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A Generalization of Cramer's Rule and Applications to Generalized Linear Differential Equations.

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Abstract

In this paper we find a formula for the solutions of the following linear equation

$$Ax = b, \quad x \in I\!R^m, \quad b \in I\!R^n, \quad m \geq n,$$

where $A = (a_{j,i})_{n\times m}$ is a $n \times m$ real matrix. We prove the following statement: For all $b \in I\!R^n$ the system is solvable if, and only if, the set of vectors $\{l_1, l_2, \dots, l_n\}$ formed by the rows of the matrix A is lineally independent in $I\!R^m$. Moreover, one solution for this equation is given by the following formula

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i}b_1 & <l_1,l_2> + a_{2i}b_1 & \cdots & <l_1,l_n> + a_{ni}b_1 \\ <l_2,l_1> + a_{1i}b_2 & \|l_2\|^2 + a_{2i}b_2 & \cdots & <l_2,l_n> + a_{ni}b_2 \\ \vdots & \vdots & \vdots & \vdots \\ <l_n,l_1> + a_{1i}b_n & <l_n,l_2> + a_{2i}b_n & \cdots & \|l_n\|^2 + a_{ni}b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & <l_1,l_2> & \cdots & <l_1,l_n> \\ <l_2,l_1> & \|l_2\|^2 & \cdots & <l_2,l_n> \\ \vdots & \vdots & \vdots & \vdots \\ <l_n,l_1> & <l_n,l_2> & \cdots & \|l_n\|^2 \end{vmatrix}} - 1,$$

$i = 1, 2, 3, \dots, m$. Finally, we apply these reults to find a solutions of the following general linear differential equation

$$B\dot{x}(t) = Ax(t) + f(t), \quad t \in I\!R \quad x \in I\!R^m,$$

where $f \in L^1_{loc}(I\!R; I\!R^n)$ and B is a $n \times m$

key words. linear equation, Cramer's rule, linear differential equation.

Resumen

En este artículo se encuentra una formula para las soluciones de la siguiente ecuación lineal

$$Ax = b, \quad x \in I\!R^m, \quad b \in I\!R^n, \quad m \geq n,$$

donde $A = (a_{j,i})_{n\times m}$ es una matriz real $n \times m$. Probaremos el siguiente resultado: Para todo $b \in I\!R^n$ el sistema soluble si, y sólo si, el conjunto de vectores $\{l_1, l_2, \dots, l_n\}$ formado por las filas de la matriz A es linealmente independiente en $I\!R^m$. Más aún, una solución para esta ecuación viene dada por la siguiente fórmula

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i}b_1 & < l_1, l_2 > + a_{2i}b_1 & \cdots & < l_1, l_n > + a_{ni}b_1 \\ < l_2, l_1 > + a_{1i}b_2 & \|l_2\|^2 + a_{2i}b_2 & \cdots & < l_2, l_n > + a_{ni}b_2 \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > + a_{1i}b_n & < l_n, l_2 > + a_{2i}b_n & \cdots & \|l_n\|^2 + a_{ni}b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & < l_1, l_2 > & \cdots & < l_1, l_n > \\ < l_2, l_1 > & \|l_2\|^2 & \cdots & < l_2, l_n > \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > & < l_n, l_2 > & \cdots & \|l_n\|^2 \end{vmatrix}} - 1,$$

$i = 1, 2, 3, \ldots, m$. Finalmente, se aplica estos resultados para hallar soluciones de la siguiente ecuación diferencial lineal general

$$B\dot{x}(t) = Ax(t) + f(t), \quad t \in I\!R \quad x \in I\!R^m,$$

donde $f \in L^1_{loc}(I\!R; I\!R^n)$ y B es un $n \times m$.

Palabras claves: Ecuación lineal, formula de Cramer, ecuación diferencial lineal.

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1 Introduction

In this paper we find a formula for the solutions of the following linear equation

$$Ax = b, \quad x \in I\!R^m, \quad b \in I\!R^n, \quad m \geq n, \tag{1}$$

or

$$\left\{ \begin{array}{c} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = b_n \end{array} \right. \tag{2}$$

Now, if we define the column vectors

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix}, \quad l_2 = \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \\ a_{2,m} \end{bmatrix}, \cdots\cdots, \quad l_n = \begin{bmatrix} a_{n,1} \\ a_{n,2} \\ \vdots \\ a_{n,m} \end{bmatrix},$$

then the system (2) also can be written as follows:

$$< l_i, x >= b_i, \quad i = 1, 2, \ldots, n, \tag{3}$$

where $< \cdot, \cdot >$ denotes the innerproduct in $I\!R^m$ and A is $n \times m$ real matrix. Usually, one can apply Gauss Elimination Method to find some solutions of this system, this method is a systematic procedure for solving systems like (1); it is based on the idea of reducing the augmented matrix

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,m} & b_n \end{bmatrix}, \tag{4}$$

to the form that is simple enough such that the system of equations can be solved by inspection. But, to my knowledge, in general there is not formula for the solutions of (1) in terms of determinants if $m \neq n$.

