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Interior Controllability of a Broad Class of Second Order Equations in $L^2(\Omega)$

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Resumen

En este trabajo se presenta una prueba simple de la controlabilidad aproximada para el interior después de amplia clase de ecuaciones de segundo orden en el espacio de Hilbert $L^2(\Omega)$.

$$\begin{cases} \ddot{y} + Ay = 1_\omega u(t), \quad t \in (0, \tau], \\ y(0) = y_0, \quad \dot{y}(0) = y_1, \end{cases}$$

dónde Ω es un dominio en $\mathbb{R}^N (N \geq 1)$, $y_0, y_1 \in L^2(\Omega)$, ω es un subconjunto no vacío abierto de Ω, 1_ω denota la función característica de la serie ω, la u distribuido de control, pertenecen a $L^2(0, \tau; L^2(\Omega))$ y $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ es un operador lineal sin límites con la siguiente descomposición espectral: $Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}$, con los valores propios $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ de A tienen multiplicidad finita γ_j igual a la dimensión de la autoespacio correspondiente, y $\{\phi_{j,k}\}$ es un conjunto ortonormal completo de vectores propios (Eigen - funciones) de A. El operador $-A$ genera un semigrupo fuertemente continuo $\{T_A(t)\}_{t\geq 0}$ dada por $T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}$. Especialmente, probar lo siguiente declaración: Si para un conjunto abierto no vacío $\omega \subset \Omega$ las restricciones $\phi^\omega_{j,k} = \phi_{j,k}|_\omega$ de $\phi_{j,k}$ para ω son funciones linealmente independientes sobre ω, entonces para todo $\tau \geq 2$ el sistema es aproximadamente controlable en $[0, \tau]$.

Como una aplicación, se prueba la capacidad de controlar el interior de la ecuación de onda nD, el módelo de la ecuación de placa vibrante, de segundo orden ecuación de Ornstein Uhlenbeck, la de segundo orden la ecuación de Laguerre y la ecuación de segundo orden de Jacobi.

Palabra claves: controlabilidad aproximada interior, ecuación de segundo orden, fuertemente continua semigrupo.

Abstract

In this paper we present a simple proof of the interior approximate controllability for the following broad class of second order equations in the Hilbert space $L^2(\Omega)$

$$\begin{cases} \ddot{y} + Ay = 1_\omega u(t), & t \in (0, \tau], \\ y(0) = y_0, & \dot{y}(0) = y_1, \end{cases}$$

where Ω is a domain in $\mathbb{R}^N (N \geq 1)$, $y_0, y_1 \in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belong to $L^2(0, \tau; L^2(\Omega))$ and $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is an unbounded linear operator with the following spectral decomposition: $Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}$, with the eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors (eigenfunctions) of A. The operator $-A$ generates a strongly continuous semigroup $\{T_A(t)\}_{t \geq 0}$ given by $T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}$. Specifically, we prove the following statement: If for an open non-empty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^{\omega} = \phi_{j,k}|_\omega$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, then for all $\tau \geq 2$ the system is approximately controllable on $[0, \tau]$.

As an application, we prove the interior controllability of the nD wave equation, the model of vibrating plate equation, the second order Ornstein-Uhlenbeck equation, the second order Laguerre equation and the second order Jacobi equation.

key words. interior approximate controllability, second order equation, strongly continuous semigroup

AMS(MOS) subject classifications. [2000]93B05, 93C25.

