu_j^*(U) \\ &= \inf_{A \subset U \in \mathcal{U}} \mu_j^*(U \backslash E) = \lim_{A \subset U \in \mathcal{U}} \mu_j^*(U \backslash E) \end{aligned} \tag{29.19.2}$$

for $j = 1, 2, 3, 4$. Consequently,

$$\mu_\theta^*(A \cap E) + \mu_\theta^*(A \backslash E) = \lim_{A \subset U \in \mathcal{U}} \mu_\theta^*(U \cap E) + \lim_{A \subset U \in \mathcal{U}} \mu_\theta^*(U \backslash E). \tag{29.19.3}$$

Proof.

$$\mu_j^*(A \cap E) = \inf_{A \cap E \subset U \in \mathcal{U}} \mu_j^*(U) \geq \inf_{A \subset U \in \mathcal{U}} \mu_j^*(U \cap E) \geq \mu_j^*(A \cap E)$$

and hence

$$\begin{aligned} \mu_j^*(A \cap E) = \inf_{A \cap E \subset U \in \mathcal{U}} \mu_j^*(U) &= \lim_{A \cap E \subset U \in \mathcal{U}} \mu_j^*(U) \\ &= \inf_{A \subset U \in \mathcal{U}} \mu_j^*(U \cap E) = \lim_{A \subset U \in \mathcal{U}} \mu_j^*(U \cap E) \end{aligned}$$

for $j = 1, 2, 3, 4$ and hence (29.19.1) holds. Similarly, (29.19.2) holds. Then (29.19.3) holds by (29.19.1) and (29.19.2).

Theorem 29.20. Let X be a quasicomplete lcHs and let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Then $M_{\boldsymbol{\mu}_u^*} = M_u$ and $\boldsymbol{\mu}_u^*(A) = \mathbf{m}_u(A) = \boldsymbol{\mu}_u(A)$ for $A \in M_u$.

Proof. In the light of Theorem 29.16 it suffices to show that $M_u \subset M_{\boldsymbol{\mu}_u^*}$. Let $E \in M_u$. Then $x^*u \in K(T)^*$ and is bounded. Moreover,

$$E \in M_{x^*u} \ \text{ for each } x^* \in X^* \tag{29.20.1}$$

since E is u-integrable and hence is x^*u-integrable for each $x^* \in X^*$. Then by Definition 29.12 we have

$$\boldsymbol{\mu}_u^*(A) = \lim_{A \subset U \in \mathcal{U}} \boldsymbol{\mu}_u(U) \text{ for } A \subset T. \qquad (29.20.2)$$

Then by (29.20.2) and by (29.16.1) we have

$$\begin{aligned} x^*\boldsymbol{\mu}_u^*(A) &= \lim_{A \subset U \in \mathcal{U}} x^*\boldsymbol{\mu}_u(U) \\ &= \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U) \\ &= \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U \cap E) + \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U \backslash E) \qquad (29.20.3) \end{aligned}$$

by (29.20.1) since $U \cap E$ and $U \backslash E$ belong to M_{x^*u}. Then by (29.20.3) and by (29.19.3) we have

$$x^*\boldsymbol{\mu}_u^*(A) = \mu_{x^*u}^*(A \cap E) + \mu_{x^*u}^*(A \backslash E). \qquad (29.20.4)$$

On the other hand, by (29.19.1) and (29.16.1) we have

$$\mu_{x^*u}^*(A \cap E) = \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U \cap E) = \lim_{A \subset U \in \mathcal{U}} x^*\boldsymbol{\mu}_u(U \cap E) \qquad (29.20.4)$$

since $U \in \mathcal{B}(T) \subset M_u$ by Theorem 29.4 and since $E \in M_u$ by hypothesis.

Similarly,

$$\mu_{x^*u}^*(A \backslash E) = \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U \backslash E) = \lim_{A \subset U \in \mathcal{U}} x^*\boldsymbol{\mu}_u(U \backslash E) \qquad (29.20.5)$$

and hence by (29.20.3), (29.20.4) and (29.20.5) and by Lemma 29.19 we have

$$\begin{aligned} x^*\boldsymbol{\mu}_u^*(A) &= \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U \cap E) + \lim_{A \subset U \in \mathcal{U}} \mu_{x^*u}(U \backslash E) \\ &= \lim_{A \cap E \subset U \in \mathcal{U}} \mu_{x^*u}(U) + \lim_{A \backslash E \subset U \in \mathcal{U}} \mu_{x^*u}(U) \\ &= x^*\boldsymbol{\mu}_u^*(A \cap E) + x^*\boldsymbol{\mu}_u^*(A \backslash E) \end{aligned}$$

for $x^* \in X^*$. Therefore, by the Hahn-Banach theorem we have

$$\boldsymbol{\mu}_u^*(A) = \boldsymbol{\mu}_u^*(A \cap E) + \boldsymbol{\mu}_u^*(A \backslash E)$$

for $A \subset T$ and hence $E \in M_{\boldsymbol{\mu}_u^*}$. Then by Theorem 29.16, $M_u = M_{\boldsymbol{\mu}_u^*}$. Moreover, $\boldsymbol{\mu}_u(E) = \mathbf{m}_u(E) = \boldsymbol{\mu}_u^*(E)$ for $E \in M_u$ by Theorems 29.6 and 29.16.

This completes the proof of the theorem.

Theorem 29.21. Let X be a quasicomplete lcHs. Let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator and let $q \in \Gamma$. Given $A \in M_u$, there exist a σ-compact set F and a G_δ set G such that $F \subset A \subset G$ and $u_q^\bullet(G \backslash F) = 0$. Conversely, given $A \subset T$, suppose for each $q \in \Gamma$ there exist a σ-compact F and a G_δ G such that $F \subset A \subset G$ and $u_q^\bullet(G \backslash F) = 0$. Then $A \in M_u$.

Proof. Let $q \in \Gamma$ be given. For $\epsilon = \frac{1}{n}$, by Theorem 29.14 there exist a compact K_n and an open set U_n in T such that $K_n \subset A \subset U_n$ with $u_q^\bullet(U_n \backslash K_n) < \frac{1}{n}$. Then $F = \bigcup_1^\infty K_n$ is σ-compact, $G = \bigcap_1^\infty U_n$ is G_δ and $F \subset A \subset G$. Clearly, $u_q^\bullet(G \backslash F) = 0$ since $u_q^\bullet(G \backslash F) \leq u_q^\bullet(U_n \backslash K_n) < \frac{1}{n}$ for all $n \in \mathbb{N}$.

Conversely, let $A \subset T$ be such that for each $q \in \Gamma$ there exist a σ-compact F and a G_δ G such that $F \subset A \subset G$ with $u_q^\bullet(G \backslash F) = 0$. Without loss of generality we shall assume that $F = \bigcup C_n$, $(C_n)_1^\infty \subset \mathcal{C}$, $C_n \nearrow F$, $G = \bigcap_1^\infty U_n$, $(U_n)_1^\infty \subset \mathcal{U}$ and $U_n \searrow G$. Since $U_n \backslash C_n \searrow G \backslash F$, $(U_n \backslash C_n)_1^\infty \subset M_u$ and $u_q^\bullet(\cdot) = ||\mathbf{m}_u||_q(\cdot)$ by Theorem 29.6(i) and since $||\mathbf{m}_u||_q$ is continuous as $\mathbf{m}_u$ is σ-additive, we have

$$0 = u_q^\bullet(G \backslash F) = ||\mathbf{m}_u||_q(G \backslash F) = \lim_n ||\mathbf{m}_u||_q(U_n \backslash C_n).$$

Thus, given $\epsilon > 0$, there exists n_0 such that $||\mathbf{m}_u||_q(U_n \backslash C_n) < \epsilon$ for $n \geq n_0$. Let $U = U_{n_0}$ and $C = C_{n_0}$. Then $u_q^\bullet(U \backslash C) = ||\mathbf{m}_u||_q(U \backslash C) < \epsilon$ and hence by Theorem 29.14, $A \in M_u$.

This completes the proof of the theorem.

