

Taxonomy of Improvement Operators and the Problem of Minimal Change

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Abstract

Improvement operators is a family of belief change operators that is a generalization of usual iterated belief revision operators. The idea is to relax the success property, so the new information is not necessarily believed after the improvement, but to ensure that its plausibility has increased in the epistemic state. In this paper we explore this large family by defining several different subclasses. In particular, as minimal change is a hallmark of belief change, we study what are the operators that produce the minimal change among several subclasses.

1 Introduction

Belief change is a key task for any any rational agent. Modeling the evolution of the beliefs of one agent when he receives new pieces of information is the aim of belief revision. The predominant approach for modeling belief revision was proposed by Alchourrón, Gärdenfors and Makinson and is known as the AGM belief revision framework [1, 6, 10].

This approach has been extended in order to cope with iterated belief revision. The main approach for iterated belief revision was proposed by Darwiche and Pearl [5] (see also [2, 9] for more recent developments and [16] for an overview of the different operators). One of the main step for addressing the iteration of the revision process was to abandon logical belief bases (because of their lack of expressive power, see e.g. [8]) for epistemic states.

In [13] a generalization of iterated belief revision operators, called improvement operators, has been proposed. The idea is to define operators on epistemic states that have a less drastic behavior than iterated belief revision operators. One of the major requirement of belief revision operators is the so-called success postulate, that imposes that the new pieces of information must be believed after the change. This is clearly required for a lot of scenarios. But there are also some cases where we would like to take the information into account in a more cautious way. So with improvement operators the plausibility of the new piece of information is increased, but it not necessarily believed after the change. They are other works on belief

revision operators that do not satisfy the success postulate, that are called non-prioritized revisions (see for instance [7] for an overview). But none of these works define such increase of the plausibility of the new information as done by improvement operators.

One major hallmark of belief change is the principle of minimal change. For belief revision this means that we do not want to allow *any* change in the beliefs of the agent in order to allow the addition of the new piece of information, we want to have a *minimal* change where the only changes are the ones really required to allow the addition. The aim of this minimal change requirement is to keep as much as possible of the old beliefs of the agent.

In this paper we explore the operators that perform a minimal change among several subclasses of improvement operators.

The rest of the paper is organized as follows: we give the preliminaries in the first section. The second section is devoted to the introduction of improvement operators. The third section is devoted to the introduction of subclasses of improvement operators. The fourth section introduce our criterion of minimality. The three next sections study different class of soft improvement operators. The last section is the conclusion.

2 Preliminaries

We consider a propositional language \mathcal{L} defined from a finite set of propositional variables \mathcal{P} and the standard connectives. Let \mathcal{L}^* denote the set of consistent formulae of \mathcal{L} .

An interpretation ω is a total function from \mathcal{P} to $\{0, 1\}$. The set of all interpretations is denoted \mathcal{W} . An interpretation ω is a model of a formula $\phi \in \mathcal{L}$ if and only if it makes it true in the usual truth functional way. $[[\alpha]]$ denotes the set of models of the formula α , i.e., $[[\alpha]] = \{\omega \in \mathcal{W} \mid \omega \models \alpha\}$. When $\{w_1, \dots, w_n\}$ is a set of models we denote by $\varphi_{w_1, \dots, w_n}$ a formula such that $[[\varphi_{w_1, \dots, w_n}]] = \{w_1, \dots, w_n\}$.

We will use epistemic states to represent the beliefs of the agent, as usual in iterated belief revision [5]. An epistemic state Ψ represents the current beliefs of the agent, but also additional conditional information guiding the revision process (usually represented by a pre-order on interpretations, a set of conditionals, a sequence of formulae, etc). Let \mathcal{E} denote the set of all epistemic states. A projection function $B : \mathcal{E} \rightarrow \mathcal{L}^*$ associates to each epistemic state Ψ a consistent formula $B(\Psi)$, that represents the current beliefs of the agent in the epistemic state Ψ .

For simplicity purpose we will only consider in this paper consistent epistemic states and consistent new information. Thus, we consider change operators as functions \circ mapping an epistemic state and a consistent formula into a new epistemic state, *i.e.* in symbols, $\circ : \mathcal{E} \times \mathcal{L}^* \rightarrow \mathcal{E}$. The image of a pair (Ψ, α)

under \circ will be denoted by $\Psi \circ \alpha$.

We adopt the following notations:

- $\Psi \circ^n \alpha$ defined as: $\Psi \circ^1 \alpha = \Psi \circ \alpha$
 $\Psi \circ^{n+1} \alpha = (\Psi \circ^n \alpha) \circ \alpha$
- $\Psi \star \alpha = \Psi \circ^n \alpha$, where n is the first integer such that $B(\Psi \circ^n \alpha) \vdash \alpha$.