1 Introduction

In this paper we present a simple proof of the interior approximate controllability of the following broad class of second order equations in the Hilbert space $L^2(\Omega)$

$$\begin{cases} \ddot{y} + Ay = 1_\omega u(t), & t \in (0, \tau], \\ y(0) = y_0, & \dot{y}(0) = y_1, \end{cases} \tag{1.1}$$

where Ω is a domain in $\mathbb{R}^N (N \geq 1)$, $y_0, y_1 \in L^2(\Omega)$, ω is an open nonempty subset of Ω, 1_ω denotes the characteristic function of the set ω, the distributed control u belong to $L^2(0, \tau; L^2(\Omega))$ and $A : D(A) \subset L^2(\Omega) \to L^2(\Omega)$ is an unbounded linear operator with the following spectral decomposition:

$$Az = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}. \tag{1.2}$$

The eigenvalues $0 < \lambda_1 < \lambda_2 < \cdots < \cdots \lambda_n \to \infty$ of A have finite multiplicity γ_j equal to the dimension of the corresponding eigenspace, and $\{\phi_{j,k}\}$ is a complete orthonormal set of eigenvectors(eigenfunctions) of A. The operator $-A$ generates a strongly continuous semigroup $\{T_A(t)\}_{t\geq 0}$ given by

$$T_A(t)z = \sum_{j=1}^{\infty} e^{-\lambda_j t} \sum_{k=1}^{\gamma_j} < z, \phi_{j,k} > \phi_{j,k}.$$

Specifically, we prove the following statement: If for an open non-empty set $\omega \subset \Omega$ the restrictions $\phi^{\omega}_{j,k} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, then for all $\tau \geq 2$ the system (1.1) is approximately controllable on $[0, \tau]$.

This result implies the interior controllability of the following well known examples of partial differential equations:

Example 1.1 *The nD Wave Equation*

$$\begin{cases} y_{tt} - \Delta y = 1_{\omega} u(t,x), \ \ in \quad (0,\tau] \times \Omega, \\ y = 0, \quad on \quad (0,\tau) \times \partial\Omega, \\ y(0,x) = y_0(x), \quad y_t(0,x) = y_1(x), \quad x \in \Omega, \end{cases} \tag{1.3}$$

where ω is an open nonempty subset of Ω, 1_{ω} denotes the characteristic function of the set ω and the distributed control $u \in L^2(0,\tau; L^2(\Omega))$.

Example 1.2 *The Model of Vibrating Plate Equation*

$$\begin{cases} w_{tt} + \Delta^2 w = 1_{\omega} u(t,x), \ \ in \quad (0,\tau] \times \Omega, \\ w = \Delta w = 0, \quad on \quad (0,\tau) \times \partial\Omega, \\ w(0,x) = \phi_0(x), \quad w_t(0,x) = \psi_0(x), \quad x \in \Omega. \end{cases} \tag{1.4}$$

where Ω is a sufficiently smooth bounded domain in $I\!R^2$, ω is an open nonempty subset of Ω, $u \in L^2([0,\tau]; L^2(\Omega))$, $\phi_0, \psi_0 \in L^2(\Omega)$.

Example 1.3 *(see [2] and [3])*

1. *The interior controllability of the Second Ornstein-Uhlenbeck Equation*

$$z_{tt} - \sum_{i=1}^{d} \left[x_i \frac{\partial^2 z}{\partial x_i^2} - x_i \frac{\partial z}{\partial x_i} \right] = 1_{\omega} u(t,x) \quad t \in [0,\tau], \quad x \in \mathbb{R}^d, \tag{1.5}$$

where $u \in L^2(0,\tau; L^2(\mathbb{R}^d, \mu))$, $\mu(x) = \frac{1}{\pi^{d/2}} \prod_{i=1}^{d} e^{-|x_i|^2}\ dx$ is the Gaussian measure in $\mathbb{R}^d$ and ω is an open nonempty subset of $\mathbb{R}^d$.

2. *The interior controllability of the Second Laguerre Equation*

$$z_{tt} - \sum_{i=1}^{d}\left[x_i\frac{\partial^2 z}{\partial x_i^2} + (\alpha_i + 1 - x_i)\frac{\partial z}{\partial x_i}\right] = 1_\omega u(t,x), \quad t \in [0,\tau], \quad x \in \mathbb{R}_+^d, \tag{1.6}$$

where $u \in L^2(0,\tau; L^2(\mathbb{R}_+^d, \mu_\alpha))$, $\mu_\alpha(x) = \prod_{i=1}^{d} \frac{x_i^{\alpha_i} e^{-x_i}}{\Gamma(\alpha_i+1)}\, dx$ *is the Gamma measure in* $\mathbb{R}_+^d$ *and* ω *is an open nonempty subset of* $\mathbb{R}_+^d$.

