Revista Notas de Matemática
Vol.5(2), No. 278, 2009, pp.20-28
http://www.saber.ula.ve/notasdematematica/
Comisión de Publicaciones
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

Almost preenvelopes of commutative rings

Carlos Parra, Rafael Parra y Juan Rada

Abstract

We study almost $\mathcal{F}$-preenvelopes in the category of rings, for a significative class $\mathcal{F}$ of commutative rings. We completely identify those rings which have an almost $\mathcal{F}$-preenvelope when $\mathcal{F}$ is the class of fields, semisimple rings, integer domains and local rings. We show that rings with Krull dimension zero have (almost) $\mathcal{V}$-preenvelopes, where $\mathcal{V}$ is the class of von Neumann regular rings.

key words. Almost preenvelopes, semiprimitive rings, local rings, von Neumann regular rings.

AMS(MOS) subject classifications. 13C05; 16S10; 18B99.

1 Introduction

Enochs [4] introduced the concept of envelope and cover with respect to an arbitrary class of objects in a given category. More precisely, let $\mathcal{C}$ be a category and $\mathcal{F}$ a class of objects in $\mathcal{C}$. A $\mathcal{F}$-preenvelope of an object M in $\mathcal{C}$ is a morphism $f : M \longrightarrow F$, where $F \in \mathcal{F}$, such that for every morphism $g : M \longrightarrow F'$, where $F' \in \mathcal{F}$, there exists a morphism $h : F \longrightarrow F'$ such that $hf = g$. Furthermore, if each morphism $k : F \longrightarrow F$ such that $kf = f$ is an isomorphism, then f is called a $\mathcal{F}$-envelope of M. The concepts of $\mathcal{F}$-precover and $\mathcal{F}$-cover are defined dually. This definition includes as particular cases the classical concepts of injective envelope and projective cover of a module, setting $\mathcal{F}$ the class of injective modules or projective modules, respectively, in the category $\mathcal{C}$ of modules.

Recently, Parra and Saorín [6] initiated the study of $\mathcal{F}$-(pre) envelopes in the category $\mathcal{C}$ of rings, when $\mathcal{F}$ is a significative class of commutative rings. They identify those rings for which there exists a $\mathcal{F}$-(pre) envelope when $\mathcal{F}$ is the class of fields, semisimple commutative rings and integer domains. Also, they study the more complicated case of $\mathcal{F}$-(pre) envelopes when $\mathcal{F}$ is the class of commutative Noetherian rings. Motivated by the articles [3] and [5]), we study the more general situation of *almost* $\mathcal{F}$-(pre) envelopes in the category $\mathcal{C}$ of rings.

We first show that every $\mathcal{F}$-(pre) envelope is an *almost* $\mathcal{F}$-(pre) envelope of a ring, but the converse does not hold (Remark 2.4 and Example 2.5). However, when $\mathcal{F}$ is the class of integer domains or $\mathcal{F}$ is contained in the class $SRings$ of semiprimitive rings, the two concepts are equivalent (Proposition 3.1 and Theorem 4.3). A particular case of semiprimitive rings is the class $\mathcal{V}$ of von Neumann regular rings. Although we were not able to identify the rings that have a $\mathcal{V}$-preenvelope, we show in Corollary 3.5 that if K-dim$(R) = 0$ then R has an almost $\mathcal{V}$-preenvelope. Finally, we identify in Theorem 5.1 those rings which have an (almost) $\mathcal{L}$-preenvelope, where $\mathcal{L}$ is the class of local rings.

2 Almost preenvelopes of commutative rings

Let R be a (possibly non-commutative) ring and $\mathcal{C}$ a class of rings. Motivated by [5] we extend the concept of $\mathcal{C}$-preenvelope of a ring R in this section. We begin by introducing the concept of superfluous ideal of a ring. We assume that all ideals are two-sided ideals.

Definition 2.1 *Let R be a ring and I an ideal of R. I is a superfluous ideal of R, denoted by $I \ll R$, if for every ideal K of R*

$$I + K = R \Longrightarrow K = R$$

A ring epimorphism $f : R \longrightarrow S$ is superfluous if $Ker\,(f) \ll R$.