Note that \star is undefined if there is no n such that $B(\Psi \circ^n \alpha) \vdash \alpha$, but for all operators \circ considered in this work, the associated operator \star will be total, that is for any pair Ψ, α there will exist n such that $B(\Psi \circ^n \alpha) \vdash \alpha$ (see postulate **(I1)**) below.

Finally, let \leq be a total pre-order, i.e a reflexive ($x \leq x$), transitive ($(x \leq y \wedge y \leq z) \rightarrow x \leq z$) and total ($x \leq y \vee y \leq x$) relation over \mathcal{W} . Then the corresponding strict relation $<$ is defined as $x < y$ iff $x \leq y$ and $y \not\leq x$, and the corresponding equivalence relation \simeq is defined as $x \simeq y$ iff $x \leq y$ and $y \leq x$. We denote $w \ll w'$ when $w < w'$ and there is no w'' such that $w < w'' < w'$. And we note $w \triangleleft w'$ when $w < w'$ and $w \not\ll w'$. We also use the notation $\min(A, \leq) = \{w \in A \mid \nexists w' \in A \ w' < w\}$. The set of total pre-orders will be noted \mathcal{TP} .

When a set \mathcal{W} is equipped with a total pre-order \leq , then this set can be splitted in different levels, that gives the ordered sequence of its equivalence classes $\mathcal{W} = \langle S_0, \dots, S_n \rangle$. So $\forall x, y \in S_i \ x \simeq y$. We say in that case that x and y are at the same level of the pre-order. And $\forall x \in S_i \ \forall y \in S_j \ i < j$ implies $x < y$. We say in this case that x is in a lower level than y . We extend straightforwardly these definitions to compare subsets of equivalence classes, i.e if $A \subseteq S_i$ and $B \subseteq S_j$ then we say that A is in a lower level than B if $i < j$.

3 Improvement operators

We recall in this section the definition of improvement operators

Definition 1 An operator \circ is said to be a weak improvement operator if it satisfies (I1) to (I6):

- (I1)** There exists n such that $B(\Psi \circ^n \alpha) \vdash \alpha$
- (I2)** If $B(\Psi) \wedge \alpha \not\vdash \perp$, then $B(\Psi \star \alpha) \equiv B(\Psi) \wedge \alpha$
- (I3)** If $\alpha \not\vdash \perp$, then $B(\Psi \circ \alpha) \not\vdash \perp$
- (I4)** For any positive integer n if $\alpha_i \equiv \beta_i$ for all $i \leq n$ then $B(\Psi \circ \alpha_1 \circ \dots \circ \alpha_n) \equiv B(\Psi \circ \beta_1 \circ \dots \circ \beta_n)$
- (I5)** $B(\Psi \star \alpha) \wedge \beta \vdash B(\Psi \star (\alpha \wedge \beta))$
- (I6)** If $B(\Psi \star \alpha) \wedge \beta \not\vdash \perp$, then $B(\Psi \star (\alpha \wedge \beta)) \vdash B(\Psi \star \alpha) \wedge \beta$

Postulates (I2-I6) are very close to postulates (R2-R6) of usual belief revision operators [1, 10, 5]. The important difference lies in postulate (I1) that is weaker than the usual success postulate (R1). So postulates (I2-I6) hold for sequences of weak improvements only (whereas for revision they require only one step).

Definition 2 A weak improvement operator is said to be an improvement operator if it satisfies (I7) to (I9)¹.

(I7) If $\alpha \vdash \mu$ then $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$

(I8) If $\alpha \vdash \neg\mu$ then $B((\Psi \circ \mu) \star \alpha) \equiv B(\Psi \star \alpha)$

(I9) If $B(\Psi \star \alpha) \not\vdash \neg\mu$ then $B((\Psi \circ \mu) \star \alpha) \vdash \mu$

These postulates correspond to the postulates for iterated revision [5, 9, 2]. Postulates (I7) and (I8) correspond to postulates (C1) and (C2) of [5], and postulate (I9) corresponds to postulate (P) of [9, 2]. As for the basic postulates, the difference lies in the fact that they hold only for sequences of improvements.

Let us now recall the corresponding representation theorems [13]. Let us first define strong faithful assignments.