Definition 29.22. Let X be a quasicomplete lcHs. Let $\mathcal{D}$ be a δ-ring containing $\mathcal{C}$ and let $\boldsymbol{\mu} : \mathcal{D} \to X$ be σ-additive. If $\boldsymbol{\mu}$ is the restriction of an X-valued weakly compact Radon vector measure $\boldsymbol{\mu}_u$, then the Lebesgue-Radon completion $\widetilde{\mathcal{D}}$ of $\mathcal{D}$ with respect to $\boldsymbol{\mu}_u$ is defined as the family $\{E \subset T :$ given $q \in \Gamma$ there exist a σ-compact F and a G_δ G such that $F \subset E \subset G$ with $u_q^\bullet(G \backslash F) = 0\}$ and the Lebesgue-Radon completion $\widetilde{\boldsymbol{\mu}_u}$ of $\boldsymbol{\mu}_u$ with respect to $\mathcal{D}$ is said to exist on $\widetilde{\mathcal{D}}$ if

$$\widetilde{\boldsymbol{\mu}_u}(E) = \lim_{K \in \mathcal{C}, K \subset E} \boldsymbol{\mu}(K)$$

exists in X for each $E \in \widetilde{\mathcal{D}}$.

The following theorem generalizes the bounded case of Theorem 4.4 of [P4].

Theorem 29.23. Let X be a quasicomplete lcHs. Let $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$ be σ-additive. Then $\boldsymbol{\mu}$ is the restriction of an X-valued weakly compact Radon vector measure $\boldsymbol{\mu}_u$ if and only if $\boldsymbol{\mu}$ is $\delta(\mathcal{C})$-regular and its range is relatively weakly compact. In that case, u is unique and is called the bounded weakly compact Radon operator determined by $\boldsymbol{\mu}$. Moreover, $M_u = \widetilde{\delta(\mathcal{C})}$, the Lebesgue-Radon completion of $\delta(\mathcal{C})$ with respect to $\boldsymbol{\mu}_u$. The Lebsesgue-Radon completion $\widetilde{\boldsymbol{\mu}_u}$ of $\boldsymbol{\mu}_u$ with respect to $\delta(\mathcal{C})$ exists on $\widetilde{\delta(\mathcal{C})}$ and coincides with $\boldsymbol{\mu}_u$.

Proof. Suppose $u : \mathcal{K}(T) \to X$ is a bounded weakly compact Radon operator and suppose $\boldsymbol{\mu} = \boldsymbol{\mu}_u|_{\delta(\mathcal{C})}$. Then $\boldsymbol{\mu}_u$ is M_u-regular by Theorem 29.9(i) and $\delta(\mathcal{C}) \subset M_u$ by Theorem 29.4(ii). Let $E \in \delta(\mathcal{C})$. As $\boldsymbol{\mu}_u$ is M_u-regular, given a neighborhood W of 0 in X, there exist a compact $K \subset E$ and an open set U in T such that $U \supset E$ and such that $\boldsymbol{\mu}_u(F) \in W$ for all $F \in M_u$ with $F \subset U \backslash K$. Hence, particularly, $\boldsymbol{\mu}_u|_{\delta(\mathcal{C})}$ is $\delta(\mathcal{C})$-regular. Therefore, $\boldsymbol{\mu}$ is $\delta(\mathcal{C})$-regular. Since M_u is a σ-algebra in T by Theorem 29.4(i) and since $\boldsymbol{\mu}_u$ is σ-additive on M_u by Theorem 29.2, the range of $\boldsymbol{\mu}_u$ and hence that of $\boldsymbol{\mu}$ is relatively weakly compact by Theorem on Extension of [K3] or by Corollary 2 of [P7].

If $\boldsymbol{\mu}$ is also equal to $\boldsymbol{\mu}_v|_{\delta(\mathcal{C})}$ for another bounded weakly compact Radon operator v on T, then by the uniqueness part of Theorem 4.4(i) of [P4], $\mu_{x^*u} = \mu_{x^*v}$ on $\delta(\mathcal{C})$ and hence $x^*u = x^*v$ on $\mathcal{K}(T)$ for each $x^* \in X^*$. Then by the Hahn-Banach theorem, $u = v$. Therefore, u is unique.

Conversely, let $\boldsymbol{\mu}$ be σ-additive and $\delta(\mathcal{C})$-regular on $\delta(\mathcal{C})$ with its range relatively weakly compact. Then by the Theorem on Extension of [K3] or by Corollary 2 of [P7], $\boldsymbol{\mu}$ has a unique σ-additive extension $\boldsymbol{\mu}_c$ on $\mathcal{B}_c(T)$ with values in X. If $\boldsymbol{\mu}_0 = \boldsymbol{\mu}_c|_{\mathcal{B}_0(T)}$, then by Theorem 1 of [DP1] , $\boldsymbol{\mu}_0$ has a unique X-valued Borel (resp. σ-Borel) regular σ-additive extension $\hat{\boldsymbol{\mu}}$ (resp. $\hat{\boldsymbol{\mu}}_c$) on $\mathcal{B}(T)$ (resp. on $\mathcal{B}_c(T)$) and $\hat{\boldsymbol{\mu}}|_{\mathcal{B}_c(T)} = \hat{\boldsymbol{\mu}}_c$. Since $\hat{\boldsymbol{\mu}}_c$ and $\boldsymbol{\mu}_c$ are σ-additive and extend $\boldsymbol{\mu}$ to $\mathcal{B}_c(T)$, by the uniqueness part of Proposition 1 of [DP1] $x^*\boldsymbol{\mu}_c = x^*\hat{\boldsymbol{\mu}}_c$ for $x^* \in X^*$ and consequently, by the Hahn-Banach theorem $\hat{\boldsymbol{\mu}}_c = \boldsymbol{\mu}_c$ on $\mathcal{B}_c(T)$. Therefore, $\boldsymbol{\mu}_c$ is $\mathcal{B}_c(T)$-regular. Thus $\boldsymbol{\mu}$ has a unique $\mathcal{B}_c(T)$-regular σ-additive extension $\boldsymbol{\mu}_c$ on $\mathcal{B}_c(T)$ and $\boldsymbol{\mu}_c = \hat{\boldsymbol{\mu}}|_{\mathcal{B}_c(T)}$.

Let

$$uf = \int_T f d\boldsymbol{\mu}_c, \quad f \in C_0(T). \qquad (29.23.1)$$

Then by Theorem 1 of [P7], u is weakly compact and let $\mathbf{m}_u$ be the representing measure of u in the sense of 18.10 of [P18]. Then $\mathbf{m}_u$ has range in X by Theorem 2 of [P9] and by Theorem 1 of [P9] and by (29.23.1) we have

$$x^* uf = \int_T f d(x^* \mathbf{m}_u) = \int_T f d(x^* \boldsymbol{\mu}_c)$$

for $f \in C_0(T)$ and for $x^* \in X^*$. Moreover, $\mathbf{m}_u|_{\mathcal{B}(T)}$ is Borel regular. Then by Theorem 20.12 of [P19], $\mathbf{m}_u|_{\mathcal{B}_c(T)}$ is $\mathcal{B}_c(T)$-regular. Consequently. by the uniqueness part of the Riesz representation theorem (σ-Borel version) we conclude that $x^*\boldsymbol{\mu}_c = x^*\mathbf{m}_u$ on $\mathcal{B}_c(T)$ for each $x^* \in X^*$ and hence by the Hahn-Banach theorem, $\boldsymbol{\mu}_c = \mathbf{m}_u|_{\mathcal{B}_c(T)}$. Since $\boldsymbol{\mu}_c|_{\delta(\mathcal{C})} = \boldsymbol{\mu}$, $\boldsymbol{\mu}$ is the restriction of $\mathbf{m}_u$ to $\delta(\mathcal{C})$. Then by Theorem 29.6(iii), $\boldsymbol{\mu}$ is the restriction of $\boldsymbol{\mu}_u$ to $\delta(\mathcal{C})$.

$M_u = \widetilde{\delta(\mathcal{C})}$, the Lebesgue-Radon completion of $\delta(\mathcal{C})$ with respect to $\boldsymbol{\mu}_u$ by Definition 29.22 and by Theorem 29.21. Then the Lebesgue-Radon completion $\widetilde{\boldsymbol{\mu}_u}$ of $\boldsymbol{\mu}_u$ exists on $\widetilde{\delta(\mathcal{C})}$ by Theorem 29.9(ii). Moreover, by the same theorem, $\widetilde{\boldsymbol{\mu}_u} = \boldsymbol{\mu}_u$.

This completes the proof of the theorem.

The following theorem generalizes the bounded case of Theorem 4.6 of [P4].

Theorem 29.24. Let X be a quasicomplete lcHs. Let $\boldsymbol{\mu}_0 : \delta(\mathcal{C}_0) \to X$ be σ-additive with range relatively weakly compact. Then $\boldsymbol{\mu}_0$ admits a unique X-valued $\delta(\mathcal{C})$-regular σ-additive extension $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$. Moreover, the following assertions hold:

(i) $\boldsymbol{\mu}_0$ is the restriction of a weakly compact Radon vector measure $\boldsymbol{\mu}_u$ and such u is unique. We say that $\boldsymbol{\mu}_0$ determines the bounded weakly compact Radon operator u.

