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Properties and relations between visibility and illumination operators

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Abstract

We define an illumination operator which is in some way related with two operators defined by Martini & Wenzel. Here we study properties of the new operator, establish relations between the existing ones and we obtain results that connect them.

key words. Convex sets, visibility, illumination, operators.

Resumen

Definimos aquí un operador de iluminación que, en algún sentido, se vincula con dos operadores utilizados por Martini y Wenzel. Presentamos propiedades del nuevo operador a la vez que establecemos relaciones con los anteriores.

Palabras Claves: Conjuntos Convexos, visibilidad, iluminación, operadores.

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1 Basic definitions and notations.

Unless otherwise stated, all the points and sets considered here are included in $\mathbb{R}^n$ the real n-dimensional euclidian space.

The open segment joining two different points x and y is (x, y), while the substitution of one or both parentheses by square ones indicates the adjunction of the corresponding endpoints. The interior, closure, boundary, and complement of a set K are denoted by: $intK$, clK, bdK and K^C respectively. The *join* of the sets A and B is the set $J\,(A, B) = \bigcup \{[a\,,\,b]\ a \in A\,,\,b \in B\}$

In particular, $J(\{x\}, K)$ is simply denoted $J(x, K)$. If K is a convex set it holds that $J(x, K) = conv(\{x\} \cup K)$, where $convK$ indicates the *convex hull* of K. The affine hull generated by the set A is $aff(A)$. We symbolize $[x, y>$ the closed ray issuing from x and going through y.

A *convex component* of S is a maximal convex subset of S. The *mirador* (*convex kernel*) of S is the set $mir\, S$ of all the points $x \in S$ that verifies $[x\,; y] \subset S$ for all $y \in S$. S is *convex* if $mir\, S = S$, and S is *starshaped* if $mir\, S$ is not empty. If K is a nonconvex set, the *convex deficiency* of K is the set $D(K) = convK \backslash K$. A *body* is a set having non empty interior.

The family of all subsets of E is denoted by $\mathcal{P}(E)$. Martini and Wenzel defined ([3]) for every $K \subset \mathbb{R}^n$ and its complement $E = K^C$, the *visibility operator* $\sigma_K : \mathcal{P}(E) \longrightarrow \mathcal{P}(E)$ by $\sigma_K(A) = A \cup \{b \in E - A : \exists a \in A$ such that $[a, b] \cap K = \varnothing$ and $[a, b > \cap K \neq \varnothing\}$. If $A = \{x\}$, the set $\sigma_K(\{x\})$ is simply denoted $\sigma_K(x)$.

2 Introduction.

Many authors have studied Visibility form different points of view. One of the lines more studied considers, as the basic definition, that a point $x \in S$ sees -via S- other point $y \in S$ if and only if $[x, y] \subset S$. Notice that the definition forces the points to lie both in the set where the visibility is stated. Martini and Wenzel in [5] refer to "visibility" in this sense but in [4] they work with this notion in another way. We describe their approach in what follows. They take some set K and state that a point $x \in K^C$ sees $y \in bdK$ if $[x, y] \cap K = \{y\}$. For a closed set K, the points involved do not lie in the same set. Of course this idea is close to the one of illumination, an area in which Martini and Wenzel have been working, and where we focus in this paper.

The visibility operator defined by Martini and Wenzel in [3] was used to characterize convex sets by means of studying properties of it in terms of set theory. Their main result states that a compact set K, such that its complement is connected, is convex if and only if σ_K is a closure operator. A closure operator verifies to be increasing, monotone and idempotent. In a recent work ([4]) they have also defined two other operators σ_0 and $\hat{\sigma}$. The first one preserves the definition of σ_K with the difference that the space E is the complement of the interior of K instead of the complement of K. The different sets $\sigma_0(A)$ contain points in the boundary of K and this fact allows the authors to relate the operator with the idea of illuminating boundary points of K from outside it. Analogously, the other operator $\hat{\sigma}$ is defined from $\mathcal{P}(R^n \backslash intK) \to \mathcal{P}(R^n \backslash intK)$ and its definition allows the authors to work with visibility in the sense considered by them.

Following these ideas, we define here an illumination operator which is related with $\sigma_{_K}$ but tries to focus only on the boundary points illuminated from the external set. We study it in terms of set theory and we also analyze properties of it in terms of Visibility Theory.