Definition 3 A function $\Psi \mapsto \leq_{\Psi}$ that maps each epistemic state Ψ to a total pre-order on interpretations \leq_{Ψ} is said to be a strong faithful assignment if and only if:

1. If $w \models B(\Psi)$ and $w' \models B(\Psi)$, then $w \simeq_{\Psi} w'$
2. If $w \models B(\Psi)$ and $w' \not\models B(\Psi)$, then $w <_{\Psi} w'$
3. For any positive integer n if $\alpha_i \equiv \beta_i$ for any $i \leq n$ then $\leq_{\Psi \circ \alpha_1 \circ \dots \circ \alpha_n} = \leq_{\Psi \circ \beta_1 \circ \dots \circ \beta_n}$

So now we can state the representation theorem for weak improvement operators:

Theorem 1 A change operator \circ is a weak improvement operator if and only if there exists a strong faithful assignment that maps each epistemic state Ψ to a total pre-order on interpretations \leq_{Ψ} such that

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_{\Psi}) \quad (1)$$

Let us now give the representation theorem for improvement operators:

Definition 4 Let \circ be a weak improvement operator and $\Psi \mapsto \leq_{\Psi}$ its corresponding strong faithful assignment. The assignment will be called a gradual assignment if the properties S1, S2 and S3 are satisfied

¹For coherence reasons we change the names of the classes of operators with respect to [13], where ‘‘Improvement operators’’ was the class of operators satisfying (I1-I11).

- (S1) If $w, w' \in \llbracket \alpha \rrbracket$ then $w \leq_{\Psi} w' \Leftrightarrow w \leq_{\Psi \circ \alpha} w'$
(S2) If $w, w' \in \llbracket \neg \alpha \rrbracket$ then $w \leq_{\Psi} w' \Leftrightarrow w \leq_{\Psi \circ \alpha} w'$
(S3) If $w \in \llbracket \alpha \rrbracket, w' \in \llbracket \neg \alpha \rrbracket$ then $w \leq_{\Psi} w' \Rightarrow w <_{\Psi \circ \alpha} w'$

Theorem 2 A change operator \circ is an improvement operator if and only if there exists a gradual assignment such that

$$\llbracket B(\Psi \star \alpha) \rrbracket = \min(\llbracket \alpha \rrbracket, \leq_{\Psi})$$

This theorem is a direct consequence of the theorem of [13] when removing (I10) and (I11) from the set of postulates and the corresponding conditions (S4) and (S5) from the assignment.

4 Belief Revision

In order to show that improvement operators are a generalization of iterated belief revision operators consider the following usual “success” postulate:

- (R1) $B(\Psi \circ \alpha) \vdash \alpha$

Note that this postulate is the one that makes a distinction between usual belief revision operators and non-prioritized revision operators. Note also that (R1) is a particular case of (I1) where $n = 1$.

Proposition 1 If \circ is a weak improvement operator (i.e. it satisfies (I1-I6)) that satisfies (R1), then it is a AGM/DP revision operator (i.e. it satisfies (R1-R6) of [5]).

Proposition 2 If \circ is an improvement operator (i.e. it satisfies (I1-I9)) that satisfies (R1), then it is an admissible revision operator (i.e. it satisfies (R1-R6) and (C1-C4) of [5] and property (P) of [2, 9]).

5 Soft Improvement

As shown in the last section usual iterated belief revision operators are a special case of weak improvement operators. This is a well known subclass. But the family of weak improvement operators is much larger than that, and we want to explore further some of its subclasses. Belief revision operators are the weak improvement operators that produces the biggest change in the epistemic state. We investigate here the opposite of the weak improvement operators spectrum, i.e. operators of soft improvement, that produce the smallest change. We propose some subclasses of soft improvement operators by providing additional

postulates and the corresponding representation theorems. And we study the minimal change operators of these subclasses.

So we are interested in soft improvement operators defined below:

Definition 5 *An improvement operator is said to be a soft improvement operator if it satisfies the following postulate*

(I10) *If $B(\Psi \star \alpha) \vdash \neg\mu$ then $B((\Psi \circ \mu) \star \alpha) \not\vdash \mu$*

This postulate says that a formula μ that is currently rejected by the agent, can not be directly accepted by a soft improvement. The only admissible change of status is that the formula can become undetermined. Then another step of soft improvement will be required to make the formula accepted by the agent. This motivates the name “soft” improvement.

We can give a representation theorem for soft improvement operators:

Definition 6 *Let \circ be a weak improvement operator and $\Psi \mapsto \leq_{\Psi}$ its corresponding strong faithful assignment. The assignment will be called a soft gradual assignment if it is gradual assignment and the following property holds*

(S4) *If $w \in [[\alpha]]$, $w' \in [[\neg\alpha]]$ then $w' <_{\Psi} w \Rightarrow w' \leq_{\Psi \circ \alpha} w$*

Theorem 3 *A change operator \circ is a soft improvement operator if and only if there exists a soft gradual assignment such that*

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_{\Psi})$$

Again, this theorem is a direct consequence of the theorem of [13] when removing (I11) from the set of postulates and the corresponding condition (S5) from the conditions of the assignment.