When $m = n$ and $\det(A) \neq 0$ the system (1) admits only one solution given by $x = A^{-1}b$, and from here one can deduce the well known Cramer Rule which says:

Theorem 1.1 *(Cramer Rule 1704-1752) If A is $n \times n$ matrix with $det(A) \neq 0$, then the solution of the system (1) is given by the formula:*

$$x_i = \frac{det((A)_i)}{det(A)}, \quad i = 1, 2, 3, \ldots, n, \tag{5}$$

where $(A)_i$ is the matrix obtained by replacing the entries in the ith column of A by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

A simple and interested generalization of Cramer Rule was done by Prof. Dr. Sylvan Burgstahler ([2]) from University of Minnesota, Duluth, where he taught for 20 years. This result is given by the following Theorem:

Theorem 1.2 *(Burgstahler 1983) If the system of equations*

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n = b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,n}x_n = b_n \end{cases} \tag{6}$$

has(unique) solution $x_1, x_2, \ldots, x_n$, *then for all* $\lambda_i \in I\!R, i = 1, 2, \ldots, n$ *one has*

$$\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_n x_n = \frac{\begin{vmatrix} a_{1,1} + \lambda_1 b_1 & a_{1,2} + \lambda_2 b_1 & \cdots & a_{1,n} + \lambda_n b_1 \\ a_{2,1} + \lambda_1 b_2 & a_{2,2} + \lambda_2 b_2 & \cdots & a_{2,n} + \lambda_n b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} + \lambda_1 b_n & a_{n,2} + \lambda_2 b_n & \cdots & a_{n,n} + \lambda_n b_n \end{vmatrix}}{\begin{vmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{n,2} & \cdots & a_{n,n} \end{vmatrix}} - 1 \tag{7}$$

In this work we prove the following Theorems:

Theorem 1.3 *For all* $b \in I\!R^n$ *the system(1) is solvable if, and only if,*

$$det(AA^*) \neq 0. \tag{8}$$

Moreover, one solution for this equation is given by the following formula

$$x = A^*(AA^*)^{-1} b, \tag{9}$$

where A^* *is the transpose of* A *(or the conjugate transpose of* A *in the complex case).*

Also, this solution coincides with the Cramer formula when $n = m$*. In fact, this formula are given as follows:*

$$x_i = \sum_{j=1}^{n} a_{j,i} \frac{det((AA^*)_j)}{det(AA^*)}, \quad i = 1, 2, 3, \ldots, m, \tag{10}$$

where $(AA^*)_j$ *is the matrix obtained by replacing the entries in the* j*th column of* AA^* *by the entries in the matrix*

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

In addition, this solution has minimum norm. i.e.,

$$\|x\| = \text{ínf}\{\|w\| : Aw = b, \quad w \in I\!R^m\}, \tag{11}$$

and $\quad \|x\| = \|w\| \quad$ *with* $\quad Aw = b \iff x = w.$

Theorem 1.4 *The solution of (1)- (3) given by (9) can be written as follows:*

$$x_i = \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i}b_1 & < l_1, l_2 > + a_{2i}b_1 & \cdots & < l_1, l_n > + a_{ni}b_1 \\ < l_2, l_1 > + a_{1i}b_2 & \|l_2\|^2 + a_{2i}b_2 & \cdots & < l_2, l_n > + a_{ni}b_2 \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > + a_{1i}b_n & < l_n, l_2 > + a_{2i}b_n & \cdots & \|l_n\|^2 + a_{ni}b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & < l_1, l_2 > & \cdots & < l_1, l_n > \\ < l_2, l_1 > & \|l_2\|^2 & \cdots & < l_2, l_n > \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > & < l_n, l_2 > & \cdots & \|l_n\|^2 \end{vmatrix}} - 1, i = 1, 2, 3, \ldots, m. \tag{12}$$

Theorem 1.5 *The system (1) is solvable for each $b \in I\!R^n$ if, and only if, the set of vectors $\{l_1, l_2, \ldots, l_n\}$ formed by the rows of the matrix A is lineally independent in $I\!R^m$.*

Moreover, a solution for the system (1) is given by the following formula:

$$x_i = \frac{\upsilon_{1i}}{\|\upsilon_1\|^2}b_1 + \frac{\upsilon_{2i}}{\|\upsilon_2\|^2}\left(b_2 - \frac{< l_2, \upsilon_1 >}{\|\upsilon_1\|^2}c_1\right) \tag{13}$$

$$+ \; + \cdots + \frac{\upsilon_{ni}}{\|\upsilon_n\|^2}\left(b_n - \sum_{i=1}^{n-1}\frac{< l_n, \upsilon_i >}{\|\upsilon_i\|^2}c_i\right), \quad i = 1, 2, \ldots, m, \tag{14}$$

where the set of vectors $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$ is obtain by the Gram-Schmidt process and the numbers $c_1, c_2, \ldots, c_n$ are given by

$$\begin{aligned} c_1 &= b_1 \\ c_2 &= b_2 - \frac{< l_2, \upsilon_1 >}{\|\upsilon_1\|^2}c_1 \\ c_3 &= b_3 - \frac{< l_3, \upsilon_1 >}{\|\upsilon_1\|^2}c_1 - \frac{< l_3, \upsilon_2 >}{\|\upsilon_2\|^2}c_2 \\ \vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ c_n &= b_n - \sum_{i=1}^{n-1}\frac{< l_n, \upsilon_i >}{\|\upsilon_i\|^2}c_i, \end{aligned} \tag{15}$$

and $\upsilon_i = [\upsilon_{i1}, \upsilon_{i2}, \upsilon_{i3}, \ldots, \upsilon_{im}]\top, \quad i = 1, 2, \ldots, n.$

Finally, we apply these results to find a solutions of the following generalized linear differential equation

$$B\dot{x}(t) = Ax(t) + f(t), \quad t \in I\!R \quad x \in I\!R^m, \tag{16}$$

where $f \in L^1_{loc}(I\!R; I\!R^n)$ and B is a $n \times m$.