(ii) If $\boldsymbol{\mu}$ is as above, then $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}$ determine the same bounded weakly compact Radon operator u (see Theorem 29.23).

(iii) If u is as in (i) and (ii), then

$$M_u \;=\; \text{the Lebesgue-Radon completion}\, \widetilde{\delta(\mathcal{C})} \text{with respect to}\, \boldsymbol{\mu}_u$$

and

$$\boldsymbol{\mu}_u(E) = \widetilde{\boldsymbol{\mu}_u}(E) \text{ for each } E \in M_u.$$

where $\widetilde{\boldsymbol{\mu}_u}$ is the Lebesgue-Radon completion of $\boldsymbol{\mu}_u$ with respect to $\delta(\mathcal{C})$.

Proof. By Theorem on Extension of [K3] or by Corollary 2 of [P7], by Proposition 1 of [DP1] and by the Hahn-Banach theorem $\boldsymbol{\mu}_0$ has a unique σ-additive X-valued Baire extension $\boldsymbol{\nu} : \mathcal{B}_0(T) \to X$ and let

$$uf = \int_T f d\boldsymbol{\nu}, \qquad f \in C_0(T).$$

Then u is weakly compact by Theorem 1 of [P7] and hence the Borel restriction of its representing measure (see 18.10 of [P18]) $\mathbf{m}_u$ is the restriction of $\boldsymbol{\mu}_u$ to $\mathcal{B}(T)$ by Theorem 29.6(iii). Moreover, by 18.10 of [P18], Then

$$x^*uf = \int_T f d(x^*\boldsymbol{\nu}) = \int_T f d(x^*\mathbf{m}_u) \qquad \text{for } f \in C_0(T) \text{ and for } x^* \in X^*.$$

Then by the uniqueness part of the Riesz representation theorem (Baire version) we have $x^*\boldsymbol{\nu} = (x^*\mathbf{m}_u)|_{\mathcal{B}_0(T)}$ for each $x^* \in X^*$. Consequently, by the Hahn-Banach theorem we have $\boldsymbol{\nu} = \mathbf{m}_u|_{\mathcal{B}_0(T)}$. Since $\mathbf{m}_u = \boldsymbol{\mu}_u$ by Theorem 29.6(iii), we have $\boldsymbol{\nu} = \boldsymbol{\mu}_u|_{\mathcal{B}_0(T)}$. Then by Theorem 29.11(iii), $\boldsymbol{\mu} = \boldsymbol{\mu}_u|_{\delta(\mathcal{C})} = \mathbf{m}_u|_{\delta(\mathcal{C})}$ is σ-additive and $\delta(\mathcal{C})$-regular and extends $\boldsymbol{\mu}_0$. If $\boldsymbol{\mu}_1 : \delta(\mathcal{C}) \to X$ is σ-additive, $\delta(\mathcal{C})$-regular and extends $\boldsymbol{\mu}_0$, then by the uniqueness part of Theorem 4.1(i) of [P4], $x^*\boldsymbol{\mu}_1 = x^*\boldsymbol{\mu}$ for each $x^* \in X^*$ and hence by the Hahn-Banach theorem, $\boldsymbol{\mu}_1 = \boldsymbol{\mu}$. Hence $\boldsymbol{\mu}_0$ admits a unique $\delta(\mathcal{C})$-regular σ-additive extension $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$.

(i) If there exists another weakly compact operator $v : C_0(T) \to X$ such that $\boldsymbol{\mu}_v|_{\delta(\mathcal{C}_0)} = \boldsymbol{\mu}_0$, then by the uniqueness part of Theorem 1 of [DP1] and by the uniqueness of σ-additive extension of $\boldsymbol{\mu}_0$ to $\mathcal{B}_0(T)$, $\boldsymbol{\mu}_u = \boldsymbol{\mu}_v$ and hence this implies that $x^*\boldsymbol{\mu}_u = x^*\boldsymbol{\mu}_v$ for $x^* \in X^*$. Consequently, by (29.16.1) and by Theorem 29.4 we have $\mu_{x^*u} = \mu_{x^*v}$ on $\mathcal{B}(T)$ for each $x^* \in X^*$. Thus $x^*u(\varphi) = x^*v(\varphi)$ for $\varphi \in \mathcal{K}(T)$ and for $x^* \in X^*$. Consequently, by the Hahn-Banach theorem, $u = v$. Hence u is unique.

(ii) Let $\boldsymbol{\omega}$ be the X-valued σ-additive extension of $\boldsymbol{\mu}$ to $\mathcal{B}_c(T)$. This exists by hypothesis and by Corollary 2 of [P7]. Then $\boldsymbol{\omega}$ also extends $\boldsymbol{\mu}_0$ to $\mathcal{B}_c(T)$. Then $uf = \int_T f d\boldsymbol{\nu} = \int_T f d\boldsymbol{\omega}$ for $f \in C_0(T)$. Then $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_0$ determine the same bounded weakly compact Radon operator u.

Hence (ii) holds.

(iii) By Theorem 29.23

$M_u = \widetilde{\delta(\mathcal{C})}$, the Lebesgue-Radon completion of $\delta(\mathcal{C})$ with respect to $\boldsymbol{\mu}_u$. Hence (iii) holds.

By Theorem 29.9(ii) and by Theorem 29.6(iii),

$$\boldsymbol{\mu}_u(E) = \mathbf{m}_u(E) = \lim_{K \in \mathcal{C}(E)} \boldsymbol{\mu}_u(K) = \widetilde{\boldsymbol{\mu}_u}(E).$$

This completes the proof of the theorem.

Theorem 29.25. Let u_1 and u_2 be prolongable Radon operators on $\mathcal{K}(T)$. If $\boldsymbol{\mu}_{u_1}|_{\delta(\mathcal{C}_0)} = \boldsymbol{\mu}_{u_2}|_{\delta(\mathcal{C}_0)}$, then $u_1 = u_2$ so that $M_{u_1} = M_{u_2}$.

Proof. Let U be a relatively compact open Baire set in T. Then by Theorem 29.8(iii), $\mathcal{B}_0(U) \subset \delta(\mathcal{C}_0) \subset M_{u_i}$, $i = 1, 2$. Then $\boldsymbol{\mu}_{u_1}|_{\mathcal{B}_0(U)} = \boldsymbol{\mu}_{u_2}|_{\mathcal{B}_0(U)} = \mathbf{m}_U$ (say). Then $\mathbf{m}_U : \mathcal{B}_0(U) \to X$ is σ-additive and the linear transformation $\boldsymbol{\omega}_U : C_0(U) \to X$ given by

$$\boldsymbol{\omega}_U f = \int_U f d\mathbf{m}_U, \qquad f \in C_0(U)$$

is weakly compact by Theorem 1 of [P7]. Then for $f \in C_0(U)$,

$$\boldsymbol{\omega}_U f = \int_U f d\mathbf{m}_U = \int_U f d\boldsymbol{\mu}_{u_i}|_{\mathcal{B}_0(U)} = \int_T f d\boldsymbol{\mu}_{u_i}|_{\mathcal{B}_0(U)} = u_i f \text{ for } i = 1, 2.$$

Thus

$$u_1 f = u_2 f, \quad f \in C_c(U).$$

Since each $f \in \mathcal{K}(T)$ belongs to $C_c(U)$ for some relatively compact open Baire set U by Theorem 50.D of [H], we conclude that $u_1 = u_2$ on $\mathcal{K}(T)$.

This completes the proof of the theorem.

Remark 29.26. One can also use Theorem 4.6(i) of [P4] to prove the above result since $\mu_{x^* u_1} = \mu_{x^* u_2}$ on $\delta(\mathcal{C}_0)$ for each $x^* \in X^*$.

The following theorem generalizes Theorem 4.4 of [P4].