Example 2.2 *Let $J\,(R)$ denote the Jacobson radical of R. If K is an ideal of R and $J\,(R)+K = R$ then $1 = a + b$, where $a \in J\,(R)$ and $b \in K$. Hence $b = 1 - a \in K$ is invertible which implies $K = R$. In other words $J\,(R) \ll R$. Hence for every ideal I of R*

$$I \subseteq J\,(R) \Rightarrow I \ll R$$

When R is commutative the converse also holds. In fact, let $a \in I$ and $r \in R$. Then $Ra + R\,(1 - ar) = R$. Since $Ra \subseteq I$, $I + R\,(1 - ar) = R$ which implies $R\,(1 - ar) = R$. Hence $1 - ar$ is invertible for every $r \in R$ and so $a \in J\,(R)$.

Definition 2.3 *Let $\mathcal{H}$ be a class of rings and R a ring. A homomorphism $\varphi : R \longrightarrow H$, where $H \in \mathcal{H}$, is an almost $\mathcal{H}$-preenvelope of R if for every homomorphism $\psi : R \longrightarrow H'$, where $H' \in \mathcal{H}$, there exists a superfluous epímorphism $\theta : H' \longrightarrow H''$, where $H'' \in \mathcal{H}$, and a homomorphism*

$\alpha : H \longrightarrow H''$ such that $\theta\psi = \alpha\varphi$.

$$\begin{array}{ccc} R & \xrightarrow{\varphi} & H \\ {}^{\psi}\downarrow & & \downarrow^{\alpha} \\ H' & \xrightarrow{\theta} & H'' \end{array}$$

φ is called an almost $\mathcal{H}$-envelope of R if each endomorphism f of H such that $f\varphi = \varphi$ is an automorphism.

Remark 2.4 *If $\varphi : R \longrightarrow H$ is a $\mathcal{H}$-preenvelope of R then φ is an almost $\mathcal{H}$-preenvelope of R. In fact, if $\psi : R \longrightarrow H'$ is a homomorphism, where $H' \in \mathcal{H}$, then there exists $\alpha : H \longrightarrow H'$ such that $\psi = \alpha\varphi$. Then $1\psi = \alpha\varphi$ where $1 : H' \longrightarrow H'$ is the identity which is clearly a superfluous epimorphism.*

Example 2.5 *Let $\mathcal{N}$ be the class of Noetherian commutative rings. We will show an example of an almost $\mathcal{N}$-preenvelope of a ring R which is not a $\mathcal{N}$-preenvelope of R. By [6, Example 5.3 (2)], there exists a non-Noetherian ring R and an ideal I of R such that $I \subseteq Nil\,(R)$, R/I is a Noetherian ring and $p : R \longrightarrow R/I$ is not a $\mathcal{N}$-preenvelope of R. However, $p : R \longrightarrow R/I$ is an almost $\mathcal{N}$-preenvelope of R. To see this, let $f : R \longrightarrow N$ be a homomorphism, where N is a Noetherian ring. Then clearly $N/f\,(I)\,N$ is a Noetherian ring, and since $I \subseteq Nil\,(R)$ then*

$$f\,(I)\,N \subseteq Nil\,(N)\,N \subseteq Nil\,(N) \subseteq J\,(N)$$

which implies that the projection $\pi : N \longrightarrow N/f\,(I)\,N$ is a superfluous epimorphism. Finally we have the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{p} & R/I \\ {}^{f}\downarrow & & {}^{\widehat{f}}\downarrow \\ N & \xrightarrow{\pi} & N/f\,(I)\,N \end{array}$$

where $\widehat{f} : N \longrightarrow N/f\,(I)\,N$ is defined by $\widehat{f}\,(x + I) = f\,(x) + f\,(I)\,N$, for every $x \in R$.