3. *The interior controllability of the Second Order Jacobi Equation*

$$z_{tt} - \sum_{i=1}^{d}\left[(1-x_i^2)\frac{\partial^2 z}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2)\, x_i)\frac{\partial z}{\partial x_i}\right] = 1_\omega u(t,x), \tag{1.7}$$

where $t \in [0,\tau]$, $x \in [-1,1]^d$, $u \in L^2(0,\tau; L^2([-1,1]^d, \mu_{\alpha,\beta}))$,
$\mu_{\alpha,\beta}(x) = \prod_{i=1}^{d}(1-x_i)^{\alpha_i}(1+x_i)^{\beta_i}\, dx$ *is the Jacobi measure in* $[-1,1]^d$ *and* ω *is an open nonempty subset of* $[-1,1]^d$.

At this point we must mention the works done by others authors to complete the exposure of this paper. To complement this requirement, we present some observations to the works of others authors, showing the difference between our results and those of them.
The approximate controllability is very well known fascinate and important subject in systems theory; there are some works done by [9], [11], [12], [13] and [14].
Particularly, in [13], the author proves the approximate controllability of the wave equation and the heat equation in two different ways. In the first one, he uses variational approach and observability condition, which is not different than the condition from Theorem 4.1.7b in [5]. Usually, people working in control for PDEs call this condition "unique continuation principle". To verify this condition, the author use the Hanh-Banach theorem, integrating by parts, the adjoint equation, the Carleman estimates and the following result due to **Holmgren Uniqueness Theorem:** Let P be a differential operator with constant coefficient in $\mathbb{R}^n$. Let z be a solution of the equation $Pz = 0$ in Q_1 where Q_1 is an open set of $\mathbb{R}^n$. Suppose that $z = 0$ in Q_2, where Q_2 is an open nonempty subset of Q_1. Then $z = 0$ on Q_3, where Q_3 is the open subset of Q_1 which contains Q_2 and such that any characteristic hyperplane of the operator P which intersects Q_3 also intersects Q_1. Here we find some differences; it is good to mention that the Carleman estimates depend on the Laplacian operator Δ, so it may not applied to those equations that do not involve the Laplacian operator, like the second order Ornstein-Uhlenbeck equation, the

second order Laguerre equation and the second order Jacobi.
The second method is based in Fourier techniques and the observability of the $1D$ wave equation. The author develops in detail some technique using Fourier analysis and more particularly on the so called Ingham type inequalities allowing to obtain several observability results for the linear $1D$ wave equations. Also, this approach can be found in [1]. They prove, for the $1D$ wave equation, the approximate controllability for $\tau > 2$.
In contrast, here we prove the controllability of the nD wave equation for $\tau \geq 2$.

The technique given here in this work is not trivial, but it is so simple that, those young mathematicians who live in remote and inhospitable places, far from major research centers in the world, can also understand and enjoy the interior controllability with a minor effort. That is one of the novelties of this work.

2 Main Results

In this section we shall prove the main result of this work; to this end, we choose denote by $X = U = L^2(\Omega)$ and the linear unbounded operator $A : D(A) \subset X \to X$ can be written as follows:

a) For all $x \in D(A)$ we have

$$Ax = \sum_{j=1}^{\infty} \lambda_j E_j \xi, \tag{2.1}$$

where

$$E_j x = \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k}. \tag{2.2}$$

So, $\{E_j\}$ is a family of complete orthogonal projections in X and $x = \sum_{j=1}^{\infty} E_j x, \ x \in X$.