Theorem 29.27. Let X be a quasicomplete lcHs. Let $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$ be σ-additive. Then $\boldsymbol{\mu}$ is the restriction of an X-valued prolongable Radon vector measure $\boldsymbol{\mu}_u$ if and only if $\boldsymbol{\mu}$ is $\delta(\mathcal{C})$-regular. In that case, u is unique and u is called the prolongable Radon operator determined by $\boldsymbol{\mu}$. Moreover, the Radon vector measure $\boldsymbol{\mu}_u$ on M_u is given by

$$\boldsymbol{\mu}_u(E) = \lim_{K\subset E, K\in\mathcal{C}} \boldsymbol{\mu}(K), \qquad E \in M_u. \qquad (29.27.1)$$

The localized Lebesgue-Radon completion of $\delta(\mathcal{C})$ with respect to $\boldsymbol{\mu}$ is defined by $\ell(\widetilde{\delta(\mathcal{C})}) = \{E \subset T :$ given $q \in \Gamma, K \in \mathcal{C}$ and $\epsilon > 0$, there exist $C \in \mathcal{C}$ and a relatively compact open set U in T such that $C \subset E \cap K \subset U$ with $u_q^\bullet(U\backslash C) < \epsilon$ and $\lim_{K\in\mathcal{C}} u_q^\bullet(U\backslash(E\cap K)) = 0\}$. The localized Lebesgue-Radon completion $\hat{\boldsymbol{\mu}}$ of $\boldsymbol{\mu}$ with respect to $\delta(\mathcal{C})$ is said to exist on $\ell(\widetilde{\delta(\mathcal{C})})$ if $\hat{\boldsymbol{\mu}}(E) = \lim_{K\subset E, K\in\mathcal{C}} \boldsymbol{\mu}(K)$ exists in X for each $E \in \ell(\widetilde{\delta(\mathcal{C})})$. Then $\ell(\widetilde{\delta(\mathcal{C})}) = M_u$ and $\hat{\boldsymbol{\mu}}(E)$ exists in X for each $E \in \ell(\widetilde{\delta(\mathcal{C})})$ and $\hat{\boldsymbol{\mu}}(E) = \boldsymbol{\mu}_u(E)$ for $E \in M_u$.

Proof. If u is a prolongable Radon operator on $\mathcal{K}(T)$, then by Theorems 29.2, 29.8 and 29.11(iii), $\delta(\mathcal{C}) \subset M_u$ and $\boldsymbol{\mu}|_{\delta(\mathcal{C})}$ is σ-additive and $\delta(\mathcal{C})$-regular. Conversely, let $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular. Let U be a relatively compact open set in T and let $\mathbf{m}_U = \boldsymbol{\mu}|_{\mathcal{B}(U)}$. Let $V_U : C_0(U) \to X$ be given by

$$V_U f = \int_U f d\mathbf{m}_U, \quad f \in C_0(U).$$

Then by Theorem 1 of [P7], V_U is weakly compact. On the other hand, if $uf = \int_T f d\boldsymbol{\mu}$ for $f \in \mathcal{K}(T)$, then $u|_{\mathcal{K}(U)} = V_U|_{\mathcal{K}(U)}$ is continuous and hence the unique continuous extension of $u|_{\mathcal{K}(U)}$ to $C_0(U)$ coincides with V_U which is weakly compact. Hence u is prolongable.

Let $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$ be σ-additive and $\delta(\mathcal{C})$-regular and let $\mathbf{m}_U = \boldsymbol{\mu}|_{\mathcal{B}(U)}$. Since $\mathbf{m}_U$ is $\mathcal{B}(U)$-regular by the hypothesis, by Lemma 18.19 $\mathbf{m}_U$ is the representing measure of the weakly compact operator V_U and since $u|_{C_0(U)} = V_U$, it follows by Theorem 29.4 that $\boldsymbol{\mu}_u|_{\mathcal{B}(U)} = \mathbf{m}_U = \boldsymbol{\mu}|_{\mathcal{B}(U)}$. Since U is an arbitrary relatively compact open set in T, it follows that $\boldsymbol{\mu}_u|_{\delta(\mathcal{C})} = \boldsymbol{\mu}$. In fact, given $E \in \delta(\mathcal{C})$, let U be a relatively compact open set such that $E \subset U$. Then $E \in \mathcal{B}(U)$ and hence $\boldsymbol{\mu}_u(E) = \boldsymbol{\mu}(E)$.

The uniqueness of u follows from Theorem 29.25 and thus u is uniquely determined by $\boldsymbol{\mu}$. Moreover, by Theorem 29.11(iv),

$$\boldsymbol{\mu}_u(E) = \lim_{K\subset E, K\in\mathcal{C}} \boldsymbol{\mu}_u(K) = \lim_{K\subset E, K\in\mathcal{C}} \boldsymbol{\mu}(K), \qquad E \in M_u \qquad (29.27.2)$$

and hence $\boldsymbol{\mu}_u$ is also determined by $\boldsymbol{\mu}$.

Let $\mathcal{R} = \{E \subset T :$ given $K \in \mathcal{C}, q \in \Gamma$ and $\epsilon > 0$, there exists $C \in \mathcal{C}$ and a relatively compact open set U in T such that $C \subset E \cap K \subset U$ with $u_q^\bullet(U \backslash C) < \epsilon$ and $\lim_{K\in\mathcal{C}} u_q^\bullet(E\backslash E\cap K) = 0$ for each $q \in \Gamma\}$.

Let $E \in M_u$. Let $K \in \mathcal{C}$, $q \in \Gamma$ and $\epsilon > 0$. Then by Lemma 29.10 there exists a relatively compact open set U in T such that $E \cap K \subset U$ and $u_q^\bullet(U\backslash(E\cap K)) < \frac{\epsilon}{2}$. Since $E\cap K \in M_u$, by Theorem 29.11(i) there exists a compact $C \subset E\cap K$ such that $u_q^\bullet((E\cap K)\backslash C) < \frac{\epsilon}{2}$. Then $C \subset E\cap K \subset U$ and $u_q^\bullet(U\backslash C) < \epsilon$.

Since $\chi_E \in \mathcal{L}_1(u)$, by the complex lcHs-version of Lemma 1.24 of [T]

$$\lim_{K\in\mathcal{C}} u_q^\bullet(\chi_{E\backslash K}) = \lim_{K\in\mathcal{C}} u_q^\bullet(E\backslash(E\cap K)) = 0$$

for each $q \in \Gamma$ and hence $M_u \subset \mathcal{R}$.

To prove the reverse inclusion, let $E \in \mathcal{R}$. Then, given $q \in \Gamma$, $K \in \mathcal{C}$ and $\epsilon > 0$, there exist $C \in \mathcal{C}$ and $U \in \mathcal{U} \bigcap \delta(\mathcal{C})$ such that $C \subset E \cap K \subset U$ with $u_q^\bullet(U\backslash C) < \epsilon$. By Urysohn's lemma there exists $\varphi \in \mathcal{K}(T)$ such that $\chi_C \le \varphi \le \chi_U$. Then $u_q^\bullet(|\varphi - \chi_{E\cap K}|) \le u_q^\bullet(\chi_U - \chi_C) = u_q^\bullet(U\backslash C) < \epsilon$. Thus $E\cap K \in M_u$ for each $K \in \mathcal{C}$. Since $\lim_{K\in\mathcal{C}} u_q^\bullet(E\backslash E\cap K) = 0$ by hypothesis, and since $E \cap K \in M_u$ for all $K \in \mathcal{C}$ and since $\mathcal{L}_1(u)$ is closed in $\mathcal{F}^0(u)$, it follows that $E \in M_u$ and hence $M_u = \mathcal{R}$. Thus $M_u = \ell(\widetilde{\delta(\mathcal{C})})$, the localized Lebesgue-Radon completion of $\delta(\mathcal{C})$. Finally, $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu}_u$ is immediate from the definition and from (29.27.2).

This completes the proof of the theorem.

The following theorem generalizes Theorem 4.6 of [P4].

Theorem 29.28. Let X be a quasicomplete lcHs. Let $\boldsymbol{\mu}_0 : \delta(\mathcal{C}_0) \to X$ be σ-additive. Then $\boldsymbol{\mu}_0$ is the restriction of a unique X-valued prolongable Radon vector measure $\boldsymbol{\mu}_u$ and u is called the prolongable Radon operator determined by $\boldsymbol{\mu}_0$. Then $\boldsymbol{\mu}_0$ admits a unique $\delta(\mathcal{C})$-regular σ-additive extension $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$ and $\boldsymbol{\mu}$ and $\boldsymbol{\mu}_0$ determine the same prolongable Radon operator u.

Proof. By Theorem of Dinculeanu and Kluvánek [DK], $\boldsymbol{\mu}_0$ has a unique σ-additive $\delta(\mathcal{C})$-regular extension $\boldsymbol{\mu} : \delta(\mathcal{C}) \to X$.