We are specially interested in studying almost $\mathcal{H}$-preenvelopes of rings when $\mathcal{H} \subseteq \mathcal{C}Rings$, the category of commutative rings. Our next result shows that we can restrict to almost $\mathcal{H}$-preenvelopes of commutative rings. By R_{com} we denote the quotient of the ring R by the ideal generated by the differences $ab - ba$, where $a, b \in R$. It is easy to see that the canonical epimorphism $\pi : R \longrightarrow R_{com}$ is a $\mathcal{C}Rings$-preenvelope of R.

Proposition 2.6 *Let R be a ring and $\mathcal{H}$ a class of rings such that $\mathcal{H} \subseteq \mathcal{C}\mathrm{Rings}$. R has an almost $\mathcal{H}$-preenvelope if and only if R_{com} has an almost $\mathcal{H}$-preenvelope.*

Proof. Let $\varphi : R \longrightarrow H$ be an almost $\mathcal{H}$-preenvelope of R. Since H is a commutative ring there exists a homomorphism $\delta : R_{com} \longrightarrow H$ such that tal que $\delta\pi = \varphi$, where $\pi : R \longrightarrow R_{com}$ is the canoncial epimorphism. We will show that $\delta : R_{com} \longrightarrow H$ is an almost $\mathcal{H}$-preenvelope of R_{com}. In fact, let $\psi : R_{com} \longrightarrow H'$ a homomorphism, where $H' \in \mathcal{H}$. Since φ is an almost $\mathcal{H}$-preenvelope of R there exists a superfluous epimorphism $\theta : H' \longrightarrow H''$, where $H'' \in \mathcal{H}$, and a homomorphism $\alpha : H \longrightarrow H''$ such that $\alpha\varphi = \theta\psi\pi$. Hence $\alpha\delta\pi = \alpha\varphi = \theta\psi\pi$ and since π is an epimorphism, $\alpha\delta = \theta\psi$ and we are done.

Conversely, assume that $\psi : R_{com} \longrightarrow H'$ is an almost $\mathcal{H}$-preenvelope of R_{com}. We will see that the composition $R \xrightarrow{\pi} R_{com} \xrightarrow{\psi} H'$ is an almost $\mathcal{H}$-preenvelope of R. In fact, let $\varphi : R \longrightarrow H$ a homomorphism where $H \in \mathcal{H}$. Since $R \xrightarrow{\pi} R_{com}$ is a $\mathcal{C}Rings$-preenvelope of R, there exist $\delta : R_{com} \longrightarrow H$ such that $\varphi = \delta\pi$. On the oher hand, since ψ is an almost $\mathcal{H}$-preenvelope of R_{com}, there exists a superfluous epimorphism $\theta : H \longrightarrow H''$, where $H'' \in \mathcal{H}$, and a homomorphism $\alpha : H' \longrightarrow H''$ such that $\alpha\psi = \theta\delta$. Consequently,

$$\alpha\psi\pi = \theta\delta\pi = \theta\varphi$$

which implies that $\psi\pi$ is an almost $\mathcal{H}$-preenvelope of R. ∎

3 Almost preenvelopes in semiprimitive rings

We denote by *SRings* the class of semiprimitive rings. In other words, $H \in SRings$ if and only if $J(H) = 0$.

Proposition 3.1 *Let R be a ring and $\mathcal{H}$ a class of rings such that $\mathcal{H} \subseteq \mathrm{SRings}$. R has an almost $\mathcal{H}$-preenvelope if, and only if, R has a $\mathcal{H}$-preenvelope.*

Proof. Assume that $\varphi : R \longrightarrow H$ is an almost $\mathcal{H}$-preenvelope of R and $\psi : R \longrightarrow H'$ a homomorphism where $H' \in \mathcal{H}$. Then there exists a superfluous epimorphism $\theta : H' \longrightarrow H''$, where $H'' \in \mathcal{H}$, and a homomorphism $\alpha : H \longrightarrow H''$ such that $\alpha\varphi = \theta\psi$. Since $Ker\theta \ll H'$, it follows from Example 2.2 that $Ker\theta \subseteq J(H') = 0$ and so θ is an isomorphism. Thus $\theta^{-1}\alpha\varphi = \psi$ which implies that $\varphi : R \longrightarrow H$ is a $\mathcal{H}$-preenvelope of R. The converse is Remark 2.4. ∎

We next apply Proposition 3.1 to several particular classes of semiprimitive rings. We denote the Krull dimension of the ring R by K-dim(R).