b) The semigroup $\{T_A(t)\}$ generated by $-A$ is given by

$$T_A(t)x = \sum_{j=1}^{\infty} e^{-\lambda_j t} E_j x. \tag{2.3}$$

c) The fractional powered spaces X^r are given by:

$$X^r = D(A^r) = \{x \in X : \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 < \infty\}, \quad r \geq 0,$$

with the norm

$$\|x\|_r = \|A^r x\| = \left\{ \sum_{n=1}^{\infty} \lambda_n^{2r} \|E_n x\|^2 \right\}^{1/2}, \; x \in X^r, \quad \text{and}$$

$$A^r x = \sum_{n=1}^{\infty} \lambda_n^r E_n x. \tag{2.4}$$

Also, for $r \geq 0$ we define $Z_r = X^r \times X$, which is a Hilbert space endowed with the norm given by

$$\left\| \begin{bmatrix} y \\ v \end{bmatrix} \right\|_{Z_r}^2 = \|y\|_r^2 + \|v\|^2.$$

Hence, with the change of variable $y' = v$, the system (1.1) can be written as a first order systems of ordinary differential equations in the Hilbert space $Z_{1/2} = X^{1/2} \times X$ as follows:

$$z' = \mathcal{A}z + B_\omega u, \quad z(0) = z_0, \quad z \in Z_{1/2}, \quad t \in (0, \tau], \tag{2.5}$$

where

$$z = \begin{bmatrix} y \\ v \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1_\omega \end{bmatrix} \text{ and } \quad \mathcal{A} = \begin{bmatrix} 0 & I_X \\ -A & 0 \end{bmatrix} \tag{2.6}$$

is an unbounded linear operator with domain $D(\mathcal{A}) = D(A) \times D(A^{1/2})$.

The proof of the following theorem follows in the same way as Theorem 3.1 from [6], by putting $c = 0$ and $d = 1$.

Theorem 2.1 *The operator $\mathcal{A}$ given by (2.6), is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t \in \mathbb{R}}$ given by*

$$T(t)z = \sum_{j=1}^{\infty} e^{A_j t} P_j z, z \in Z, t \geq 0 \tag{2.7}$$

where $\{P_j\}_{j \geq 1}$ is a complete family of orthogonal projections in the Hilbert space $Z_{1/2}$ given by

$$P_j = diag[E_j, E_j], j \geq 1 \tag{2.8}$$

and

$$A_j = R_j P_j, \quad R_j = \begin{bmatrix} 0 & 1 \\ -\lambda_j & 0 \end{bmatrix}. \tag{2.9}$$

Also,

$$A_j^* = R_j^* P_j, \quad R_j^* = \begin{bmatrix} 0 & -1 \\ \lambda_j & 0 \end{bmatrix}.$$

Moreover, $e^{A_j s} = e^{R_j s} P_j$ and the eigenvalues of R_j are: $\sqrt{\lambda_j} i$ and $-\sqrt{\lambda_j} i$.

Now, before proving the main Theorem, we shall give the definition of approximate controllability for this system. To this end, for all $z_0 \in Z_{1/2}$ and $u \in L^2(0, \tau; U)$ the initial value problem

$$\begin{cases} z' = \mathcal{A} z + B_\omega u(t), \quad z \in Z, \quad t \in (0, \tau], \\ z(0) = z_0, \end{cases} \tag{2.10}$$

where the control function u belong to $L^2(0, \tau; U)$, admits only one mild solution given by

$$z(t) = T(t) z_0 + \int_0^t T(t-s) B_\omega u(s) ds, \;\; t \in [0, \tau]. \tag{2.11}$$