Since each $f \in \mathcal{K}(T)$ is $\boldsymbol{\mu}_0$-integrable in T,

$$uf = \int_T f d\boldsymbol{\mu}_0, \qquad f \in \mathcal{K}(T)$$

is well defined, linear and has values in X. Moreover, for a relatively compact open set U in T, by Theorem 50.D of [H] there exists a relatively compact open Baire set U_0 such that $U \subset U_0$. Then for $f \in C_c(U)$ and for $q \in \Gamma$

$$q(uf) = q(\int_T f d\boldsymbol{\mu}_0) \leq ||f||_U ||\boldsymbol{\mu}_o||_q(U_0)$$

and hence u is a Radon operator. Moreover, the operator

$$u : C_0(U) \to X$$

given by

$$uf = \int_U f d\boldsymbol{\mu}_0, \qquad f \in C_0(U)$$

is continuous and is the restriction of

$$u_U : C_0(U_0) \to X$$

given by

$$u_U f = \int_U f d\boldsymbol{\mu}_0, = \int_U f d\boldsymbol{\mu} \qquad f \in C_0(U_0)$$

which is weakly compact by Lemma 18.19. Then by the proof of Theorem 29.28, u is prolongable. Since $\boldsymbol{\mu}$ is uniquely determined by $\boldsymbol{\mu}_0$ and since $\boldsymbol{\mu}$ determines u, $\boldsymbol{\mu}_0$ and $\boldsymbol{\mu}$ determine u uniquely.

This completes the proof of the theorem.

30. $\mathcal{L}_p(u)$ AS $\mathcal{L}_p(\mathbf{m}_u)$, $1 \le p < \infty$, u WEAKLY COMPACT AND $\mathcal{L}_1(v)$ AS $\mathcal{L}_1(\mathbf{m}_v)$, v PROLONGABLE

Let X be a Banach space (resp. a quasicomplete lcHs)and let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Then by Convention 25.8, $u : C_0(T) \to X$ is continuous and weakly compact. Let $\mathbf{m}_u : \mathcal{B}(T) \to X$ be the representing measure of u in the sense of 18.10 of Ch. IV. Then $\mathbf{m}_u$ is σ-additive, its restriction to $\mathcal{B}(T)$ is Borel regular and

$$u(\varphi) = \int_T \varphi d\mathbf{m}_u, \quad \varphi \in C_0(T)$$

where the integral is a (BDS)-integral. See Definition 3 and Theorems 2 and 6 of [P9]. Hereafter, $\mathbf{m}_u$ will denote the Borel restriction of the representing measure of u unless otherwise stated. Let u be a bounded weakly compact Radon operator with values in X. In the first part of this section, we show that $f \in \mathcal{L}_1(u)$ if and only if $f \in \mathcal{L}_1((\mathbf{m}_u)$ and in that case, $\int f du = \int_T f d\mathbf{m}_u$. For such u, we also show that $\mathcal{L}_p(u)$ is the same as $\mathcal{L}_p(\mathbf{m}_u)$ for $1 \le p < \infty$.

Let $v : \mathcal{K}(T) \to X$ be a prolongable Radon operator, X being a Banach space or a quasicomplete lcHs. Let $\mathbf{m}_v : \delta(\mathcal{C}) \to X$ be the representing measure of v (see Definition 19.5 and Theorem 19.9 of Ch. IV). In the second half of this section, we show that $f \in \mathcal{L}_1(v)$ if and only if $f \in \mathcal{L}_1(\mathbf{m}_v)$ and in that case, $\int f dv = \int_T f d\mathbf{m}_v$. Thus the questions (Q5) and (Q6) mentioned in Introduction of Ch. I are answered in the affirmative. See Remark 30.23.

Let $g : T \to \boldsymbol{K}$ be $\mathbf{m}_u$-measurable. Then by Theorem 5.3 we have

$$(\mathbf{m}_u)^\bullet_p(g, T) = \sup_{|x^*| \le 1} \left(\int_T |g|^p dv(x^* \mathbf{m}_u)\right)^{\frac{1}{p}} \qquad (*)$$

for $1 \le p < \infty$.

Definition 30.1. Let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Let $g : T \to \boldsymbol{K}$ be u-measurable. For $1 \le p < \infty$, let

$$u^\bullet_p(g) = \sup_{|x^*| \le 1} \left(\int_T |g|^p d|x^* u|\right)^{\frac{1}{p}} \qquad (30.1.1)$$

where $|x^* u|$ is given by (12) on p.55 of [B].

The following theorem gives the relation between $(\mathbf{m}_u)^\bullet_p(g,T)$ and $u^\bullet_p(g)$ for $1 \le p < \infty$.

Theorem 30.2. Let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator and let $\mathbf{m}_u$ be the representing measure of u as in 18.10 of Ch. IV. Then a function $g : T \to \boldsymbol{K}$is u-measurable if and only if g is $\mathbf{m}_u$-measurable. Moreover, for a u-measurable scalar function g,

$$u^\bullet_p(g) = (\mathbf{m}_u)^\bullet_p(g,T) \qquad (30.2.1)$$

for $1 \le p < \infty$. Also we have

$$u^\bullet_p(f+g) \le u^\bullet_p(f) + u^\bullet_p(g) \qquad (30.2.2)$$

$$u^\bullet_p(\alpha f) = |\alpha| u^\bullet_p(f),\ \alpha \in \boldsymbol{K} \qquad (30.2.3)$$

and

$$u^\bullet_1(fg) \le u^\bullet_{p_1}(f) \cdot u^\bullet_{p_2}(g) \qquad (30.2.4)$$

for u-measurable scalar functions f and g on T if $1 \le p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$.

Proof. By Theorem 29.6(iv), g is $\mathbf{m}_u$-measurable if and only if it is u-measurable.

By (30.1.1) and by 18.10 we have

$$\begin{aligned} u^\bullet_p(g) &= \sup_{|x^*|\le 1} \Big(\int_T |g|^p d|x^*u|\Big)^{\frac{1}{p}} \\ &= \sup_{|x^*|\le 1} \Big(\int_T |g|^p d|u^*x^*|\Big)^{\frac{1}{p}} \\ &= \sup_{|x^*|\le 1} \Big(\int_T |g|^p d|x^* \circ \mathbf{m}_u|\Big)^{\frac{1}{p}} \\ &= \sup_{|x^*|\le 1} \Big(\int_T |g|^p dv(x^* \circ \mathbf{m}_u)\Big)^{\frac{1}{p}} \end{aligned}$$

by Notation 4.4, Theorem 4.7(vi) and Theorem 4.11 of [P3] and by Theorem 3.3 of [P4] where $v(x^* \circ \mathbf{m}_u) = v(x^* \circ \mathbf{m}_u, \mathcal{B}(T))$ on $\mathcal{B}(T)$. Hence

$$u^\bullet_p(g) = (\mathbf{m}_u)^\bullet_p(g,T).$$

Now by (30.2.1) and by Theorem 5.13 we have

$$\begin{aligned} u_p^\bullet(f+g) &= (\mathbf{m}_u)_p^\bullet(f+g,T) \\ &\leq (\mathbf{m}_u)_p^\bullet(f,T) + (\mathbf{m}_u)_p^\bullet(g,T) \\ &= u_p^\bullet(f) + u_p^\bullet(g) \end{aligned}$$

and

$$u_p^\bullet(\alpha f) = |\alpha| u_p^\bullet(|f|)$$

for $1 \leq p < \infty$ and for $\alpha \in \boldsymbol{K}$ whenever $f, g : T \to \boldsymbol{K}$ are u-measurable. Moreover, if $1 \leq p_1, p_2 < \infty$ with $\frac{1}{p_1} + \frac{1}{p_2} = 1$, then by Theorem 5.13 and by (30.2.1) we have

$$\begin{aligned} u_1^\bullet(fg) &= (\mathbf{m}_u)_1^\bullet(fg,T) \\ &\leq (\mathbf{m}_u)_{p_1}^\bullet(f,T) \cdot (\mathbf{m}_u)_{p_2}^\bullet(g,T) \\ &= u_{p_1}^\bullet(f) \cdot u_{p_2}^\bullet(g). \end{aligned}$$

This completes the proof of the theorem.

Definition 30.3. Let X be a Banach space and let $1 \leq p < \infty$. Let $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Let $\mathcal{F}_p^0(u) = \{f : T \to \boldsymbol{K}, f\, u\text{-measurable and } u^\bullet(|f|^p) < \infty\}$. Let $I_p(u) = \{f : T \to \boldsymbol{K}, f\, u\text{-measurable and } |f|^p \in \mathcal{L}_1(u)\}$. Let

$$I(u) = \{f : T \to \boldsymbol{K}, f\, u\text{-measurable and } u\text{-integrable}\}.$$

Theorem 30.4. Under the hypothesis of Definition 30.3, $I_1(u) = I(u)$.