2 Proof of the Main Theorems

In this section we shall prove Theorems 1.3, 1.4, 1.5 and more. To this end, we shall denote by $< x, y >$ the Euclidian innerproduct in $I\!R^k$ and the associated norm by $\| x\| = \sqrt{< x, x >}$. Also, we shall use some ideas from [3] and the following result from [1], pp 55.

Lemma 2.1 *Let W and Z be Hilbert space, $G \in L(W, Z)$ and $G^* \in L(Z, W)$ the adjoint operator, then the following statements holds,*

(i) $\text{Rang}(G) = Z \iff \exists \gamma > 0$ *such that*

$$\|G^* z\|_W \geq \gamma \|z\|_Z, \quad z \in Z.$$

(ii) $\overline{\text{Rang}(G)} = Z \iff Ker(G^*) = \{0\} \iff G^* \ \ is \ \ 1-1.$

Proof of Theorem 1.3. The matrix A may also viewed as a linear operator $A : I\!R^m \to I\!R^n$; therefore $A \in L(I\!R^m, I\!R^n)$ and its adjoint operator A^* is the transpose of A and $A^* : I\!R^n \to I\!R^m$.

Then, system (1) is solvable for all $b \in I\!R^n$ if, and only if, the operator A is surjective. Hence, from the Lemma 2.1 there exists $\gamma > 0$ such that

$$\|A^* z\|_{I\!R^m} \geq \gamma \|z\|_{I\!R^n}, \quad z \in I\!R^n.$$

Therefore,

$$< AA^* z, z > \geq \gamma^2 \|z\|^2_{I\!R^n}, \quad z \in I\!R^n.$$

This implies that AA^* is one to one. Since AA^* is a $n \times n$ matrix, then $\det(AA^*) \neq 0$.

Suppose now that $\det(AA^*) \neq 0$. Then $(AA^*)^{-1}$ exists and given $b \in I\!R^n$ we can see that $x = A^*(AA^*)^{-1} b$ is a solution of $Az = b$.

Now, since $z = (AA^*)^{-1} b$ is the only solution of the equation

$$(AA^*) w = b,$$

then from Theorem 1.1 (Cramer Rule) we obtain That:

$$z_1 = \frac{\det((AA^*)_1)}{\det(AA^*)}, z_2 = \frac{\det((AA^*)_2)}{\det(AA^*)}, \cdots, z_n = \frac{\det((AA^*)_n)}{\det(AA^*)},$$

where $(AA^*)_i$ is the matrix obtained by replacing the entries in the ith column of AA^* by the entries in the matrix

$$\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}.$$

Then, the solution $x = A^*(AA^*)^{-1}b$ of (1) can be written as follows

$$x = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,m} & a_{2,m} & \cdots & a_{n,m} \end{bmatrix} \begin{bmatrix} \frac{\det((AA^*)_1)}{\det(AA^*)} \\ \frac{\det((AA^*)_2)}{\det(AA^*)} \\ \vdots \\ \frac{\det((AA^*)_n)}{\det(AA^*)} \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n a_{j,1} \frac{\det((AA^*)_j)}{\det(AA^*)} \\ \sum_{j=1}^n a_{j,2} \frac{\det((AA^*)_j)}{\det(AA^*)} \\ \vdots \\ \sum_{j=1}^n a_{j,m} \frac{\det((AA^*)_j)}{\det(AA^*)} \end{bmatrix}.$$

Now, we shall see that this solution has minimum norm. In fact, consider w in $I\!R^m$ such that $Aw = b$ and

$$\|w\|^2 = \|x + (w - x)\|^2 = \|x\|^2 + 2\mathrm{Re} < x, w - x > + \|w - x\|^2.$$

On the other hand,

$$< x, w - x >=< A^*(AA^*)^{-1}b, w - x =< (AA^*)^{-1}b, Aw - Ax >=< (AA^*)^{-1}b, b - b >= 0.$$

Hence, $\|w\|^2 - \|x\|^2 = \|w - x\|^2 \geq 0$.
Therefore, $\|x\| \leq \|w\|$, and $\|x\| = \|w\|$ iff $x = w$. □

Proof of Theorem 1.4. From Theorem 1.3 this solution is given by

$$x_i = a_{1,i}z_1 + a_{2,i}z_2 + \cdots + a_{n,i}z_n, \quad i = 1, 2, \ldots, m,$$

where

$$z_j = \frac{\det((AA^*)_j)}{\mathrm{del}(AA^*)}, \quad j = 1, 2, \ldots, n,$$

is the only solution of the system

$$AA^*z = b,$$

given by Theorem 1.1(Cramer Rule).