There are many operators in the class of soft improvement, that have quite different behaviors.

So we will try to identify some specific behaviors for soft improvement operators, and to study what are the minimal change operators in each of these classes.

There is one important difference between soft improvement operators : some of them can be defined locally, by looking only at the information of similar plausibility, while some of them are defined globally, i.e. they require to look at the whole epistemic state. We call this locality property , and we express it on the soft gradual assignment:

Definition 7 A soft gradual assignment is modular if: let $N_w = \{w'' \mid w'' \simeq_\Psi w\}$, let $R_{\Psi\circ\alpha} \in \{<, \simeq, \emptyset^2\}$ and $R_\Psi \in \{\triangleleft, \ll, \simeq\}$:

$$(Sg) \ w' R_{\Psi\circ\alpha} w = f(w' R_\Psi w, N_w, \alpha)$$

So this modularity property states that the plausibility relation between two interpretations after the improvement is a function of (i.e. the only information required are) the relation between the two interpretations before the improvement, the set of interpretations that are equivalent to the less plausible interpretation before the improvement, and of course the new piece of information.

Intuitively this property expresses the fact that to know the change of plausibility of interpretations at a given level after the improvement by a formula it is enough to look at this level and the immediately lower one.

One can state another property that identify an important difference on the behavior of soft improvement operators:

Definition 8 A soft gradual assignment is a systematic enhancement if:

$$(Sse) \text{ If } w \models \neg\alpha, w' \models \alpha \text{ and } w \ll_\Psi w', \text{ then } w \not\ll_{\Psi\circ\alpha} w'$$

This property states that (the plausibility of) every model of the new piece of information α is systematically improved. That means that if a model of the negation of α was just a little more plausible before the improvement than a model of α , then it is no longer the case after the improvement (the model of α will be at least as plausible as the model of the negation).

So we will identify three different classes of soft improvement operators:

Systematic Soft Improvement (SSI) operators that correspond to systematic enhancement assignments.

Modular Soft Improvement (MSI) operators that correspond to modular soft gradual assignments.

Soft Improvement (SI) operators that correspond to soft gradual assignments.

In the following sections we will define minimal change operators for each of these classes. But we have to define first what is our minimality criterion. That is the object of the next Section.

² $w' \emptyset_{\Psi\circ\alpha} w$ means that the relationship between w and w' is not determined by the function f for $w' R_{\Psi\circ\alpha} w$. So this relationship will be defined by $w R_{\Psi\circ\alpha} w'$ since the pre-order is a complete one.

6 Minimality

One major objective of belief change theories is to define operators that produces minimal change in the beliefs of the agent. This is a natural requirement because beliefs are valuable, so we want to keep as much as possible the old beliefs of the agent (no unnecessary forgetting), and because we want the agent to be rational by not adding exotic beliefs (no unjustified addition).

For improvement operators, as the representation theorem states that each operator corresponds to a gradual assignment (and if we consider this representation as the canonical one), we can consider these operators as transitions between total pre-orders. In this case there is a natural measure of change: the Kemeny distance [11] between the old pre-order (the pre-order associated to the old epistemic state) and the new one.

Definition 9 *The Kemeny Distance is the function $d_K : \mathcal{TP} \times \mathcal{TP} \rightarrow \mathbb{N}$ defined as: given \leq_1, \leq_2 two total pre-orders, $d_K(\leq_1, \leq_2)$ is the cardinal of the symmetrical difference of the pre-orders, i.e. the number of elements in \leq_1 which are not in \leq_2 plus the number of elements in \leq_2 which are not in \leq_1 . In symbols we have*

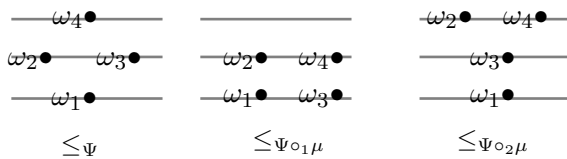
$$d_K(\leq_1, \leq_2) = | \leq_1 \Delta \leq_2 |$$

Definition 10 *Let \circ_1 and \circ_2 be two improvement operators. We say that \circ_1 produces less change than \circ_2 if for any epistemic state Ψ and any formula μ :*

$$d_K(\leq_\Psi, \leq_{\Psi \circ_1 \mu}) \leq d_K(\leq_\Psi, \leq_{\Psi \circ_2 \mu})$$

So this definition means that an operator produces less change than another one if on all possible improvements the first one produces less change (with respect to Kemeny distance) than the second one.