Proof. If $f \in I(u)$, then $f \in \mathcal{L}_1(u)$. Thus, given $\epsilon > 0$, there exists $\varphi \in \mathcal{K}(T)$ such that $u^\bullet(|f - \varphi|) < \epsilon$. Since $u^\bullet(|f| - |\varphi|) \leq u^\bullet(|f - \varphi|) < \epsilon$, $|f| \in \mathcal{L}_1(u)$ and hence $f \in I_1(u)$. Conversely, if $f \in I_1(u)$, then f is u-measurable and $|f|$ is u-integrable. Then by the complex analogue of Theorem 1.22 of [T], f is u-integrable. Hence $I_1(u) = I(u)$.

Definition 30.5. Let X be a Banach space, $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator and $1 \le p < \infty$. Let $\mathcal{L}_p(u) = \{f \in I_p(u) : u^\bullet(|f|^p) < \infty\}$.

Theorem 30.6. Let X, u and p be as in Definition 30.5. Then $\mathcal{L}_p(u) = I_p(u) \subset \mathcal{F}_p^0(u)$.

Proof. If $f \in I_p(u)$, then f is u-measurable and $|f|^p \in \mathcal{L}_1(u)$. Then by the complex analogue of Definition 1.6 and by that of Lemma 1.5 of [T], $u^\bullet(|f|^p) < \infty$.

Theorem 30.7. Let X be a Banach space and $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Then a function $f : T \to \boldsymbol{K}$is u-integrable if and only if f is $\mathbf{m}_u$-integrable in T and in that case

$$\int f du = \int_T f d\mathbf{m}_u.$$

Moreover, for $f \in \mathcal{L}_1(u)$, $u_1^\bullet(f) = (\mathbf{m}_u)_1^\bullet(f)$.

Proof. Let $\varphi \in C_c(T)$. Then by the proof of Lemma 18.19, φ is $\mathbf{m}_u$-integrable in T. Then by 18.10 we have

$$x^* u(\varphi) = \int_T \varphi d(x^* u) = \int_T \varphi d(u^* x^*) = \int_T \varphi d(x^* \circ \mathbf{m}_u) = x^*(\int_T \varphi d\mathbf{m}_u)$$

for $x^* \in X^*$. Hence by the Hahn-Banach theorem

$$u(\varphi) = \int \varphi du = \int_T \varphi d\mathbf{m}_u \qquad (30.7.1)$$

for $\varphi \in C_0(T)$.

Let f be u-integrable. Then there exists $(\varphi_n)_1^\infty \subset C_c(T)$ such that $u^\bullet(|f - \varphi_n|) \to 0$ as $n \to \infty$ and hence by the complex analogue of **1.10** of [T] we have

$$|\int f du - u(\varphi_n)| = |\int (f - \varphi_n) du| \le u^\bullet(|f - \varphi_n|) \to 0$$

as $n \to \infty$ and hence

$$\int f du = \lim_n u(\varphi_n) = \lim_n \int \varphi_n du. \qquad (30.7.2)$$

Then by Proposition 25.10, by Lemma 25.11 and by 18.10 we have

$$\begin{aligned}
u^\bullet(|f-\varphi_n|) &= \sup_{|x^*|\le 1} |u_{x^*}|^\bullet(|f-\varphi_n|)\\
&= \sup_{|x^*|\le 1} |u_{x^*}|(|f-\varphi_n|)(\text{by } 25.11)\\
&= \sup_{|x^*|\le 1} |x^*u|(|f-\varphi_n|)\\
&= \sup_{|x^*|\le 1} |u^*x^*|(|f-\varphi_n|)\\
&= \sup_{|x^*\le 1} \int |f-\varphi_n| d|x^*\circ \mathbf{m}_u|\\
&= \sup_{|x^*|\le 1} \int |f-\varphi_n| dv(x^*\circ \mathbf{m}_u) \qquad (*)
\end{aligned}$$

by Notation 4.4 and by Theorems 4.7(vi) and 4.11 of [P3] and by Theorem 3.3 of [P4] where $v(x^*\circ\mathbf{m}_u) = v(x^*\circ\mathbf{m}_u, \mathcal{B}(T))$.

Therefore, by (*) we have

$$u^\bullet(|f-\varphi_n|) = \sup_{|x^*|\le 1} \int_T |f-\varphi_n| dv(x^*\circ\mathbf{m}_u) = (\mathbf{m}_u)_1^\bullet(f-\varphi_n, T). \qquad (30.7.3)$$

As $u^\bullet(|f-\varphi_n|)\to 0$, by (30.7.3) we have $(\mathbf{m}_u)_1^\bullet(f-\varphi_n,T)\to 0$. Consequently, by Theorem 20.10, $f\in\mathcal{L}_1(\mathbf{m}_u)$. Moreover, by (30.7.1), (30.7.3), (5.3.1) and (30.7.2) we have

$$\int_T f d\mathbf{m}_u = \lim_n \int_T \varphi_n d\mathbf{m}_u = \lim_n \int \varphi_n du = \int f du.$$

Thus f is $\mathbf{m}_u$-integrable in T if f is u-integrable and

$$\int f du = \int_T f d\mathbf{m}_u. \qquad (30.7.4)$$

Conversely, let f be $\mathbf{m}_u$-integrable in T. Then by Theorem 20.10 there exists $(\varphi_n)_1^\infty \subset C_c(T)$ such that $(\mathbf{m}_u)_1^\bullet(f-\varphi_n,T)\to 0$ so that by (5.3.1) we have

$$\int_T f d\mathbf{m}_u = \lim_n \int_T \varphi_n d\mathbf{m}_u.$$

But by (30.7.1) we have

$$\int_T \varphi_n d\mathbf{m}_u = \int \varphi_n du$$

and hence

$$\int f d\mathbf{m}_u = \lim_n \int \varphi_n du. \qquad (30.7.5)$$

As $(\mathbf{m}_u)^\bullet_1(f-\varphi_n,T) = u^\bullet(|f-\varphi_n|)$ by (30.2.1), $u^\bullet(|f-\varphi_n|)\to 0$ and hence $f\in\mathcal{L}_1(u)$ by the complex version of Definition 1.6 of [T]. Then by the complex analogue of **1.10** of [T] we have

$$|\int f du - \int \varphi_n du| \leq u^\bullet(|f-\varphi_n)|) \to 0$$

as $n\to\infty$ and hence by (30.7.1) and (30.7.5) we have

$$\int f du = \lim_n \int \varphi_n du = \lim_n \int_T \varphi_n d\mathbf{m}_u = \int_T f d\mathbf{m}_u.$$

Thus $\mathcal{L}_1(u) = \mathcal{L}_1(\mathbf{m}_u)$ and for $f\in\mathcal{L}_1(u)$,

$$\int f du = \int_T f d\mathbf{m}_u$$

whenever u is a bounded weakly compact Radon operator on $\mathcal{K}(T)$ with values in a Banach space X. Moreover, for $f\in\mathcal{L}_1(u)$, $u^\bullet_1(f) = (\mathbf{m}_u)^\bullet_1(f,T)$ by (30.2.1).

This completes the proof of the theorem.

Theorem 30.8. Let X, u and p be as in Definition 30.5. Then $\mathcal{L}_p(u)$ is a seminormed space.

Proof. Let f, $g\in\mathcal{L}_p(u)$ and α be a scalar. Then $|f|^p$, $|g|^p\in\mathcal{L}_1(u)$. Since $|f+g|^p\leq 2^p\max(|f|^p,|g|^p)\leq 2^p(|f|^p+|g|^p)$, since $|f+g|^p$ is u-measurable and since $|f|^p+|g|^p\in\mathcal{L}_1(u)$ by the complex version of Theorem 1.22 of [T], $|f+g|^p\in\mathcal{L}_1(u)$ and hence $f+g\in\mathcal{L}_p(u)$. Clearly, $|\alpha f|^p\in\mathcal{L}_1(u)$ for $\alpha\in\boldsymbol{K}$and hence $\mathcal{L}_p(u)$ is a vector space over $\boldsymbol{K}$ Moreover, by (30.2.2) and (30.2.3) and by Theorem 30.6, $\mathcal{L}_p(u)$ is a seminormed space.

Theorem 30.9. Let X, u and p be as in Definition 30.5. Then $\mathcal{L}_p(u) = \mathcal{L}_p(\mathbf{m}_u)$ and hence is complete for $1\leq p<\infty$.