On the other hand, we have the following expression for AA^*

$$
\begin{aligned}
AA^* &= \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n,1} & a_{2,2} & \cdots & a_{n,m} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{n,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{n,2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1,m} & a_{12,m} & \cdots & a_{n,m} \end{bmatrix} \\
&= \begin{bmatrix} \|l_1\|^2 & < l_1, l_2 > & \cdots & < l_1, l_n > \\ < l_2, l_1 > & \|l_2\|^2 & \cdots & < l_2, l_n > \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > & < l_n, l_2 > & \cdots & \|l_n\|^2 \end{bmatrix}.
\end{aligned}
$$

Then, applying Theorem 1.2 we obtain:

$$
\begin{aligned}
x_i &= a_{1,i}z_1 + a_{2,i}z_2 + \cdots + a_{n,i}z_n = \\
&= \frac{\begin{vmatrix} \|l_1\|^2 + a_{1i}b_1 & < l_1, l_2 > + a_{2i}b_1 & \cdots & < l_1, l_n > + a_{ni}b_1 \\ < l_2, l_1 > + a_{1i}b_2 & \|l_2\|^2 + a_{2i}b_2 & \cdots & < l_2, l_n > + a_{ni}b_2 \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > + a_{1i}b_n & < l_n, l_2 > + a_{2i}b_n & \cdots & \|l_n\|^2 + a_{ni}b_n \end{vmatrix}}{\begin{vmatrix} \|l_1\|^2 & < l_1, l_2 > & \cdots & < l_1, l_n > \\ < l_2, l_1 > & \|l_2\|^2 & \cdots & < l_2, l_n > \\ \vdots & \vdots & \vdots & \vdots \\ < l_n, l_1 > & < l_n, l_2 > & \cdots & \|l_n\|^2 \end{vmatrix}} - 1.
\end{aligned}
$$

This complete the proof of the Theorem.

□

Proof of Theorem 1.5. Suppose the system is solvable for all $b \in I\!R^n$. Now, assume the existence of real numbers $c_i, i = 1, 2, \ldots, n$ such that

$$c_1 l_1 + c_2 l_2 + c_3 l_3 + \cdots + c_n l_n = 0.$$

Then, there exists $x \in I\!R^m$ such that

$$
\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = c_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = c_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = c_n \end{cases}
$$

In other words,

$$< l_i, x > = c_i, \quad i = 1, 2, \ldots, n.$$

Hence,

$$< c_i l_i, x > = c_i^2, \quad i = 1, 2, \ldots, n.$$

So,

$$< c_1l_1 + c_2l_2 + c_3l_3 + \cdots + c_nl_n, x >= c_1^2 + c_2^2 + c_3^2 + \cdots + c_n^2 = 0.$$

Therefore, $c_1 = c_2 = \cdots = c_n = 0$, which prove the independence of $\{l_1, l_2, \ldots, l_n\}$.

Now, suppose that the set $\{l_1, l_2, \ldots, l_n\}$ is linearly independent in $\mathbb{R}^m$. Using the Gram-Schmidt process we can find a set $\{\upsilon_1, \upsilon_2, \ldots, \upsilon_n\}$ of orthogonal vectors in $\mathbb{R}^m$ given by the formula:

$$\begin{aligned}
\upsilon_1 &= l_1 \\
\upsilon_2 &= l_2 - \frac{< l_2, \upsilon_1 >}{\|\upsilon_1\|^2}\upsilon_1 \\
\upsilon_3 &= l_3 - \frac{< l_3, \upsilon_1 >}{\|\upsilon_1\|^2}\upsilon_1 - \frac{< l_3, \upsilon_2 >}{\|\upsilon_2\|^2}\upsilon_2 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
\upsilon_n &= l_n - \sum_{i=1}^{n-1} \frac{< l_n, \upsilon_i >}{\|\upsilon_i\|^2}\upsilon_i
\end{aligned} \tag{17}$$

Then, system (1) will be equivalent to the following system

$$\left\{ \begin{array}{l}
< \upsilon_1, x >= c_1 \\
< \upsilon_2, x >= c_2 \\
< \upsilon_3, x >= c_3 \\
\vdots \quad \vdots \quad \vdots \\
< \upsilon_n, x >= c_n
\end{array} \right. \tag{18}$$

where

$$\begin{aligned}
c_1 &= b_1 \\
c_2 &= b_2 - \frac{< l_2, \upsilon_1 >}{\|\upsilon_1\|^2}c_1 \\
c_3 &= b_3 - \frac{< l_3, \upsilon_1 >}{\|\upsilon_1\|^2}c_1 - \frac{< l_3, \upsilon_2 >}{\|\upsilon_2\|^2}c_2 \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
c_n &= b_n - \sum_{i=1}^{n-1} \frac{< l_n, \upsilon_i >}{\|\upsilon_i\|^2}c_i
\end{aligned} \tag{19}$$

If we denote the vectors υ_i's by

$$\upsilon_i = \begin{bmatrix} \upsilon_{i1} \\ \upsilon_{i2} \\ \upsilon_{i3} \\ \vdots \\ \upsilon_{im} \end{bmatrix}, \quad i = 1, 2, \ldots, n,$$

and the $n \times m$ matrix Υ by

$$\Upsilon = \begin{bmatrix} \upsilon_{1,1} & \upsilon_{1,2} & \cdots & \upsilon_{1,m} \\ \upsilon_{2,1} & \upsilon_{2,2} & \cdots & \upsilon_{2,m} \\ \vdots & \vdots & \vdots & \vdots \\ \upsilon_{n,1} & \upsilon_{n,2} & \cdots & \upsilon_{n,m} \end{bmatrix}$$