Corollary 3.2 *Let R be a ring, $\mathcal{F}$ the class of fields, $\mathcal{S}$ the class of semisimple commutative rings and $\mathcal{V}$ the class of regular von Neumann commutative rings.*

1. *R has an almost $\mathcal{F}$-(pre) envelope if, and only if, R has a $\mathcal{F}$-(pre) envelope if, and only if, R is local and K-dim$(R) = 0$. In this case, the projection $\pi : R \longrightarrow R/\mathfrak{m}$ is the $\mathcal{F}$-envelope where $\mathfrak{m}$ is the maximal ideal of R.*
2. *R has an almost $\mathcal{S}$-(pre) envelope if, and only if, R has a $\mathcal{S}$-(pre) envelope if, and only if, $Spec\,(R)$ is finite. In this case, the canonical map $R \longrightarrow \prod_{\mathfrak{p}\in Spec(R)} k\,(\mathfrak{p})$ is the $\mathcal{S}$-envelope, where $k\,(\mathfrak{p})$ is the quotient field of $R/\mathfrak{p}$ for every $\mathfrak{p} \in Spec\,(R)$.*
3. *R has an almost $\mathcal{V}$-(pre) envelope if, and only if, R has a $\mathcal{V}$-(pre)envelope.*

Proof. This is an immediate consequence of Proposition 3.1 and [6, Theorem 2.2]. ■

Although we do not know which rings exactly have $\mathcal{V}$-preenvelopes, in our next results we show that every ring R such that K-dim$(R) = 0$ has a $\mathcal{V}$-preenvelope. Recall that a ring N is von Neumann regular if, and only if, K-dim$(N) = 0$ and $Nil\,(N) = 0$.

Lemma 3.3 *Let R be a commutative ring. If $f : R \longrightarrow N$ is a ring homomorphism, where $N \in \mathcal{V}$, then $Nil\,(R) \subseteq Ker\,(f)$. Furthermore, if f is a $\mathcal{V}$-preenvelope of R then $Nil\,(R) = Ker\,(f)$.*

Proof. If $x \in Nil\,(R)$ then there exists a positive integer n such that $x^n = 0$. Hence,

$$0 = f\,(0) = f\,(x^n) = [f\,(x)]^n$$

and so $f\,(x)$ is a nilpotent element of N. Since $N \in \mathcal{V}$, then $Nil\,(N) = 0$ and so $x \in Ker\,(f)$.

Now assume that f is a $\mathcal{V}$-preenvelope of R and let $\mathfrak{p}$ be a prime ideal of R. Consider the inclusion $i : R/\mathfrak{p} \longrightarrow k\,(\mathfrak{p})$ and the canonical epimorphism $\pi : R \longrightarrow R/\mathfrak{p}$. Since $k\,(\mathfrak{p}) \in \mathcal{V}$ and f is a $\mathcal{V}$-preenvelope of R, there exists a homomorphism $\varphi : N \longrightarrow k\,(\mathfrak{p})$ such that $\varphi f = i\pi$.Consequently,

$$0 = \varphi f\,(Kerf) = i\pi\,(Kerf)$$

Since i is a monomorphism, $\pi\,(Kerf) = 0$ which implies $Kerf \subseteq Ker\pi = \mathfrak{p}$. Thus, $Kerf \subseteq \bigcap_{\mathfrak{p}\in Spec(R)} \mathfrak{p} = Nil\,(R)$. ■

We next show that the existence of $\mathcal{V}$-preenvelopes for a ring reduces to the existence of $\mathcal{V}$-preenvelopes for reduced rings. Recall that a ring R is reduced if $Nil\,(R) = 0$.