Proof. By Definition 30.5 and Theorem 30.6, $\mathcal{L}_p(u) = I_p(u)$ for $1\leq p<\infty$. Moreover, by Theorem 30.2, for $f\in\mathcal{L}_p(u)$,

$$u^\bullet_p(f) = (\mathbf{m}_u)^\bullet_p(f,T). \qquad (30.9.1)$$

Then by Theorems 30.6, 30.7 and 7.5, $f\in\mathcal{L}_p(u)$ if and only if $f\in\mathcal{L}_p(\mathbf{m}_u)$. Consequently, by (30.9.1) and by Theorem 6.8 of Ch. II, $\mathcal{L}_p(u)$ is complete.

This completes the proof of the theorem.

Definition 30.10. Let X be a quasicomplete lcHs and $u : K(T) \to X$ be a bounded weakly compact Radon operator. A u-measurable function $f : T \to \boldsymbol{K}$is said to be u-integrable in T if it is $u_q = \Pi_q \circ u$-integrable in T with values in $\widetilde{X_q}$ (considering $u_q : \mathcal{K}(T) \to X_q \subset \widetilde{X_q}$) for each $q \in \Gamma$ (see Definition 25.18). In that case, using Notation 10.16 of Ch. III, we define

$$\int f du = \lim_{\longleftarrow} \int f du_q.$$

Definition 30.11. Let X be a quasicomplete lcHs and $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Let $1 \le p < \infty$. For $q \in \Gamma$ and $g : T \to \boldsymbol{K}u$-measurable, let

$$(u_q)_p^\bullet(g) = \sup_{x^* \in U_q^0} (\int_T |g|^p dv(x^* u))^{\frac{1}{p}}$$

where $U_q^0 = \{x^* \in X^* : |x^*(x)| \le 1 \,\mathrm{for}\, x \in U_q\}$.

Theorem 30.12. Under the hypothesis of Definition 30.11,

$$(u_q)_p^\bullet(g) = ((\mathbf{m}_u)q)_p^\bullet(g, T).$$

Proof. By Proposition 10.14(ii)(b) and by the definition of Ψ_{x^*} as given in Proposition 10.14(ii)(a) of Ch. III, $\{\Psi_{x^*} : x^* \in U_q^0\}$ is a norm determining subset of the closed unit ball of $(X_q)^*$ and for $x^* \in U_q^0$, $x^*(\Pi_q \circ u) = \Psi_{x^*} u_q = x^* u_q = x^* u$ by (ii)(a) of the said proposition. Then by Lemma 5.2(ii) of Ch. II and by (30.2.1) we have

$$(\Psi_{x^*} u_q)_p^\bullet(g) = (\int_T |g|^p dv(\Psi_{x^*} u_q))^{\frac{1}{p}} = (\int_T |g|^p dv(x^* \circ \mathbf{m}_u))^{\frac{1}{p}}$$

and hence

$$(u_q)_p^\bullet(g) = \sup_{x^* \in U_q^0} (\Psi_{x^*} u_q)_p^\bullet(g) = \sup_{x^* \in U_q^0} (\int_T |g|^p dv(x^* \circ \mathbf{m}_u))^{\frac{1}{p}} = ((\mathbf{m}_u)_q)_p^\bullet(g, T)$$

by Theorem 13.2 of Ch. III.

Theorem 30.13. Let X be a quasicomplete lcHs and $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Let $f : T \to \boldsymbol{K}$be u-measurable. If f is u-integrable, then $\int f du \in X$.

Proof. By Theorem 30.12, $(u_q)_1^\bullet(g) = ((\mathbf{m}_u)_q)_1^\bullet(g, T)$. If f is u_q-integrable, then given $\epsilon > 0$, there exists $\varphi \in C_c(T)$ such that $u_q^\bullet(|\varphi - f|) < \epsilon$. Then by Theorem 30.12, $((\mathbf{m}_u)_q)_1^\bullet(f - \varphi, T) < \epsilon$ and hence there exists $x_q \in \widetilde{X_q}$ such that $\int f du_q = \int_T f d\mathbf{m}_q = x_q$ for each $q \in \Gamma$. Thus

$$\int f du = \lim_{\longleftarrow} \int f du_q = \lim_{\longleftarrow} x_q = \lim_{\longleftarrow} \int f d\mathbf{m}_q \in X$$

by Definition 12.1 and Theorem 12.3 of Ch. III. Hence the theorem holds.

Remark 30.14 By using the complex version of Lemma 2.21 of [T] and Theorem 25.24, one can give an alternative proof of the above theorem.

Definition 30.15. Let X be a quasicomplete lcHs and $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator. Let $1 \le p < \infty$. Let $\mathcal{F}_p^0 = \{f : T \to \boldsymbol{K}, f\, u\text{-measurable and}\, (u_q)_p^\bullet(f) < \infty \,\text{for each}\, q \in \Gamma\}$. Then we define $\mathcal{L}_p(u) = \{f \in \mathcal{F}_p^0(u) \,\text{and}\, |f|^p u\text{-integrable (with values in}\, X)\}$.

Using Theorem 30.12 and adapting the proof of Theorem 15.3(i) of Ch. III one can prove the following theorem. The details are left to the reader.

Theorem 30.16. Let X be a quasicomplete lcHs, $u : \mathcal{K}(T) \to X$ be a bounded weakly compact Radon operator and $1 \le p < \infty$. Let $f_n^{(q)}$, $n \in \boldsymbol{N}$ be u_q-measurable scalar functions on T for $q \in \Gamma$. Let $K^{(q)}$ be a finite constant such that $|f_n^{(q)}| \le K^{(q)}$ u_q-a.e. in T. If $f_n^{(q)} \to f$ u_q-a.e. in T where f is a scalar function on T, then f, $f_n^{(q)}, n \in \boldsymbol{N}$ belong to $\mathcal{L}_p(u_q)$ and $\lim_n (u_q)_p^\bullet(f_n^{(q)} - f) = 0$ for $q \in \Gamma$. Consequently, $f \in \mathcal{L}_p(u)$. When $p = 1$, f is u-integrable and

$$\lim_n |q(\int f du) - |\int f_n^{(q)} du_q|_q| = 0.$$

Lemma 30.17. Let X be a Banach space and let $u : \mathcal{K}(T) \to X$ be a prolongable Radon operator with $\mathbf{m}_u$ as its representing measure (see Theorem 19.9 of Ch. IV). For a relatively compact open set ω in T, let $v = u|_{\mathcal{K}(\omega)}$. If $f \in \mathcal{L}_1(u)$, then $f\chi_\omega \in \mathcal{L}_1(v)$. If $\mathbf{m}_v = \mathbf{m}_u|_{\mathcal{B}(\omega)}$, then $f\chi_\omega \in \mathcal{L}_1(\mathbf{m}_v)$ and

$$\int f\chi_\omega dv = \int f\chi_\omega du = \int f\chi_\omega d\mathbf{m}_v = \int f\chi_\omega d\mathbf{m}_u.$$

Proof. By hypothesis, f is u-integrable. Since u is prolongable and ω is a relatively compact open set in T, $v = u|_{\mathcal{K}(\omega)}$ is a bounded weakly compact Radon operator. Since $f \in \mathcal{L}_1(u)$, given

$\epsilon > 0$, there exists $\varphi \in \mathcal{K}(T)$ such that $u^{\bullet}(|f-\varphi|) < \epsilon$. Since $|\varphi\chi_\omega| \le |\varphi| \in \mathcal{L}_1(v)$, by the complex version of Theorem 1.22 of [T] the function $\varphi\chi_\omega \in \mathcal{L}_1(v)$. Moreover, in the notation of Lemma 27.3 we have $\widehat{f\chi_\omega} = f\chi_\omega$ and $\widehat{\varphi\chi_\omega} = \varphi\chi_\omega$ and hence by Lemma 27.3(ii)

$$v^{\bullet}(|f\chi_\omega - \varphi\chi_\omega|) = u^{\bullet}(|\widehat{f\chi_\omega} - \widehat{\varphi\chi_\omega}|) = u^{\bullet}(|f\chi_\omega - \varphi\chi_\omega|) \le u^{\bullet}(|f-\varphi|) < \epsilon$$

and hence $f\chi_\omega \in \mathcal{L}_1(v)$ by the complex version of the argument given in the last lines on p. 67 of [T] since $\varphi\chi_\omega \in \mathcal{L}_1(v)$.

Since $v = u|_{\mathcal{K}(\omega)}$, $\mathbf{m}_v = \mathbf{m}_u|_{\mathcal{B}(\omega)}$. Since v is a bounded weakly compact Radon operator, and since $f\chi_\omega \in \mathcal{L}_1(v)$, by Theorem 30.7 we have $f\chi_\omega \in \mathcal{L}_1(\mathbf{m}_v)$ and

$$\int f\chi_\omega dv = \int f\chi_\omega du = \int f\chi_\omega d\mathbf{m}_v = \int f\chi_\omega d\mathbf{m}_u.$$

Hence the lemma holds.