Then, applying Theorem 1.3 we obtain that, system (18) has solution for all $C \in I\!R^n$ if, and only if, $\det(\Upsilon\Upsilon^*) \neq 0$. But,

$$\Upsilon\Upsilon^* = \begin{bmatrix} \|\upsilon_1\|^2 & <\upsilon_1, \upsilon_2> & \cdots & <\upsilon_1, \upsilon_n> \\ <\upsilon_2, \upsilon_1> & \|\upsilon_2\|^2 & \cdots & <\upsilon_2, \upsilon_n> \\ \vdots & \vdots & \vdots & \vdots \\ <\upsilon_n, \upsilon_1> & <\upsilon_n, \upsilon_2> & \cdots & \|\upsilon_n\|^2 \end{bmatrix} = \begin{bmatrix} \|\upsilon_1\|^2 & 0 & 0 & \cdots & 0 \\ 0 & \|\upsilon_2\|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \|\upsilon_n\|^2 \end{bmatrix}$$

So,

$$\det(\Upsilon\Upsilon^*) = \|\upsilon_1\|^2\|\upsilon_2\|^2 \cdots \|\upsilon_n\|^2 \neq 0.$$

From here and using the formula (9) we complete the proof of this Theorem. □

2.1 Examples and Particular Cases

In this section we shall consider some particular cases and examples to illustrate the results of this work.

Example 2.1 *Consider the following particular case of system (1)*

$$a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b. \tag{20}$$

In this case $n = 1$ and $A = [a_{1,1}, a_{1,2}, \cdots, a_{1,m}]$. Then, if we define the column vector

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix},$$

$$AA^* = \begin{bmatrix} a_{1,1} & a_{1,2} & \vdots & a_{1,m} \end{bmatrix} \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix} = \|l_1\|^2.$$

Then, $(AA^*)^{-1}b = b\|l_1\|^{-2}$ and

$$x = A^*(AA^*)^{-1}b = \begin{bmatrix} a_{1,1}b\|l_1\|^{-2} \\ a_{1,2}b\|l_1\|^{-2} \\ \vdots \\ a_{1,m}b\|l_1\|^{-2} \end{bmatrix}.$$

Therefore, a solution of the system (20) is given by:

$$x_i = \frac{a_{1i}b}{\|l_1\|^2} = \frac{a_{1i}b}{\sum_{j=1}^m a_{1j}^2}, \quad i = 1, 2, \dots, m. \tag{21}$$

Example 2.2 *Consider the following particular case of system (1)*

$$\begin{matrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = b_2 \end{matrix} \tag{22}$$

In this case $n = 2$ *and*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2}x_2 & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \end{bmatrix},$$

Then, if we define the column vectors

$$l_1 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \\ \vdots \\ a_{1,m} \end{bmatrix}, \quad l_2 = \begin{bmatrix} a_{2,1} \\ a_{2,2} \\ \vdots \\ a_{2,m} \end{bmatrix},$$

then

$$AA^* = \begin{bmatrix} a_{1,1} & a_{1,2}x_2 & \cdots & a_{1,m} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,m} \end{bmatrix} \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ \vdots & \vdots \\ a_{1,m} & a_{12,m} \end{bmatrix} = \begin{bmatrix} \|l_1\|^2 & < l_1, l_2 > \\ < l_2, l_1 > & \|l_2\|^2 \end{bmatrix}.$$

Hence, from the formula (36) we obtain that:

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = A^*(AA^*)^{-1}b = \begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \\ \vdots & \vdots \\ a_{1,m} & a_{12,m} \end{bmatrix} \begin{bmatrix} \frac{det((AA^*)_1)}{det(AA^*)} \\ \frac{det((AA^*)_2)}{det(AA^*)} \end{bmatrix}$$

Therefore, a solution of the system (22) is given by:

$$x_1 = a_{11}\frac{b_1\|l_2\|^2 - b_2 < l_1, l_2 >}{\|l_1\|^2\|l_2\|^2 - | < l_1, l_2 > |^2} + a_{21}\frac{b_2\|l_1\|^2 - b_1 < l_2, l_1 >}{\|l_1\|^2\|l_2\|^2 - | < l_1, l_2 > |^2} \tag{23}$$

$$x_2 = a_{12}\frac{b_1\|l_2\|^2 - b_2 < l_1, l_2 >}{\|l_1\|^2\|l_2\|^2 - | < l_1, l_2 > |^2} + a_{22}\frac{b_2\|l_1\|^2 - b_1 < l_2, l_1 >}{\|l_1\|^2\|l_2\|^2 - | < l_1, l_2 > |^2} \tag{24}$$

$$\vdots = \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \tag{25}$$

$$x_m = a_{1m}\frac{b_1\|l_2\|^2 - b_2 < l_1, l_2 >}{\|l_1\|^2\|l_2\|^2 - | < l_1, l_2 > |^2} + a_{2m}\frac{b_2\|l_1\|^2 - b_1 < l_2, l_1 >}{\|l_1\|^2\|l_2\|^2 - | < l_1, l_2 > |^2} \tag{26}$$

Now, we shall apply the foregoing formula to find the solution of the following system

$$\begin{cases} x_1 + x_2 = 1 \\ -x_1 + x_2 + x_3 = -1 \end{cases} \tag{27}$$

If we define the column vectors

$$l_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \quad l_2 = \begin{bmatrix} - \\ 1 \\ 1 \end{bmatrix},$$

then $det(AA^*) = \|l_1\|^2\|l_2\|^2 - |< l_1, l_2 >|^2 = \|l_1\|^2\|l_2\|^2 = 6$ *and* $x_1 = \frac{5}{6}$, $x_2 = \frac{1}{6}$, *and* $x_3 = \frac{-2}{6}$.