Definition 2.1 *(***Approximate Controllability***) The system (2.10) is said to be approximately controllable on $[0, \tau]$ if for every z_0, $z_1 \in Z_{1/2}$, $\varepsilon > 0$ there exists $u \in L^2(0, \tau; U)$ such that the solution $z(t)$ of (2.11) corresponding to u verifies:*

$$\|z(\tau) - z_1\| < \varepsilon.$$

The next theorem can be found in [4] and [5] for the following general evolution

$$z' = \mathcal{A} z + B u(t), \quad z \in Z, \quad u \in U, \quad t \in [0, \tau], \tag{2.12}$$

where Z, U are Hilbert spaces, $\mathcal{A} : D(\mathcal{A}) \subset Z \longrightarrow Z$ is the infinitesimal generator of strongly continuous semigroup $\{T(t)\}_{t \geq 0}$ in $Z, B \in L(U, Z)$ and the control function u belong to $L^2(0, \tau; U)$.

Theorem 2.2 *The system (2.12) is approximately controllable on $[0, \tau]$ if, and only if,*

$$B^* T^*(t) z = 0, \quad \forall t \in [0, \tau] \Rightarrow z = 0. \tag{2.13}$$

Now, we are ready to formulate and prove the main theorem of this work.

Theorem 2.3 *(***Main Theorem** *)If for an open non-empty set $\omega \subset \Omega$ the restrictions $\phi_{j,k}^{\omega} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, then for all $\tau \geq 2$ the system (1.1) is approximately controllable on $[0, \tau]$.*

Proof We shall apply Theorem 2.2 to prove the controllability of system (2.10) or (2.5). To this end, we observe that the adjoint of operator B_ω is by

$$B_\omega^* = \begin{bmatrix} 0 & 1_\omega \end{bmatrix}$$

and

$$T^*(t)z = \sum_{j=1}^{\infty} e^{A_j^* t} P_j z, \quad z \in Z, \quad t \geq 0.$$

Therefore,

$$B_\omega^* T^*(t)z = \sum_{j=1}^{\infty} B_\omega^* e^{R_j^* t} P_j z.$$

On the other hand, we have that

$$e^{R_j^* t} = \begin{bmatrix} \cos j\pi t & -\frac{1}{j\pi}\sin j\pi t \\ j\pi \sin j\pi t & \cos j\pi t \end{bmatrix}.$$

Suppose for all $z \in Z_{1/2}$ that:

$$B_\omega^* T^*(t)z = \sum_{j=1}^{\infty} \{ j\pi \sin(j\pi t)(1_\omega E_j z_1) + \cos(j\pi t)(1_\omega E_j z_2)\} = 0, \quad \forall t \in [0, \tau].$$

Then, if we make the change of variable $s = \pi t$, we obtain that

$$\sum_{j=1}^{\infty} \{ j\pi \sin(js)(1_\omega E_j z_1) + \cos(js)(1_\omega E_j z_2)\} = 0, \quad \forall s \in [0, \pi\tau].$$

Since $\tau \geq 2$ we get that

$$\sum_{j=1}^{\infty} \{ j\pi \sin(js)(1_\omega E_j z_1) + \cos(js)(1_\omega E_j z_2)\} = 0, \quad \forall s \in [0, 2\pi].$$

On the order hand, it is well known that $\{1, \cos(js), \sin(js) : j = 1, 2, 3, \dots\}$ is an orthogonal base of $L^2[0, 2\pi]$, which implies that

$$(1_\omega E_j z_1)(x) = 0 \text{ and } (1_\omega E_j z_2)(x) = 0, \quad \forall x \in \Omega, \quad j = 1, 2, 3, \dots.$$

i.e.,

$$(E_j z_1)(x) = 0 \text{ and } (E_j z_2)(x) = 0, \quad \forall x \in \omega, \quad j = 1, 2, 3, \dots.$$

i.e.,

$$\sum_{k=1}^{\gamma_j} < z_1, \phi_{j,k} > \phi_{j,k}(x) = 0 \text{ and } \sum_{k=1}^{\gamma_j} < z_2, \phi_{j,k} > \phi_{j,k}(x) = 0, \quad \forall x \in \omega, \quad j = 1, 2, 3, \dots.$$

Since the restrictions $\phi^{\omega}_{j,k} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, we get that

$$< z_1, \phi_{j,k} >= 0 \quad \text{and} \quad < z_2, \phi_{j,k} >= 0, \quad j = 1, 2, 3, \dots; k = 1, 2, \dots, \gamma_j.$$

Therefore,

$$P_j(z) = \begin{bmatrix} E_j(z_1) \\ E_j(z_2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence, $z = \sum_{j=1}^{\infty} P_j z = 0$, and the proof of the Theorem is completed.