We begin extending to non convex sets the definition of illumination presented by Boltyanski (see [1]) for convex sets.

Definition 1 *Let K be a closed body and $E = K^C$. A point $y \in bdK$ is illuminated by $x \in E$ if $[x; y) \cap K = \varnothing$ and $[x; y > \cap\, intK \neq \varnothing$.*

We define an illumination operator $il_K : \mathcal{P}(E) \longrightarrow \mathcal{P}(cl(E))$. If $A \subset E$ then $il_K(A)$ consists of all those points in bdK which can be illuminated from at least some point of A, i.e.:

Definition 2 *Let K be a subset in $\mathbb{R}^n$ and $E = K^C$. The illumination operator $il_K : \mathcal{P}(E) \longrightarrow \mathcal{P}(cl(E))$ verifies that if $A \subset E$ then $il_K(A)$ is the set*

$il_K(A) = \{y \in bdK$: there exists some $x \in A$ such that $[x; y) \cap K = \varnothing$ and $[x; y > \cap$ $intK \neq \varnothing\}$

In the case that $A = \{x\}$, we denote $il_K(x) = il_K(\{x\})$.

The relation between $\sigma_{_K}$, $il_K(A)$ and σ_0 is clear: for every set $A \subset E$, it holds that $\sigma_{_K}(A) \cup il_K(A) = \sigma_0(A)$.

3 Results.

The first two items of the next result show two analogous descriptions of visibility and illumination operators. The third states the monotony of the illumination operator.

Proposition 3 *Let K be a subset in $\mathbb{R}^n$ and $E = K^C$. If $A \subset E$ then*

1. $\sigma_{_K}(A) = \bigcup_{a \in A} \sigma_{_K}(a)$

2. *If* K *is closed and* $int(K) \neq \varnothing$, *then* $il_K(A) = \bigcup_{a \in A} il_K(a)$

3. *If* $A \subset B \subset E$ *then* $il_K(A) \subseteq il_K(B)$.

Proof. *Since the two first proofs are similar, we include here the second one.*

A certain point $y \in il_K(A)$ *if and only if there exists* $a \in A$ *such that* $y \in il_K(a)$. *This is equivalent to state* $y \in \bigcup_{a \in A} il_K(a)$.

The third item is immediate from the first one. ■

Remark 4 *The converse of the monotony does not hold as this example shows.* $K = \{(x,y) \in \mathbb{R}^2 : 4 \leq x \leq 6; -5 \leq y \leq 5\}$, $A = \{(x,y) \in \mathbb{R}^2 : x = 2,\ -1 \leq y \leq 1\}$, $B = \{(x,y) \in \mathbb{R}^2 : x = 1; -2 \leq y \leq 2\}$

Proposition 5 *Let* K *be a closed convex body,* $E = K^C$, $y \in bdK$ *and* $x \in E$. *If* $(x;y) \subset int(\sigma_K(x))$ *then* $y \in il_K(x)$.

Proof. We prove that the segment $[x,y)$ does not meet K and that the closed ray $[x,y>$ meets the interior of K.

For the first one, recall that by the choice of x, $x \notin K$ and $(x,y) \subset int(\sigma_K(x)) \subset \sigma_K(x) \subset K^C$.

To prove the second assertion, suppose that $[x,y> \cap intK = \varnothing$. Let us consider $z \in (x;y) \subset int(\sigma_K(x))$ and B, a ball with center x, included in $\sigma_K(x)$. By our assumption and the fact that $y \in bdK$, we can take H a support hyperplane of K through y such that $(x,y) \subset H$. We denote H^+ the closed half-space that contains K and H^- its complementary half-space.

If $t \in B \cap H^-$ then $(x,t) \subset H^-$ and thus $[x,t> \cap K = \varnothing$ which contradicts that $t \in \sigma_K(x)$. Thus $[x,y> \cap intK \neq \varnothing$. ■

Remark 6 *In the previous proposition, the condition of convexity of* K *cannot be removed. To see this, consider* $K = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(x,y) \in \mathbb{R}^2 : y = 0\ ,\ 1 \leq x \leq 3\}$. *Taking the points* $x = (0,-3) \in \mathbb{R}^2$ *and* $y = (2,0) \in K$ *we obtain that* $(x;y) \subset int(\sigma_K(x))$ *but* $y \notin il_K(x)$.