Example 2.3 *Consider the following general case of system (1)*

$$\begin{cases} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,m}x_m = b_1 \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,m}x_m = b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ a_{n,1}x_1 + a_{n,2}x_2 + \cdots + a_{n,m}x_m = b_n \end{cases} \tag{28}$$

Then, if $\{l_1, l_2, \ldots, l_n\}$ is an orthogonal set in $I\!R^m$, we get

$$AA^* = \begin{bmatrix} \|l_1\|^2 & 0 & 0 & \cdots & 0 \\ 0 & \|l_2\|^2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \|l_n\|^2 \end{bmatrix},$$

and the solution of the system (1) is very simple and given by:

$$x_i = \sum_{j=1}^{n} a_{j,i} b_j \|l_j\|^{-2}, \quad i = 1, 2, \ldots, m. \tag{29}$$

Now, we shall apply the formula (29) to find solution of the following system

$$\begin{cases} -x_1 - x_2 + x_3 + x_4 = 1 \\ -x_1 + x_2 - x_3 + x_4 = 1 \\ x_1 - x_2 - x_3 + x_4 = 1 \end{cases} \tag{30}$$

If we define the column vectors

$$l_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \quad l_2 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \quad l_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}.$$

Then, $\{l_1, l_2, L_3\}$ is an orthogonal set in $I\!R^3$ and the solution of this system is given by: $x_1 = \frac{-1}{4}$, $x_2 = \frac{-1}{4}$, $x_3 = \frac{-1}{4}$ and $x_4 = \frac{3}{4}$.

3 Variational Method to Obtain Solutions

The Theorems 1.3, 1.4 and 1.5 give a formula for one solution of the system (1) which has minimum norma. But, it is no the only way allowing to build solutions of this equation. Next, we shall present a variational method to obtain solutions of (1) as a minimum of a quadratic functional $\jmath : I\!R^n \to I\!R$,

$$\jmath(\xi) = \frac{1}{2}\|A^*\xi\|^2 - < b, \xi >, \quad \forall \xi \in I\!R^n. \tag{31}$$

Proposition 3.1 *For a given $b \in I\!R^n$ the equation (1) has a solution $x \in I\!R^m$ if, and only if,*

$$< x, A^*\xi > - < b, \xi >= 0, \quad \forall \xi \in I\!R^n. \tag{32}$$

It is easy to see that (32) is in fact an optimality condition for the critical points of the quadratic functional $\jmath$ define above.

Lemma 3.1 *Suppose the quadratic functional $\jmath$ has a minimizer $\xi_b \in I\!R^n$. Then,*

$$x_b = A^*\xi_b, \tag{33}$$

is a solution of (1).

Proof . First, observe that $\jmath$ has the following form

$$\jmath(\xi) = \frac{1}{2} < AA^*\xi, \xi > - < b, \xi >, \quad \forall \xi \in I\!R^n.$$

Then, if ξ_b is a point where $\jmath$ achieves its minimum value, we obtain that

$$\frac{d}{d\xi}\{\jmath\}(\xi_b) = AA^*\xi_b - b = 0.$$

So, $AA^*\xi_b = b$ and $x_b = A^*\xi_b$ is a solution of (1).

□

Remark 3.1 *Under the condition of Theorem 1.3, the solution given by the formulas (33) and (9) coincide.*

Theorem 3.1 *The system (8) is solvable if, and only if, the quadratic functional $\jmath$ define by (31) has a minimum for all $b \in I\!R^n$*

Proof . Suppose (8) is solvable. Then, the matrix A viewed as an operator from $\mathbb{R}^m$ to $\mathbb{R}^n$ is surjective. Hence, from Lemma 2.1 there exists $\gamma > 0$ such that

$$\|A^*\xi\|^2 \geq \gamma^2 \|\xi\|^2, \quad \xi \in \mathbb{R}^n.$$

Then,

$$\jmath(\xi) \geq \frac{\gamma^2}{2}\|\xi\|^2 - \|b\|\|\xi\|, \quad \xi \in \mathbb{R}^n.$$

Therefore,

$$\lim_{\|\xi\|\to\infty} \jmath(\xi) = \infty.$$

Consequently, $\jmath$ is coercive and the existence of a minimum is ensured.

The other way of the proof follows as in proposition 3.1.

□

Now, we shall consider an example where Theorems 1.3, 1.4 and 1.5 can not be applied, but proposition 3.1 does.