Theorem 3.4 *Let R be a commutative ring. R has a $\mathcal{V}$-preenvelope if, and only if, $R/Nil\,(R)$ has a $\mathcal{V}$-preenvelope.*

Proof. Assume that $f : R \longrightarrow N$ is a $\mathcal{V}$-preenvelope of R. By Lemma 3.3, $Nil\,(R) = Ker\,(f)$. Hence $g : R/Nil\,(R) \longrightarrow N$ given by $g\,(\overline{x}) = f\,(x)$, where $x \in R$, is a well defined monomorphism. We will show that g is a $\mathcal{V}$-preenvelope of $R/Nil\,(R)$. Assume that $h : R/Nil\,(R) \longrightarrow N'$ is a homomorphism, where $N' \in \mathcal{V}$. Since f is a $\mathcal{V}$-preenvelope of R, there exists a homomorphism $\varphi : N \longrightarrow N'$ such that $\varphi f = h\pi$, where $\pi : R \longrightarrow R/Nil\,(R)$ is the canonical epimorphism. Further, it is clear that $g\pi = f$. Hence

$$h\pi = \varphi f = \varphi g\pi$$

and since π is a epimorphism, $h = \varphi g$.

Conversely, suppose that $f : R/Nil\,(R) \longrightarrow N$ is a $\mathcal{V}$-preenvelope of $R/Nil\,(R)$. We will show that the composition

$$R \xrightarrow{\pi} R/Nil\,(R) \xrightarrow{f} N$$

is a $\mathcal{V}$-preenvelope of R. In fact, let $g : R \longrightarrow N'$ be a homomorphism, where $N' \in \mathcal{V}$. By Lemma 3.3, $Nil\,(R) \subseteq Ker\,(g)$. Then we can define the homomorphism $\overline{g} : R/Nil\,(R) \longrightarrow N'$ by $\overline{g}\,(\overline{u}) = g\,(u)$, for every $u \in R$. Note that $\overline{g}\pi = g$. Since f is a $\mathcal{V}$-preenvelope of $R/Nil\,(R)$, there exists a homomorphism $\varphi : N \longrightarrow N'$ such that $\varphi f = \overline{g}$. Consequently,

$$\varphi f\pi = \overline{g}\pi = g$$

which shows that $f\pi$ is a $\mathcal{V}$-preenvelope of R. ■

Corollary 3.5 *If R is a commutative ring such that K-dim$(R) = 0$ then R has a $\mathcal{V}$-preenvelope.*

Proof. Since K-dim$(R/Nil\,(R)) = K$-dim$(R) = 0$ and $R/Nil\,(R)$ is a reduced ring, then $R/Nil\,(R) \in \mathcal{V}$. Consequently, the identity $i : R/Nil\,(R) \longrightarrow R/Nil\,(R)$ is a $\mathcal{V}$-preenvelope of $R/Nil\,(R)$. By Theorem 3.4, R has a $\mathcal{V}$-preenvelope. ■

4 Almost preenvelopes in integer domains

In this section $\mathcal{D}$ will denote the class of integer domains.

Proposition 4.1 *Let R be a commutative ring. If $\varphi : R \longrightarrow D$ is an almost $\mathcal{D}$-preenvelope of R then $Ker\varphi = Nil\,(R)$.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R and $R/\mathfrak{p} \stackrel{i}{\longrightarrow} k(\mathfrak{p})$ the inclusion. If $R \stackrel{\pi}{\longrightarrow} R/\mathfrak{p}$ is the canonical epimorphism then since $\varphi : R \longrightarrow D$ is an almost $\mathcal{D}$-preenvelope of R and $k(\mathfrak{p}) \in \mathcal{D}$, there exists a superfluous epimorphism $\theta : k(\mathfrak{p}) \longrightarrow D'$, where $D' \in \mathcal{D}$, and $\delta : D \longrightarrow D'$ a homomorphism such that $\theta i \pi = \delta\varphi$. Since $k(\mathfrak{p})$ is a field, $Ker\theta = \{0\}$ or $Ker\theta = k(\mathfrak{p})$. But the second possibility cannot occur because $Ker\theta \ll k(\mathfrak{p})$. Thus $Ker\theta = \{0\}$ and θ is an isomorphism. In particular, $i\pi = \theta^{-1}\delta\varphi$. Therefore, $i\pi(Ker\varphi) = 0$ and since i is a monomorphism, $\pi(Ker\varphi) = 0$ so that $Ker\varphi \subseteq Ker\pi = \mathfrak{p}$. This shows that $Ker\varphi \subseteq \bigcap_{\mathfrak{p}\in Spec(R)} \mathfrak{p} = Nil(R)$.