Example 1 *Suppose $\mathcal{W} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Consider two improvement operators \circ_1 and \circ_2 . Let Ψ be an epistemic state where \leq_Ψ is his respective pre-order given by the gradual assignment (see the figure below). Let μ be a formula such that $[[\mu]] = \{\omega_3, \omega_4\}$. Suppose that the pre-orders $\leq_{\Psi \circ_1 \mu}$ and $\leq_{\Psi \circ_2 \mu}$ are the results of the improvement of Ψ by the new information μ with respect the operators \circ_1 and \circ_2 respectively.*



It is not hard to see that $d_K(\leq_\Psi, \leq_{\Psi \circ_1 \mu}) = 3$

and that $d_K(\leq_\Psi, \leq_{\Psi \circ_2 \mu}) = 2$. So on this example \circ_2 produces less change than \circ_1 (but to conclude that \circ_2 produces less change than \circ_1 this has to be checked for all cases).

7 Systematic Soft Improvement

Before giving the postulate that characterize this behavior we need some notations.

Definition 11 *Let \circ be a change operator satisfying (II). Let α , β and Ψ be two formulae and an epistemic state respectively. We say that α is below β with respect to Ψ , given \circ , denoted $\alpha \prec_{\Psi}^{\circ} \beta$ (or simply $\alpha \prec_{\Psi} \beta$ if there is no ambiguity about \circ) if and only if $\alpha \not\vdash \perp$, $\beta \not\vdash \perp$, $B(\Psi \star \alpha) \vdash B(\Psi \star (\alpha \vee \beta))$ and $B(\Psi \star \beta) \not\vdash B(\Psi \star (\alpha \vee \beta))$.*

The pair (α, β) is Ψ -consecutive, denoted $\alpha \prec_{\Psi}^{\circ} \beta$ (or simply $\alpha \prec_{\Psi} \beta$ if there is no ambiguity about \circ) if and only if $\alpha \prec_{\Psi} \beta$ and there is no formula γ such that $\alpha \prec_{\Psi} \gamma \prec_{\Psi} \beta$.

So now, let us introduce an additional postulate in order to characterize operators of systematic soft improvement:

(I11) If $B(\Psi \star \alpha) \vdash \neg\mu$ and $\alpha \prec_{\Psi} \alpha \wedge \mu$ then $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$

And we can state a corresponding representation theorem:

Theorem 4 *A change operator \circ is a systematic soft improvement operator if and only if there exists a systematic enhancement such that*

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_{\Psi})$$

This is approximatively the main Theorem in [13]. The only difference is that condition (Sse) is used in the assignment instead of condition (S5) in [13]:

(S5) If $w \in [[\alpha]]$, $w' \in [[\neg\alpha]]$ then $w' \ll_{\Psi} w \Rightarrow w \leq_{\Psi \circ \alpha} w'$

Clearly (Sse) and (S5) are equivalent in the presence of others (S1-S4) conditions.

As explained in [13], there is only one operator of systematic soft improvement (once the pre-order associated to the initial epistemic state is fixed). We will call this operator one-improvement, and note it \odot .

The fact that this is the only operator of this class implies straightforwardly that it is the one that produces the minimal change.

So condition (Sse) can be considered as very strong, since it defines a class of soft improvement operators that contains only one operator. But, first we consider that (Sse) is very sensible, so it is interesting to study its consequences. And secondly recall that the class of weak improvement operators is wider than soft

improvement operators, and (Sse) can also prove valuable to discriminate operators in other classes. For instance we can remark that it allows to discriminate belief revision operators, since Boutilier's natural revision [3] and Darwiche and Pearl \bullet operator [5] do not satisfy (Sse), while Nayak's lexicographic operator [15, 12] does.

8 Modular Soft Improvement

Modular soft improvement are operators that can be defined locally, by looking at beliefs of similar plausibility.

(H1) If $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \ll_{\Psi} \alpha \wedge \mu$ and $\neg\exists\beta(\beta \vdash \neg\mu \text{ and } \alpha \ll_{\Psi} \beta)$, then $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$

This postulate means that when the revision (i.e. sequence of improvements until success) by α implies the negation of μ , if μ is just a little less plausible than its negation given α , then an improvement by μ will be enough to remove its negation from the beliefs of the agent. Note that this postulate is weaker than (H1).

(H2) If $B(\Psi \star \alpha) \vdash \neg\mu$, $\alpha \ll_{\Psi} \alpha \wedge \mu$ and $\exists\beta(\beta \vdash \neg\mu \text{ and } \alpha \ll_{\Psi} \beta)$, then $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$

This postulate is very close from (H1), and deals with the case where the revision (i.e. sequence of improvements until success) by α implies the negation of μ , but μ and $\neg\mu$ are both a little less plausible than $\neg\mu$, then an improvement by μ will not be enough to remove its negation from the beliefs of the agent.