Example 3.1 *consider the system with linearly independent rows*

$$\begin{cases} x_1 + x_2 + x_3 = 1 \\ 2x_1 + 2x_2 + 2x_3 = 2 \end{cases}$$

In this case $n = 2$ *and*

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \quad and \quad b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Then

$$AA^* = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 6 & 12 \end{bmatrix}.$$

Therefore, the critical points of the quadratic functional $\jmath$ *given by (31) satisfy the equation*

$$AA^*\xi = b.$$

i.e.,

$$\begin{array}{c} 3\xi_1 + 6\xi_2 = 1 \\ 6\xi_1 + 12\xi_2 = 2 \end{array}$$

So, there are infinitely many critical points given by

$$\xi = \begin{bmatrix} \frac{1}{3} - 2a \\ a \end{bmatrix}, \quad a \in \mathbb{R}.$$

Hence, a solution of the system is given by

$$x = A^*\xi = \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \frac{1}{3} - 2a \\ a \end{bmatrix} = \begin{bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{bmatrix}$$

4 The Case $m < n$

The case $m < n$ is undetermined since the equation $Ax = b$ has solution only when $b \in \text{Rang}(A)$. But, nevertheless we can produce the following Theorems:

Theorem 4.1 *Suppose $m < n$ and $A : I\!R^m \to I\!R^n$ is one to one. Then, for all $b \in Rang(A)$ the equation*

$$Ax = b, \quad x \in I\!R^m, \quad b \in I\!R^n, \quad m < n, \tag{34}$$

admits only one solution given by

$$x = (A^*A)^{-1}A^*b. \tag{35}$$

Moreover, this solution is given by the following formula:

$$x_i = \frac{det((A^*A)_i)}{det(A^*A)}, \quad i = 1, 2, 3, \ldots, m, \tag{36}$$

where $(A^*A)_i$ is the matrix obtained by replacing the entries in the ith column of AA^* by the entries in the matrix

$$\begin{bmatrix} \sum_{j=1}^n a_{j,1}b_j \\ \sum_{j=1}^n a_{j,2}b_j \\ \vdots \\ \sum_{j=1}^n a_{j,m}b_j \end{bmatrix}$$

Proof . Since A is one to one, from Lemma 2.1 we have that $A^* : I\!R^n \to I\!R^m$ is surjective, and consequently $\det(A^*A) \neq 0$. Therefore, $(A^*A)^{-1}$ exists and

$$x = (A^*A)^{-1}A^*b, \tag{37}$$

is the only solution of (34). In fact, if $x \in I\!R^m$ is the only solution of (34), then $A^*Ax = A^*b$ and $(A^*A)^{-1}A^*Ax = (A^*A)^{-1}A^*b$. So, $x = (A^*A)^{-1}A^*b$. The remainder of the proof follows in the same way as in Theorem 1.3 □

Theorem 4.2 *If the set of vectors $\{l_1, l_2, \ldots, l_n\}$ formed by the columns of the matrix A is an orthogonal set in $I\!R^m$, then*

$$A^*A \;=\; \begin{bmatrix} \|l_1\|^2 & 0 & \cdots & 0 \\ 0 & \|l_2\|^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \|l_n\|^2 \end{bmatrix}.$$

and the solution $x = (A^*A)^{-1}A^*b$ of the system (34) takes the following simple form

$$x = \begin{bmatrix} \frac{1}{\|l_1\|^2}\sum_{j=1}^n a_{j,1}b_j \\ \frac{1}{\|l_2\|^2}\sum_{j=1}^n a_{j,2}b_j \\ \vdots \\ \frac{1}{\|l_m\|^2}\sum_{j=1}^n a_{j,m}b_j \end{bmatrix}.$$

5 Generalized Linear Equations

Let A and B be $n \times m$ matrixes with $m \geq n$ such that $\det(AA^*) \neq 0$ and the eigenvalues of $BA^*(AA^*)^{-1}$ are not zero. Under these conditions we consider the following generalized linear system of differential equations with implicit derivative

$$B\dot{x}(t) = Ax(t) + f(t), \quad t \in I\!R, \quad f \in L^1_{loc}(I\!R, I\!R^n). \tag{38}$$

Before study the non homogeneous equation (38) we shall look for the solution of the homogeneous equation

$$B\dot{x}(t) = Ax(t), \quad t \in I\!R. \tag{39}$$

We shall look for the solution of (39) in the following form:

$$x(t) = e^{\lambda t}\xi \ \text{ with } \ \lambda \neq 0, \quad \xi = A^*(AA^*)^{-1}b, \quad b \in I\!R^n, \quad \xi \in I\!R^m.$$

Then, if we putt $S = A^*(AA^*)^{-1}$ we get the following algebraic equations for λ and b:

$$(\lambda BS - I)b = (\lambda BA(AA^*)^{-1} - I)b = 0, \quad b \neq 0. \tag{40}$$

$$\det(\lambda BS - I) = 0. \tag{41}$$

Lemma 5.1 *If α is an eigenvalue of the matrix $BS = BA^*(AA^*)^{-1}$ with corresponding eigenvector $b \in I\!R^n$, then $x(t) = e^{\lambda t}\xi$, with $\lambda = \alpha^{-1}$ and $\xi = Sb$, is a solution of (39).*

Proof . If $x(t) = e^{\lambda t}\xi$, then

$$\begin{aligned} Ax(t) &= Ae^{\lambda t}\xi = e^{\lambda t}ASb \\ &= e^{\lambda t}b = e^{\lambda t}\lambda BSb \\ &= \lambda e^{\lambda t}B\xi = B(\lambda e^{\lambda t}\xi) = B\dot{x}(t). \end{aligned}$$

□

Corollary 5.1 *If BS possess n linearly independent eigenvectors $b_1, b_2, \ldots, b_n$ and $\frac{1}{\lambda_j}$ be the real eigenvalue that corresponds to b_j (The numbers $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ need not all be distinct). Then, for all $c_j \in I\!R, j = 1, 2, \ldots, n$*

$$x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \xi_j, \quad \xi_j = S b_j, \tag{42}$$

is a general solution of the equation (39).