Now we show that $Nil(R) \subseteq Ker\varphi$. Let $x \in Nil(R)$, then there exists $n \in \mathbb{N}$ such that $x^n = 0$. Hence

$$[\varphi(x)]^n = \varphi(x^n) = \varphi(0) = 0$$

and since D is an integer domain, $\varphi(x) = 0$ and so $x \in Ker\varphi$. In concluison, $Ker\varphi = Nil(R)$. ■

Proposition 4.2 *Let R be a commutative reduced ring. R has an almost $\mathcal{D}$-preenvelope if and only if R is an integer domain.*

Proof. Assume that $\varphi : R \longrightarrow D$ is an almost $\mathcal{D}$-preenvelope of R. By Proposition 4.1, $Ker\varphi = Nil(R) = \{0\}$. Hence φ is a monomorphism. Consequently, $R \cong \varphi(R) \subseteq D$ is an integer domain. ■

It was shown in [6, Proposition 2.4] that a ring R has a $\mathcal{D}$-(pre)envelope if, and only if, $Nil(R)$ is a prime ideal of R. As we can see in our next result, this is equivalent to the existence of an almost $\mathcal{D}$-preenvelope of R.

Theorem 4.3 *Let R be a commutative ring. R has an almost $\mathcal{D}$-preenvelope if, and only if, R has a $\mathcal{D}$-preenvelope if, and only if, $Nil(R)$ is a prime ideal. In that case, the projection $p : R \longrightarrow R/Nil(R)$ is the $\mathcal{D}$-envelope.*

Proof. We only need to show that if $\varphi : R \longrightarrow D$ is an almost $\mathcal{D}$-preenvelope of R then $Nil(R)$ is a prime ideal of R. By Proposition 4.1, $Ker\varphi = Nil(R)$. If $a, b \in R$ satisfy $ab \in Ker\varphi$ then $0 = \varphi(ab) = \varphi(a)\varphi(b)$. Since D is an integer domain, $a \in Ker\varphi$ or $b \in Ker\varphi$. Consequently, $Ker\varphi = Nil(R)$ is a prime ideal of R. ■

5 Almost preenvelopes in local rings

In this section $\mathcal{L}$ denotes the class of local rings.

Theorem 5.1 *Let R be a commutative ring. The following conditions are equivalent:*

1. *R has a $\mathcal{L}$-(pre) envelope;*
2. *R has an almost $\mathcal{L}$-preenvelope;*
3. *R is local.*

Proof. 1. $\Rightarrow$ 2. This is a consequence of Remark 2.4.

2. $\Rightarrow$ 3. Let $\varphi : R \longrightarrow L$ be an almost $\mathcal{L}$-preenvelope of R and $\mathfrak{m}$ the unique maximal ideal of L. Then $\mathfrak{p} = \varphi^{-1}(\mathfrak{m})$ is a prime ideal of R. On the other hand, $J(L) = \{x \in L : x \text{ is not invertible}\}$ since L is a local ring. Thus, if $s \in R\backslash\mathfrak{p}$ then $\varphi(s) \notin \mathfrak{m} = J(L)$ and so $\varphi(s)$ is invertible. It follows from the universal property of localization that there exists a unique homomorphism $h : R_{\mathfrak{p}} \longrightarrow L$ such that $\varphi = hf$, where $f : R \longrightarrow R_{\mathfrak{p}}$ is the canonical homomorphism. We will show that $f : R \longrightarrow R_{\mathfrak{p}}$ is an almost $\mathcal{L}$-preenvelope of R. In fact, let $\alpha : R \longrightarrow L'$ be a homomorphism, where $L' \in \mathcal{L}$. Then, since φ is an almost $\mathcal{L}$-preenvelope of R, there exists a superfluous epimorphism $\theta : L' \longrightarrow L''$, where $L'' \in \mathcal{L}$, and a homomorphism $\delta : L \longrightarrow L''$ such that $\theta\alpha = \delta\varphi$. Consequently

$$\delta h f = \delta\varphi = \theta\alpha$$

which implies that $f : R \longrightarrow R_{\mathfrak{p}}$ is an almost $\mathcal{L}$-preenvelope of R.