□

The following basic theorem will be used to prove an important consequence of the foregoing theorem.

Theorem 2.4 *(see Theorem 1.23 from [10], pg. 20) Suppose $\Omega \subset I\!R^n$ is open, non-empty and connected set, and f is real analytic function in Ω with $f = 0$ on a non-empty open subset ω of Ω. Then, $f = 0$ in Ω.*

Corollary 2.1 *If $\phi_{j,k}$ are analytic functions on Ω, then for all open non-empty set $\omega \subset \Omega$ and all $\tau \geq 2$ the system (1.1) is approximately controllable on $[0, \tau]$.*

Proof . It is enough to prove that, for all open non-empty set $\omega \subset \Omega$ the restrictions $\phi^{\omega}_{j,k} = \phi_{j,k}|_{\omega}$ of $\phi_{j,k}$ to ω are linearly independent functions on ω, which follows directly from Theorem 2.4. □

3 Applications

For the applications, we shall use corollary 2.1 and the following fact.

Theorem 3.1 *The eigenfunctions of the operator $-\Delta$ with Dirichlet boundary conditions on Ω are real analytic functions in Ω.*

In this section we shall prove the approximate controllability of the equations (1.3), (1.4),(1.5), (1.6) and (1.7); specifically, we will prove the following theorems.

Theorem 3.2 *For all open non-empty set $\omega \subset \Omega$ and all $\tau \geq 2$ the systems (1.3) and (1.4) are approximately controllable on $[0, \tau]$.*

Proof Let $X = L^2(\Omega)$ and consider the linear unbounded operator $-\Delta : D(-\Delta) \subset X \to X$ with $D(-\Delta) = H_0^1(\Omega) \cap H^2(\Omega)$. The operator $-\Delta$ has the following very well known properties: The spectrum of $-\Delta$ consists of only eigenvalues

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j \to \infty,$$

each one with multiplicity γ_j equal to the dimension of the corresponding eigenspace.

a) There exists a complete orthonormal set $\{\phi_{j,k}\}$ of eigenvectors (eigenfunctions) of $-\Delta$.

b) For all $x \in D(-\Delta)$ we have

$$-\Delta x = \sum_{j=1}^{\infty} \lambda_j \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j E_j \xi. \tag{3.1}$$

The result follows from Theorem 2.4 and Corollary 2.1 if we put $A = -\Delta$ for The nD Wave Equation (1.3) and

$$A = (\Delta)^2 x = \sum_{j=1}^{\infty} \lambda_j^2 \sum_{k=1}^{\gamma_j} < \xi, \phi_{j,k} > \phi_{j,k} = \sum_{j=1}^{\infty} \lambda_j^2 E_j \xi,$$

for The Model of Vibrating Plate Equation (1.4).

□

Theorem 3.3 *For all open non-empty set $\omega \subset \Omega$ and all $\tau \geq 2$ the equations (1.5), (1.6) and (1.7) are approximately controllable on $[0, \tau]$.*

Proof It is enough to prove that the operators

i) Ornstein-Uhlenbeck operator: $-A = \frac{1}{2}\nabla - \langle x, \Delta_x \rangle$, defined on $\Omega = \mathcal{R}^d$, with $\Delta_x = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$ in the space $X = L^2(\mathcal{R}^d, \mu)$.