Proposition 7 *Let* K *be a closed set,* $intK \neq \varnothing$ *and* $x \in E = K^C$, *then it holds that* $il_K(x) \subset il_K(\sigma_K(x))$.

Proof. Using the fact that $\{x\} \subset \sigma_K(x)$, it is immediate by item 3 of the proposition 3. ■

We explore here under what conditions the equality holds.

Proposition 8 *If K is a closed convex body and $x \in E = K^C$, then, $il_K(\sigma_K(x)) \subset il_K(x)$.*

Proof. Consider $y \in il_K(\sigma_K(x))$, then there exists $z \in \sigma_K(x)$ such that $y \in il_K(z)$. Let us suppose that $[x; y) \cap K \neq \varnothing$, then we can take $w \in [x; y) \cap K$. Since $z \in \sigma_K(x)$, it holds that $[x; z] \cap K = \varnothing$ and $[x; z > \cap K \neq \varnothing$. In this situation, if $t \in [x; z > \cap K$ then $(z; y) \cap (w; t) \neq \varnothing$. This fact provides a contradiction because $[w; t] \subset K$ and $(z; y) \subset K^C$. Hence $[x; y) \cap K = \varnothing$. On the other hand, since $z \in \sigma_K(x)$, there exists $p \in [x; z > \cap K$. It is clear that $p \neq z$. Furthermore, since $y \in il_K(z)$, we can pick $w \in [z; y > \cap intK \neq \varnothing$, then there exists $u \in (w; p) \subset intK$ such that $y \in (x; u)$, then $[x; y >$ meets $intK$ because this ray meets u. ∎

Corollary 9 *Let $K \subset \mathbb{R}^n$ be a closed convex body, then*

1. *If $x \in E = K^C$, $il_K(x) = il_K(\sigma_K(x))$.*
2. *If $A \subset E = K^C$, $il_K(A) = il_K(\sigma_K(A))$.*

Proof. *The first item is a trivial consequence of the two previous propositions.*

To prove the second notice that by proposition 3.2 $il_K(A) = \bigcup_{a \in A} il_K(a)$

and $\bigcup_{a \in A} il_K(a) = \bigcup_{a \in A} il_K(\sigma_K(a))$ *because K is convex*

$\bigcup_{a \in A} il_K(\sigma_K(a)) = il_K\left(\bigcup_{a \in A} \sigma_K(a)\right)$ *by proposition 3.2*

$il_K\left(\bigcup_{a \in A} \sigma_K(a)\right) = il_K(\sigma_K(A))$ *by proposition 3.1*

Finally we get $il_K(A) = il_K(\sigma_K(A))$. ∎

Proposition 10 *Let K be a convex body and $x \in E = K^C$. If B is a convex body and $B \subset \sigma_K(x)$ then:*

1. *$\sigma_B(x) \subset \sigma_K(x)$.*
2. *$\sigma_K(B) \subset \sigma_K(x)$.*

Proof.

1. Let $y \in \sigma_B(x)$, we can pick $z \in [x; y > \cap B$. By hypothesis this point z lies in $\sigma_K(x)$ and therefore $[x; z] \cap K = \varnothing$ and $[x; z > \cap K \neq \varnothing$. It is clear that both rays $[x; y >$ and $[x; z >$ coincide, then $[x; y] \cap K = \varnothing$ and the thesis follows.

2. Let $y \in \sigma_K(B)$.

   If $y \in B$ we have nothing to prove because $B \subset \sigma_K(x)$.

   If $y \notin B$ there exists $b \in B$ such that $[b; y] \cap K = \varnothing$ and $[b; y > \cap K \neq \varnothing$. As $b \in B \subset \sigma_K(x)$ results $[x; b] \cap K = \varnothing$ and $[x; b > \cap K \neq \varnothing$. Then there exist $x_1 \in [x; b > \cap K$ and $y_1 \in [b; y > \cap K$. There exists $z \in [x_1; y_1]$ such that $y \in [x; z]$. As K is convex, this point z belongs to K. Then $[x; y > \cap K \neq \varnothing$. On the other hand, suppose that there exists $t \in [x; y] \cap K$. Thus $[t; z] \subset K$ and this is absurd because $y \in [t; z]$ but $y \notin K$.