Definition 12 A soft improvement operator which satisfies (H1) and (H2) is called a half improvement operator.

We can also define these operators semantically:

Definition 13 Let \circ be a soft improvement operator and $\Psi \mapsto \leq_{\Psi}$ its corresponding soft gradual assignment. The assignment will be called a half gradual assignment if the following properties (SH1) and (SH2) are satisfied:

(SH1) If $\omega \in [[\mu]]$, $\omega' \in [[\neg\mu]]$, $\omega' \ll_{\Psi} \omega$ and $\nexists\omega'' \in [[\neg\mu]]$ such that $\omega'' \simeq_{\Psi} \omega$, then, $\omega \leq_{\Psi \circ \mu} \omega'$.

(SH2) If $\omega \in [[\mu]]$, $\omega' \in [[\neg\mu]]$, $\omega' \ll_{\Psi} \omega$ and $\exists\omega'' \in [[\neg\mu]]$ such that $\omega'' \simeq_{\Psi} \omega$ then, $\omega' <_{\Psi \circ \mu} \omega$.

Note that both (SH1) and (SH2) use to conclude on $\leq_{\Psi \circ \mu}$, only information on the new formula, the old relation \leq_{Ψ} between the two interpretations, and the interpretations that was at the same level of ω . This means that the half-gradual assignement is a modular assignement.

We can now state the representation theorem:

Theorem 5 *A change operator \circ is a half improvement operator if and only if there exists a half gradual assignement such that*

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_{\Psi})$$

In fact, just as for one-improvement we can prove that:

Proposition 3 *Once the pre-order associated to the first epistemic state is fixed, there is a unique half-improvement operator. Let us note \circledast this operator.*

Unlike one-improvement, the sole operator in the class of systematic soft improvement, half-improvement is not the only operator in the class of modular soft improvement.

But we can show that half-improvement is a particular modular soft improvement operator. In fact, we can show that half-improvement produces less changes than one-improvement:

Proposition 4 *Let Ψ be an epistemic state (a total preorder). Then for all formula μ ,*

$$d_K(\leq_{\Psi}, \leq_{\Psi \circledast \mu}) \leq d_K(\leq_{\Psi}, \leq_{\Psi \circ \mu})$$

That is, the operator \circledast produces less changes than the operator \circ .

More generally, we have the following result:

Proposition 5 *Half-improvement operator is the modular soft improvement that produces the smallest change (i.e. it produces less change than any other modular soft improvement operator).*

9 Looking for the Best Soft Improvement

Now we move to the general class of soft improvement operators. The fact of not satisfying modularity allows to define much more different operators. This allows also to define an interesting soft improvement operator producing the smallest change under certain conditions.

In order to simplify the presentation of the postulates we introduce the following definition:

Definition 14 *μ is separated in Ψ iff $\forall \beta (B(\Psi \star \beta) \vdash \mu$ or $B(\Psi \star \beta) \vdash \neg \mu$).*

This definition of separation of a formula in an epistemic state means that any revision (and improvement) of this epistemic state will always give epistemic states that are informed about this formula (i.e. the formula or its negation can be inferred).

Definition 15 *A soft improvement operator which satisfies the following two postulates is called a best improvement operator*

(B1) *If μ is separated in Ψ , $B(\Psi \star \alpha) \vdash \neg\mu$ and $\alpha \ll_{\Psi} \alpha \wedge \mu$, then $B((\Psi \circ \mu) \star \alpha) \not\vdash \neg\mu$*

(B2) *If μ is not separated in Ψ and $B(\Psi \star \alpha) \vdash \neg\mu$, then $B((\Psi \circ \mu) \star \alpha) \vdash \neg\mu$*

Postulate (B1) is close to postulates (H1) and (I11), but it holds only when the formula is separated in the epistemic state.

Postulate (B2) states that, when the formula is not separated in the epistemic state (which is the general case), the change is the same one than with (H2).

We can give a semantical counterpart to these postulates.

Definition 16 *μ is s-separated in \leq_{Ψ} iff $\nexists \omega_1 \in [[\mu]], \omega_2 \in [[\neg\mu]]$ s.t. $\omega_1 \simeq_{\Psi} \omega_2$*

Definition 17 *Let \circ be a soft improvement operator and $\Psi \mapsto \leq_{\Psi}$ its corresponding soft gradual assignment. The assignment will be called a best gradual assignment if the following properties*

(SB1) *If μ is s-separated in \leq_{Ψ} , $\omega \in [[\mu]], \omega' \in [[\neg\mu]]$ and $\omega' \ll_{\Psi} \omega$ then $\omega \leq_{\Psi \circ \mu} \omega'$.*

(SB2) *If μ is not s-separated in \leq_{Ψ} , $\omega \in [[\mu]], \omega' \in [[\neg\mu]]$ and $\omega' <_{\Psi} \omega$ then $\omega' <_{\Psi \circ \mu} \omega$.*

The following lemma will be useful in the proofs:

Lemma 1 *Let \circ be a weak improvement operator. Then, μ is separated in Ψ iff μ is s-separated in \leq_{Ψ} .*

Let us give now the corresponding representation theorem.