Example 5.1 *Consider the following system of differential equations*

$$\begin{cases} \dot{x}_1 + \dot{x}_2 + \dot{x}_3 = x_1 - x_2 + x_3 \\ \dot{x}_1 - \dot{x}_2 = -x_1 + x_2 + 2x_3 \end{cases} \tag{43}$$

Here,

$$B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}.$$

Then,

$$AA^* = \begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}, \quad A^*(AA^*)^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{-1}{3} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{3} \end{bmatrix} \quad \text{and} \quad BS = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$$

Hence, the eigenvalues and corresponding eigenvectors of BS are respectively:

$$\alpha_1 = \frac{1}{\sqrt{3}}, \quad b_1 = \begin{bmatrix} \frac{\sqrt{3}+1}{2} \\ 1 \end{bmatrix}, \quad \alpha_2 = \frac{-1}{\sqrt{3}}, \quad b_2 = \begin{bmatrix} \frac{-(\sqrt{3}-1)}{2} \\ 1 \end{bmatrix}.$$

On the other hand,

$$\xi_1 = Sb_1 = \begin{bmatrix} \frac{\sqrt{3}}{6} \\ \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}+3}{6} \end{bmatrix} \quad \text{and} \quad \xi_2 = Sb_2 = \begin{bmatrix} \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}-3}{6} \end{bmatrix}.$$

Then,

$$x_1(t) = e^{\lambda_1 t} \xi_1 = e^{\sqrt{3}_1 t} \begin{bmatrix} \frac{\sqrt{3}}{6} \\ \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}+3}{6} \end{bmatrix} \quad \text{and} \quad x_2(t) = e^{\lambda_2 t} \xi_2 = e^{-\sqrt{3}_1 t} \begin{bmatrix} \frac{-\sqrt{3}}{6} \\ \frac{\sqrt{3}}{6} \\ \frac{\sqrt{3}-3}{6} \end{bmatrix}$$

are two solutions of (43).

Theorem 5.1 *(Variation Constant Formula) If BS possess n linearly independent eigenvectors $b_1, b_2, \ldots, b_n$ and $\frac{1}{\lambda_j}$ be the real eigenvalue that corresponds to b_j (The numbers $\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}$ need not all be distinct). Then, for all $c_j \in I\!R, j = 1, 2, \ldots, n$*

$$x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \xi_j + \int_0^t \left(\sum_{j=1}^{n} e^{\lambda_j (t-s)} \beta_j(s) \xi_j \right) ds, \quad \xi_j = S b_j, \tag{44}$$

where $f(t) = \sum_{j=1}^{n} \beta_j(t) b_j$ is a general solution of the equation (38).

Corollary 5.2 *Under the conditions of the foregoing Theorem, if $b_1, b_2, \ldots, b_n$ are orthogonal, then a general solution of (38) is given by*

$$x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} \xi_j + \int_0^t \left(\sum_{j=1}^{n} e^{\lambda_j (t-s)} < b_j, f(s) > \xi_j \right) ds, \quad \xi_j = S b_j, \tag{45}$$

Proof of Theorem 5.1.. Clearly that a solution $x(t)$ of (38) has the form $x(t) = x_h(t) + x_p(t)$, where $x_h(t)$ is a solution of the homogeneous equation (39) and $x_p(t)$ is a particular solution of the non homogeneous equation (38). So, we look for a particular solution of (38) by the variation constant method; that is to say, we need to find functions $c_j(t), j = 1, 2, \ldots, n$ such that

$$x(t) = \sum_{j=1}^{n} c_j(t) e^{\lambda_j t} \xi_j,$$

is a solution of (38). To this end, let us consider the following expression:

$$\begin{aligned} B\dot{x} &= \sum_{j=1}^{n} \dot{c}_j(t) e^{\lambda_j t} B \xi_j + \sum_{j=1}^{n} c_j(t) B \lambda_j e^{\lambda_j t} \xi_j \\ &= \sum_{j=1}^{n} \dot{c}_j(t) e^{\lambda_j t} B \xi_j + \sum_{j=1}^{n} c_j(t) B \frac{d}{dt} \left(e^{\lambda_j t} \xi_j \right) \\ &= \sum_{j=1}^{n} \dot{c}_j(t) e^{\lambda_j t} B \xi_j + \sum_{j=1}^{n} c_j(t) A e^{\lambda_j t} \xi_j \\ &= f(t) + \sum_{j=1}^{n} c_j(t) A e^{\lambda_j t} \xi_j. \end{aligned}$$

Then, we have to solve the equation for the unknown $c_j(t)$,

$$\sum_{j=1}^{n} \dot{c}_j(t) e^{\lambda_j t} B \xi_j = f(t),$$

which is equivalent to the equation

$$\sum_{j=1}^{n} \dot{c}_j(t) e^{\lambda_j t} b_j = f(t) = \sum_{j=1}^{n} \beta_j(t) b_j.$$

Since $b_1, b_2, \ldots, b_n$ are linearly independent, then

$$c_j(t) = \int_0^t e^{-\lambda_j s} \beta_j(s) ds.$$

This completes the proof of the Theorem.

$\square$

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