■

Proposition 11 *Let K be a convex body, $A \subset E = K^C$ and $B \subset E$ such that $\sigma_K(A) \subseteq \sigma_K(B)$. Then $il_K(A) \subseteq il_K(B)$.*

Proof. Corollary 9.2 and Proposition 3.3. ■

Remark 12 *The converse does not hold, as this example shows.*

$K = \{(x, y) \in \mathbb{R}^2 : 4 \leq x \leq 6; -5 \leq y \leq 5\}$, $A = \{(x, y) \in \mathbb{R}^2 : x = 2, \ -1 \leq y \leq 1\}$, $B = \{(x, y) \in \mathbb{R}^2 : x = 1; -2 \leq y \leq 2\}$. *Then $il_K(A) = il_K(B)$ but $\sigma_K(A) \nsubseteq \sigma_K(B)$.*

The next proposition is related with Prop. 2.3. of [5] in the sense that both states starshapedness of sets related, in some sense, with $\sigma_K(A)$. In Martini‘s work, the mirador of $K \cup \sigma_K(A)$ is K (in the case that K is convex and non empty), while in this work we study the possibilities of A to be the mirador of $\sigma_K(A)$.

Proposition 13 *Let $K \subset \mathbb{R}^n$ a closed set and $x \in E = K^C$.The following properties hold:*

1. *If $intK \neq \varnothing$ and $y \in il_K(x)$ then $[x; y) \subset \sigma_K(x)$.*

2. *The set $\sigma_K(x)$ is starshaped and $x \in mir(\sigma_K(x))$.*

Proof.

1. Let $z \in [x; y)$. The result is immediate because $[x; z) \subset [x; y)$, and $y \in il_K(x)$.

2. Let $z \in \sigma_K(x)$ and $y \in [x; z]$. the inclusion of the segments $[x; y] \subset [x; z]$ implies that $y \in \sigma_K(x)$, thus x sees z via $\sigma_K(x)$.

■

Remark 14 *This second result cannot be extended to any set A instead of $\{x\}$. Even assuming the convexity of A and K is it not enough to make this statement valid. Consider, for example, $K = conv\{(0,0),(2,1),(2,-1)\}$ and $A = \{(x,y) \in \mathbb{R}^2 : x = -2; -4 \leq y \leq 4\}$. In this case A is not included in $mir(\sigma_K(A))$. Recall that $mir(\sigma_K(A)) = conv\{(0,0),(-2,1),(-2,-1)\}$).*

Proposition 15 *Let $K \subset \mathbb{R}^n$ be a closed set and $x \in E = K^C$. If $y \in int(\sigma_K(x))$ then there exists $\varepsilon > 0$ such that $J(x; B(\varepsilon; y)) \subset \sigma_K(x)$.*

Proof. If $y \in int(\sigma_K(x))$ then there exists $\varepsilon > 0$ such that $B(y;\varepsilon) \subset \sigma_K(x)$. From the previous proposition $x \in mir(\sigma_K(x))$ implies that $J(x; B(y;\varepsilon)) \subset \sigma_K(x)$. ■

The following theorem is connected with one of the well known results by Boltyanski about illumination by sources (see [1]). The authors assert that the smaller number of sources needed to illuminate a convex compact body in $\mathbb{R}^n$ is $n+1$. We explore here the position of the sources to be able to illuminate such a set. The result is a necessary condition, but not sufficient, for a set A to be able to illuminate a convex compact body.

Theorem 16 *Let K be a compact convex body and $A \subset E = K^C$. If $il_K(A) = bdK$ then $aff(A) = \mathbb{R}^n$.*

Proof. Suppose that $aff(A) \subsetneq \mathbb{R}^n$. Therefore there exists an hyperplane H such that $aff(A) \subset H$. Let H_1 be the support hyperplane of K parallel to H. We denote H_1^+ the half-space which verifies $A \subset H_1^+$ and $K \subset H_1^+$. It is clear that there exists a point $x \in H_1 \cap K = H_1 \cap bdK$. We assert that such point of the set is not illuminated by any point of A. To prove this we take any $a \in A$. Then the open half-line with origin in x and going in the same direction of $[a, x>$ does not meet K, then such $x \in bdK$ is not illuminated by A which is an absurd. Thus $aff(A) = \mathbb{R}^n$.
■