Theorem 6 *A change operator \circ is a best improvement operator if and only if there exists a best gradual assignment such that*

$$[[B(\Psi \star \alpha)]] = \min([[\alpha]], \leq_{\Psi})$$

And, as for one-improvement and half-improvement, we can show that there is only one best-improvement operator:

Proposition 6 *Once the pre-order associated to the first epistemic state is fixed, there is a unique best-improvement operator. We note \oplus this operator.*

These uniqueness results are important since few change operators are axiomatically define (usual characterizations in belief revision defined families of operators).

Let us now turn to the minimality issue.

Proposition 7 *Let Ψ be an epistemic state (a total preorder). Then for all formula μ ,*

$$d_K(\leq_{\Psi}, \leq_{\Psi \oplus \mu}) \leq d_K(\leq_{\Psi}, \leq_{\Psi \circ \mu})$$

That is, the operator \oplus produces less changes than the operator \circ .

As a corollary from the previous propositions we have the following result:

Proposition 8 *Among the operators \circ , \otimes and \oplus the operator \oplus is the operator that produces minimal change.*

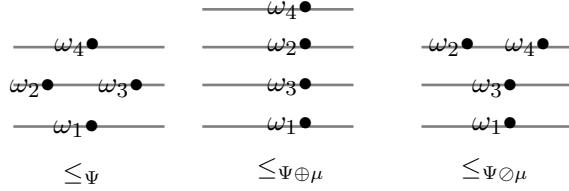
It is easy to figure out soft-improvement operators that produces less change than best-improvement. In fact the soft-improvement operator that produces the smallest change is the one that increases the plausibility of only one level of models of the new formula (this level is not randomly chosen, but is the one which produces the less change for the Kemeny distance). It seems to us that defining this minimal change operator is not of great interest, since this operator will not have a clear meaning from a logical point of view. It is of no use to look at absolute minimization if it costs too many logical properties (recall that Boutilier's natural revision operator [3], although achieving the minimal change for a belief revision operator, make it at a price of bad logical properties [5]).

So, amongst improvement operators that do not add arbitrary choices in the choice of the models of the new information to be improved, best-improvement is the one that produces the smallest change:

Proposition 9 *Best-improvement operator is the soft improvement operator satisfying (B1) that produces the smallest change.*

Example 1 shows that in some cases \otimes produces strictly less changes than \circ (\circ_2 was \otimes and \circ_1 was \circ). The following example will shows that in some cases \oplus produces strictly less changes than \otimes .

Example 2 Suppose $\mathcal{W} = \{\omega_1, \omega_2, \omega_3, \omega_4\}$. Consider the improvement operators \oplus and \circledast . Let Ψ be an epistemic state where \leq_Ψ is his respective pre-order given by the gradual assignment (see the figure below). Let be μ a formula such that $\llbracket \mu \rrbracket = \{\omega_3, \omega_4\}$. The pre-orders $\leq_{\Psi \circledast \mu}$ and $\leq_{\Psi \oplus \mu}$ are the results of the improvement of Ψ by the new information μ with respect the operators \oplus and \circledast respectively.



It is not hard to see that $d_K(\leq_\Psi, \leq_{\Psi \oplus \mu}) = 1$ and that $d_K(\leq_\Psi, \leq_{\Psi \circledast \mu}) = 2$.

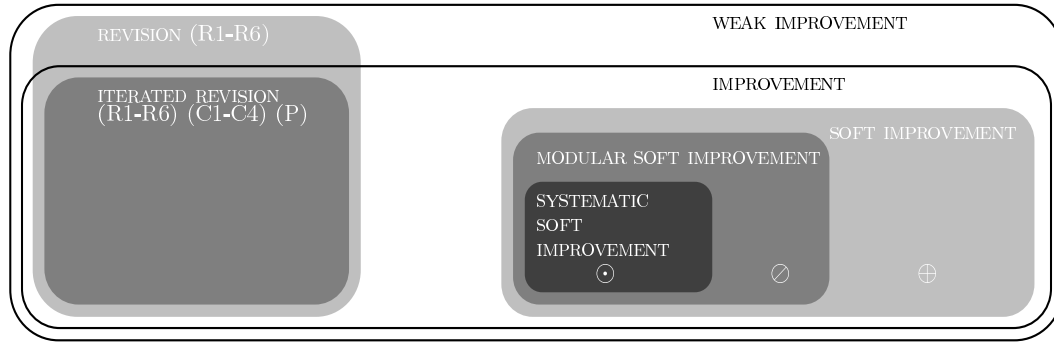


Figure 1: A map of weak improvement operators

10 Example

We provide some examples of improvements in this section, in order to illustrate the behavior (and the differences) of one-improvement, half-improvement and best-improvement.