ii) Laguerre operator: $A = -\sum_{i=1}^d \left[x_i \frac{\partial^2 z}{\partial x_i^2} + (\alpha_i + 1 - x_i) \frac{\partial z}{\partial x_i} \right]$, defined on $\Omega = (0, \infty)^d$, with $\alpha_i > -1$, $i = 1, \dots, d$ in the space $X = L^2(\mathcal{R}_+^d, \mu_\alpha)$

ii) Jacobi operator: $A = -\sum_{i=1}^d \left[(1 - x_i^2) \frac{\partial^2 z}{\partial x_i^2} + (\beta_i - \alpha_i - (\alpha_i + \beta_i + 2)\, x_i) \frac{\partial z}{\partial x_i} \right]$, $\Omega = (-1, 1)^d$, with $\alpha_i, \beta_i > -1$, $i = 1, \dots, d$ in the space $X = L^2([-1, 1]^d, \mu_{\alpha,\beta})$, can be represented in the form of (1.2), which was done in [2] and [3], where they also prove that the eigenfunctions of these operators are polynomial functions in multiple variables. Hence, the eigenfunctions are trivially analytic. So, applying Corollary 2.1 we get the result.

□

4 Conclusion

The novelty of this result is based in the fact that, it is general, rigorous, applicable and easily comprehensible by those young mathematician who are located in places away from majors research center.

References

[1] S. A. AVDONIN AND S.A. IVANOV, "Families of exponentials. The method of moments in controllability problems for distributed parameter systems",. Cambridge Univ. Press, 1995.

[2] D.BARCENAS, H. LEIVA AND W. URBINA, *Controllability of the Ornstein-Uhlenbeck Equation.* IMA J. Math. Control Inform. **23** no. 1, (2006), 1–9.

[3] D. BARCENAS, H. LEIVA, Y. QUINTANA AND W. URBINA, *Controllability of Laguerre and Jacobi Equations.* International Journal of Control, Vol. 80, No. 8, August 2007, 1307-1315.

[4] R.F. CURTAIN and A.J. PRITCHARD, “Infinite Dimensional Linear Systems”, Lecture Notes in Control and Information Sciences, Vol. 8. Springer Verlag, Berlin (1978).

[5] R.F. CURTAIN and H.J. ZWART, “An Introduction to Infinite Dimensional Linear Systems Theory”, Texts in Applied Mathematics, Vol. 21. Springer Verlag, Berlin (1995).

[6] H. LEIVA, “Exact controllability of the suspension bridge model proposed by Lazer and McKenna”, J. Math. Anal. Appl. 309 (2005), 404-419.

[7] H. LEIVA, “A Lemma on C_0-Semigroups and Applications ” Quaestiones Mathematicae **26** (2003), 247–265.

[8] S. AXLER, P. BOURDON AND W. RAMEY, *Harmonic Fucntion Theory.* Graduate Texts in Math., **137**. Springer Verlag, New york (1992).

[9] D. L. RUSSELL, *Controllability and Stabilizability Theory for Linear Partial Differential Equations: Recent Progress and Open Questions.* SIAM Rev. **20** No. 4 (1978), 636–739.

[10] S. AXLER, P. BOURDON AND W. RAMEY, *Harmonic Fucntion Theory.* Graduate Texts in Math., **137**. Springer Verlag, New york (1992).

[11] R. TRIGGIANI, *Extensions of Rank Conditions for Controllability and Observability to Banach Spaces and Unbounded Operators.* SIAM J. Control Optimization **14** No. 2 (1976), 313–338.

[12] XU ZHANG, *A Remark on Null Exact Controllability of the Heat Equation.* SIAM J. CONTROL OPTIM. Vol. 40, No. 1(2001), pp. 39-53.

[13] E. ZUAZUA, *Controllability of a System of Linear Thermoelasticity*, J. Math. Pures Appl., **74**, (1995), 291-315.

[14] E. ZUAZUA, *Control of Partial Differential Equations and its Semi-Discrete Approximation.* Discrete and Continuous Dynamical Systems, vol. **8**, No. 2. April (2002), 469-513.

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