Lemma 30.18. Let $f : T \to \boldsymbol{K}$ and let $u : \mathcal{K}(T) \to X$ be a prolongable Radon operator where X is a Banach space. Then f is u-integrable if and only if f is $\mathbf{m}_u$-integrable in T.

Proof. Let f be u-integrable. Then by Theorem 27.9, f is x^*u-integrable in T for each $x^* \in X^*$ and for each open Baire set ω in T there exists $x_\omega \in X$ such that

$$x^*(x_\omega) = \int_\omega f d(x^*u)$$

for $x^* \in X^*$. By 18.10 of Ch. IV, $x^*u = u^{**}x^* = x^* \circ \mathbf{m}_v$ since $v = u|_{C_0(\omega)} : C_0(\omega) \to X$ is weakly compact. Hence

$$x^*(x_\omega) = \int_\omega f d(x^* \circ \mathbf{m}_v) \qquad (30.18.1)$$

for $x^* \in X^*$. Let $H = \{x^* \in X^* : |x^*| \le 1\}$. Then H is a norm determining set for X and by the Orlicz-Pettis theorem, H has the Orlicz property. Hence by (30.18.1), by the arbitrariness of the open Baire set ω in T and by Theorem 22.4 of Ch. V, f is $\mathbf{m}_u$-integrable.

Conversely, let f be $\mathbf{m}_u$-integrable in T. Then clearly f is $x^* \circ \mathbf{m}_u$-integrable in T for $x^* \in X^*$. Moreover, by 18.10 of Ch. IV, $x^* \circ \mathbf{m}_u = x^*u|_{C_0(\omega)}$ and hence f is x^*u-integrable in T for each $x^* \in X^*$. As observed above, H is a norm determining set for X with the Orlicz property. Then by Theorem 22.4 of Ch. V, for each open Baire set U in T, there exists a vector $x_U \in X$ such that

$$x^*(x_U) = \int f d(x^* \circ \mathbf{m}_u) = \int f d(x^* u)$$

for $x^* \in X^*$. Consequently, by Theorem 27.9 f is u-integrable. Hence the lemma holds.

Theorem 30.19. Let X be a Banach space and let $v : \mathcal{K}(T) \to X$ be a prolongable Radon operator. If $f \in \mathcal{L}_1(v)$, then $f \in \mathcal{L}_1(\mathbf{m}_v)$ and

$$\int f dv = \int_T f d\mathbf{m}_v.$$

Conversely, if $f \in \mathcal{L}_1(\mathbf{m}_v)$, then $f \in \mathcal{L}_1(v)$ and

$$\int_T f d\mathbf{m}_v = \int f dv.$$

Proof. Let $f : T \to \mathbb{K}$ be v-integrable. Then f is x^*v-integrable for each $x^* \in X^*$ and hence $N(f)$ is σ-bounded. Let $(K_n)_1^\infty \subset \mathcal{C}$ such that $N(f) \subset \bigcup_1^\infty K_n$. Then by Theorem 50.D of [H], there exist relatively compact open sets $(\omega_n)_1^\infty$ in T such that $K_n \subset \omega_n$, $n \in \mathbb{N}$. Then $N(f) \subset \bigcup_1^\infty \omega_n$. Let $U_n = \bigcup_{k=1}^n \omega_k$. Then $U_n \nearrow$ and U_n is relatively compact and open in T for each n. Moreover, $f\chi_{U_n} \to f$ pointwise in T. As $|f\chi_{U_n}| \le |f| \in \mathcal{L}_1(v)$, by the complex analogue of Theorems 1.22 and 4.7 of [T] we have

$$\int f dv = \lim_n \int f\chi_{U_n} dv. \qquad (30.19.1)$$

Let $v_n = v|_{\mathcal{K}(U_n)}$. As $U_n, n \in \mathbb{N}$, are relatively compact open sets in T, by the hypothesis on v, $v_n, n \in \mathbb{N}$, are bounded weakly compact Radon operators on $\mathcal{K}(U_n)$ and by Lemma 30.17, $f\chi_{U_n} \in \mathcal{L}_1(v_n))$, $f\chi_{U_n} \in \mathcal{L}_1(\mathbf{m}_{v_n})$ and

$$\int f\chi_{U_n} dv_n = \int f\chi_{U_n} dv = \int f\chi_{U_n} d\mathbf{m}_{v_n} = \int f\chi_{U_n} d\mathbf{m}_v.$$

Then by (30.19.1), by Lemma 30.18 and by LDCT given by Theorem 3.7 and Remark 4.3 of Ch. I we have

$$\int f dv = \lim_n \int f\chi_{U_n} dv = \lim_n \int f\chi_{U_n} d\mathbf{m}_v = \int f d\mathbf{m}_v.$$

Conversely, let $f \in \mathcal{L}_1(\mathbf{m}_v)$. Then clearly $N(f)$ is σ-bounded and hence there exists a sequence of $(\omega_n)_1^\infty$ of relatively compact open sets in T such that $N(f) \subset \bigcup_1^\infty \omega_n$. Take $U_n = \cup_{k=1}^n \omega_k$. Then

$f\chi_{U_n} \to f$ pointwise in T. Then by Theorem 3.5(vii) and Remark 4.3 of Ch. I, $f\chi_{U_n} \in \mathcal{L}_1(\mathbf{m}_v)$ for each n. Then by LDCT given by Theorem 3.7 and Remark 4.3 of Ch. I we have

$$\int f d\mathbf{m}_v = \lim_n \int f\chi_{U_n} d\mathbf{m}_v. \qquad (30.19.2)$$

Let $v_n = v|_{\mathcal{K}(U_n)}$. Then by Lemma 30.17 we have

$$\int f\chi_{U_n} dv_n = \int f\chi_{U_n} d(\mathbf{m}_{v_n})$$

and

$$\lim_n \int f\chi_{U_n} dv = \lim_n \int f\chi_{U_n} d(\mathbf{m}_{v_n}) = \lim_n \int f\chi_{U_n} d\mathbf{m}_v = \int_T f d\mathbf{m}_v \qquad (30.19.3)$$

by (30.19.2).

Since $(f\chi_{U_n})_1^\infty \subset \mathcal{L}_1(v)$ and $f\chi_{U_n} \to f$ pointwise in T and since f is v-integrable by hypothesis, by Theorem 4.7 of [T] we have

$$\int f dv = \lim_n \int f\chi_{U_n} dv.$$

Then by (30.19.3) we conclude that

$$\int f dv = \int_T f d\mathbf{m}_v.$$

This completes the proof of the theorem.

Definition 30.20. Let X be a quasicomplete lcHs and $v : \mathcal{K}(T) \to X$ be a prolongable Radon operator. A v-measurable scalar function is said to be v-integrable if it is $v_q = \Pi_q \circ v$-integrable with values in $\widetilde{X_q}$ (considering $v_q : \mathcal{K}(T) \to X_q \subset \widetilde{X_q}$) for each $q \in \Gamma$. In that case, using Notation 10.16 of Ch. III, we define

$$\int f dv = \underset{\longleftarrow}{\lim} \int f dv_q.$$

Theorem 30.21. Let X be a quasicomplete lcHs and $v : \mathcal{K}(T) \to X$ be a prolongable Radon operator. Let $f : T \to \boldsymbol{K}$ be v-measurable. If f is v-integrable, then $\int f dv \in X$.

Proof. This follows from Definition 30.20, the complex version of Lemma 2.21 of [T] and Theorem 25.24.

Theorem 30.22. Let X be a quasicomplete lcHs and $v : \mathcal{K}(T) \to X$ be a prolongable Radon operator. If $f \in \mathcal{L}_1(v)$, then $f \in \mathcal{L}_1(\mathbf{m}_v)$ and

$$\int f dv = \int f d\mathbf{m}_v.$$

Conversely, if $f \in \mathcal{L}_1(\mathbf{m}_v)$, then $f \in \mathcal{L}_1(v)$ and

$$\int f d\mathbf{m}_v = \int f dv.$$

Proof. This is immediate from Definitions 12.1 (of Ch. III) and 30.20 and from Theorem 30.19.

Remark 30.23. By Theorems 30.7 and 30.21, the questions (Q5) and (Q6) mentioned in Introduction of Ch. I are answered in the affirmative.

Remark 30.24. It is not known whether the results analogous to Theorems 30.9 and 30.12 hold for prolongable Radon operators on $\mathcal{K}(T)$.

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