Figure 2 shows how three epistemic states, whose associated pre-orders are $\leq_{\Psi_1}, \leq_{\Psi_2}, \leq_{\Psi_3}$, are changed through the three soft-improvement operators studied in the previous sections (one-improvement \circledast , half-improvement \circledcirc , and best-improvement \oplus).

Interpretations are not represented on the figures, we just represent the “levels” where the interpretation are located. Orange lines represent the new formula μ , so models of this formula are located on these orange levels. Black lines represent the levels with models of $\neg\mu$.

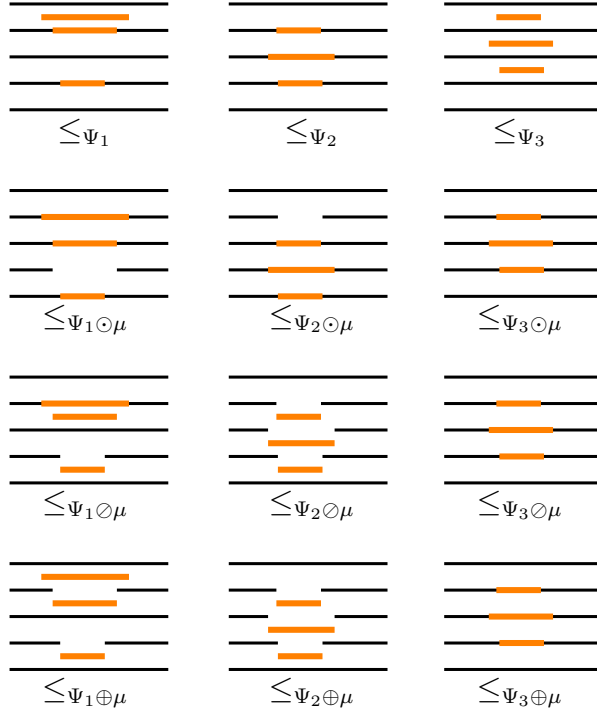


Figure 2: Examples of Soft Improvements

In the \leq_{Ψ_3} case, the three operators lead to the same result. This is the case where μ is separated in Ψ_3 .

The \leq_{Ψ_2} case shows a situation where every model of the new formula μ is equivalent to a model of its negation. In this case half-improvement and best-improvement give the same result, that produces less change than the result obtained with one-improvement. So the change is smaller. Remark that to obtain the same result as one-improvement it will just require another iteration, i.e. $\leq_{\Psi_2 \odot \mu} = \leq_{\Psi_2 \odot \mu \odot \mu} = \leq_{\Psi_2 \oplus \mu \oplus \mu}$.

The \leq_{Ψ_1} case is the most interesting. It is a more usual case, and it shows the difference of behaviors of the three operators. It clearly shows that in the general case best-improvement produces less change than half-improvement, that produces less change than one-improvement. To explain intuitively the change obtained by the three operators: one-improvement increase the plausibility of each model of the new formula by moving it to the first (w.r.t its current position) lower level of models of its negation. Half-improvement increase the plausibility of each model, but only by a “half-level” (i.e. if the model of the new formula was equivalent to a model of its negation then now it is strictly more plausible, but not as plausible as the lower models of the negation. And if the model of the new formula was not equivalent to a model of its negation, then now it is moved to the first lower level of models of the negation). For Best-improvement, as there are some models of the new formula that was equivalent to model of its negation, then only these models have

their plausibility improved (from a “half-level”).

11 Conclusion and Related Work

In this paper we have started the investigation of soft improvement operators. Soft improvement operators are a subclass of weak improvement operators, just as belief revision operators are. See figure 1 for a map of weak improvement subclasses. We defined two subclasses of soft improvement operators: modular soft improvement operators and systematic soft improvement operators. For each of this class we provide a prototypical operator, that we characterize logically and for which we provide a representation theorem. We also study these operators with respect to minimal change, when this minimality is computed using the Kemeny distance between the pre-orders obtained through the assignments.

Ideas close to the ones behind the definition of the one-improvement operator already appeared in some works such as [4, 17, 14], but there was no logical characterization in all these works. As far as we know there is no work mentioning ideas close to half-improvement or best-improvement.

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