Now let $\mathfrak{p}'$ be a prime ideal of R and consider the canonical homomorphism $f' : R \longrightarrow R_{\mathfrak{p}'}$. Then there exists a superfluous epimorphism $\alpha_1 : R_{\mathfrak{p}'} \longrightarrow L_1$, where $L_1 \in \mathcal{L}$ and a homomorphism $\delta_1 : R_{\mathfrak{p}} \longrightarrow L_1$ such that $\delta_1 f = \alpha_1 f'$. Let $s \in R\backslash\mathfrak{p}$. Then $f(s)$ is invertible and so $\alpha_1 f'(s) = \delta_1 f(s)$ is also invertible. Therefore, there exists $b \in L_1$ such that $\alpha_1 f'(s) b = 1$. Since α_1 is an epimorphism, there exists $a \in R_{\mathfrak{p}'}$ such that $b = \alpha_1(a)$ and so

$$\alpha_1\left(f'(s)\,a\right) = 1 = \alpha_1(1)$$

It follows that $1 - f'(s)\,a \in Ker(\alpha_1) \subseteq J\left(R_{\mathfrak{p}'}\right)$. Since $R_{\mathfrak{p}'}$ is local, $f'(s)\,a = 1 - (1 - f'(s)\,a)$ is invertible, consequently, $f'(s) = \frac{s}{1}$ is invertible in $R_{\mathfrak{p}'}$. Assume that $\frac{s}{1}\frac{u}{v} = 1$, where $u \in R$ and $v \in R\backslash\mathfrak{p}'$. Then there exists $s' \in R\backslash\mathfrak{p}'$ such that $s'(us - v) = 0$. Since $s' \in R\backslash\mathfrak{p}'$ and $v \in R\backslash\mathfrak{p}'$ we deduce that $s'us = s'v \in R\backslash\mathfrak{p}'$ and so $s \in R\backslash\mathfrak{p}'$. We have shown in this way that $\mathfrak{p}' \subseteq \mathfrak{p}$. Since $\mathfrak{p}'$ eas an arbitrary prime ideal of R, we conclude that $\mathfrak{p}$ is the unique maximal ideal of R, and so R is a local ring.

3. $\Rightarrow$ 1. The identity $i : R \longrightarrow R$ is a $\mathcal{L}$-preenvelope of R. ■

References

[1] F. Anderson and K. Fuller, Rings and Categories of Modules. New York: Springer-Verlag. 1974.

[2] M. Atiyah and I. Macdonald, Introducción al Álgebra Conmutativa. Editorial Reverté. 1973.

[3] L. Bican and B. Torrecillas, Almost precovers. Comm Algebra, 2002, 30: 477 - 487.

[4] E. Enochs, Injective and flat covers, envelopes and resolvents, Israel J. Math., 1981, 39, 189 - 209.

[5] M. Lixin, D. Nanqing and T. Wenting, On almost precovers and almost preenvelopes, Acta Mathematica Scientia 2006, 26B (3): 395-400.

[6] R. Parra and M. Saorín, Envelopes of commutative rings, arXiv:0906.4357v1 [math.AC].

CARLOS PARRA
Departamento de Matemáticas, Facultad de Ciencias,
Universidad de Los Andes
Mérida 5101 - A, Venezuela
e-mail: carlosparra@ula.ve

RAFAEL PARRA
Departamento de Cálculo, Escuela Básica,
Facultad de Ingeniería
Universidad de Los Andes
Mérida 5101 - A, Venezuela
e-mail: rafaelparra@ula.ve

JUAN RADA
Departamento de Matemáticas
Universidad Simón Bolívar
Caracas 1080 - A, Venezuela
e-mail: juanrada@usb.ve