Remark 17 *The previous theorem cannot be extend to an unbounded convex body as the example shows. $A = \{(-1;0),(1;0)\}$ and $K = \{(x,y) \in \mathbb{R}^2 : -1 < x < 1;\ y < \frac{1}{x^2-1}\}$ then bdK is illuminated by A but $aff(A) = \{(x,y) \in \mathbb{R}^2 : y = 0\} \neq \mathbb{R}^2$.*

Lemma 18 *Let $K \subset \mathbb{R}^n$ be a compact convex set and K_0 a connected component of the convex deficiency $D(K)$. If $K_0 \subset int(convK)$ and $x \in K_0$ then $\sigma_K(x) \subset K_0$.*

Proof. Let $p \in \sigma_K(x)$. We can take a point $y \in [x, p > \cap K$. As $x \in K_0$ and $y \in K \subset convK$ then $[x, y] \subset convK$ and therefore $[x, p] \subset convK$. Thus $[x, p] \subset D(K)$ and $p \in K_0$. ■

The following theorem is a characterization of convex sets in terms of σ_K.

Theorem 19 *For a compact body K, the following statements are equivalent:*

(i) K is a convex set

(ii) If $x \in K^C$ then $\sigma_K(x) \cap convK = \varnothing$.

Proof. (i)⇒(ii) it is immediate because if $x \in K^C$ then $\sigma_K(x) \cap convK = \sigma_K(x) \cap K$ which is empty by the definition of $\sigma_K(x)$.

(ii)⇒(i) Let us suppose that K is not a convex set. Then, there exists a connected component K_0 of the convex deficiency $D(K)$. We split the prove in two cases:

a) $K_0 \subset int(convK)$.

Let us take any $x \in K_0$. In this case $x \in convK$ and always $x \in \sigma_K(x)$ by definition of $\sigma_K(x)$, then $\sigma_K(x) \cap convK \neq \varnothing$.

b) $K_0 \not\subset int(convK)$.

In this case there exists $x \in K_0$ such that $x \notin int(convK)$ and therefore $x \in bd(convK)$. The fact that $x \in K_0$ and K is compact implies that $x \notin bdK$. Then $x \in bdK_0$ and therefore $bdK_0 \nsubseteq bdK$. Thus there exist points a, $b \in convK$ that verify $[a, b] \subset bd(convK)$ ([2]). Let us take $x_0 \in (a, b)$ and let H be a support hyperplane of $convK$ through x_0. We call H^+ and H^- to the half-spaces determined by H where $K \subset H^+$. Let us consider $t \in intK$ and let L be the line through x_0 and t. Any point $y \in L \cap H^-$ verifies that $\sigma_K(y) \cap convK \neq \varnothing$. To prove this, notice that $x_0 \in convK$ (by the choice of x_0). Furthermore $[y, x_0] \cap K = \varnothing$ (because $[y, x_0) \subset H^-$ and $x_0 \in H \setminus K$) and $[y, x_0 > \cap K \neq \varnothing$ (because $t \in [y, x_0 > \cap K$). Hence this point $y \in K^C$ verifies $\sigma_K(y) \cap convK \neq \varnothing$ and the thesis follows.

■

4 References.

[1] Boltyanski, S., Martini, H., Soltan, P. S.: Excursions into combinatorial geometry, Springer-Verlag (1996).

[2] González, V.; Rodríguez, M.: Some geometrical results about the convex deficiency of a compact set, sent to the Int. Jour. of Pure and Appl. Math., Bulgaria (2006).

[3] Martini, H. y Wenzel, W.: A characterization of convex sets via visibility, Technische Universität Chemnitz, Fakultät für Mathematik, Preprint (2000).

[4] Martini, H; Wenzel, W.: Illumination and visibility problems in terms of closure operators, Beitr. Algebra Geom., **45**, N^{o}2, 607-614, (2004).

[5] Martini, H; Wenzel, W.: An analogue of the Krein-Milman theorem for starshaped sets, Beitr. Algebra Geom., **44**, N^{o}2, 441-449, (2003).

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