Separation axioms and the prime spectrum of commutative semirings

Alirio Peña, Luz M. Ruza and Jorge Vielma

Abstract

In this work by a *semiring* we mean a commutative semiring with nonzero identity and we study the prime spectrum of a semiring with its Zariski topology.

We prove that this space is spectral and we characterize when it belong to one of the following classes: irreducible, supercompact (in the lattice-theoretic sense given in [7]), connected, nested, zero-dimensional, regular, normal, $T_{1/4}$, $T_{1/2}$ and $T_{3/4}$. We describe the regular, kerneled, isolated and open-regular points of this space and we show that it is always either supercompact or weak R_0 (in the sense of [17]). In our treatment we introduce the *absolutely (prime-) irreducible* ideals of a semiring which are the versions for semirings of the ideals studied by Picavet ([23]) and Gilmer ([9]), and the isolated points of the prime spectrum of a semiring are the absolutely prime-irreducible minimal prime ideals.

key words. Spectral space, supercompact element of a complete lattice, weak R_0-space, separation axiom, nested space, completely irreducible ideal.

AMS subject classifications. Primary 54D10, 54F65, 13C05; Secondary 13C05

1   Introduction

We present a systematic study of the prime spectrum of a semiring with nonzero identity from an (affine) classical view point, following the ideas of Zariski, as a unified approach to the cases of commutative rings ([4]) and bounded distributive lattices ([6]). We show some of the basic properties of the topological space $(\mathrm{Spec}(R), t_Z)$, called the *prime spectrum of R*, where $\mathrm{Spec}(R)$ is the family of prime ideals of a semiring R and t_Z is the Zariski topology on $\mathrm{Spec}(R)$. We characterize when $\mathrm{Spec}(R)$ satisfies one of the following properties: irreducible, supercompact (in the lattice-theoretic sense of [7]), connected, nested, zero-dimensional, regular and normal, as well as when this satisfies the separation axioms T_2, T_1, $T_{1/2}$, $T_{1/4}$, $T_{3/4}$, R_1, R_0, weak R_0 (in the sense of [3]) and weak R_0 (in the sense of [17], and that we call R_0^*).

In section 1 we give some terminology and basic results. In section 2 we study the prime spectrum $(\mathrm{Spec}(R), t_Z)$ of a semiring R which we denote by $\mathrm{Spec}(R)$ only. In general, $\mathrm{Spec}(R)$ is a spectral space (Theorem 3.1) and it is irreducible if and only if R has a unique minimal prime ideal (Corollary 3.2). Also, $\mathrm{Spec}(R)$ is a supercompact space if and only if R is a local semiring (Theorem 3.2), we describe the clopen subsets of $\mathrm{Spec}(R)$ (Lemma 3.1) and we use this result to characterize the connectedness of this space (Corollary 3.3). Again, we characterize when $\mathrm{Spec}(R)$ is a nested space (Theorem 3.3), and for this space the following conditions are equivalent: T_1, T_2, every prime ideal is maximal, R_0, weak R_0 (in the sense of [3]) and R_1 (Theorem 3.4). In section 3 we show the regular points of $\mathrm{Spec}(R)$ (Theorem 4.1) and in this space, every regular point is closed and every non weak η-redundant maximal ideal of R is a regular point of $\mathrm{Spec}(R)$ (Corollary 4.1). Also, $\mathrm{Spec}(R)$ is a regular space if and only if every prime ideal of R is maximal (Corollary 4.2), and it is a normal space if and only if R is a Gelfand semiring (Corollary 4.3 and Theorem 4.2). This last characterization generalizes a well-known result for commutative rings (Theorem 2.1 in [19]). In section 4 we describe the kerneled, isolated and open-regular points of $\mathrm{Spec}(R)$ (Lemma 5.1 and Theorems 5.1-5.2) and we apply these results to characterize when it is a $T_{1/4}$, $T_{1/2}$ and $T_{3/4}$-space, respectively (Corollaries 5.1-5.2 and 5.6). Thus, $\mathrm{Spec}(R)$ is a discrete space if and only if every prime ideal of R is both absolutely prime-irreducible and minimal prime (Corollary 5.3). Finally, every prime ideal of a semiring is either weak η-redundant or absolutely prime-irreducible (Lemma 5.2), we characterize when a minimal prime ideal is absolutely prime-irreducible, and when a prime ideal is an isolated point of the prime spectrum (Corollary 5.4).

2 Terminology and basic results

In this work $\mathbb{N} := \{0, 1, 2, \dots\}$ denotes the set of natural numbers, $|X|$ is the cardinal of a set X and $\wp(X)$ is the power-set of X. For every nonempty subset $\mathcal{F}$ of $\wp(X)$, $\mathcal{F}^{\cap}$ denotes the family formed by all the intersections of sets in $\mathcal{F}$. By a *space* we mean a topological space, (X, τ) always denote a space and τ^* is the family of τ-closed sets. For every subset Y of X, we denote by $\tau(Y)$ and $\overline{Y}^{\tau}$ the family of τ-open neighborhoods of Y and the τ-closure of Y, respectively. Also, $\widehat{Y}^{\tau} := \bigcap \tau(Y)$ is the τ-*kernel* of Y, and if $Y = \widehat{Y}^{\tau}$ then Y is called τ-*kerneled* ([18]). Further, $\tau(x) := \tau(\{x\})$, $\overline{x}^{\tau} := \overline{\{x\}}^{\tau}$ and $\widehat{x}^{\tau} := \widehat{\{x\}}^{\tau}$ for every $x \in X$.

A *semiring* (commutative with nonzero identity) is an algebra $(R, +, \cdot, 0, 1)$ where R is a set with $0, 1 \in R$, and $+$ and $\cdot$ are binary operations on R called *sum* and *multiplication*, respectively,

which satisfies the following conditions:

(S_1) $(R, +, 0)$ and $(R, \cdot, 1)$ are commutative monoids with $1 \neq 0$.

(S_2) The multiplication *distributes* with the sum, this is, $\forall\, a, b, c \in R,\ a\,(b + c) = a\,b + a\,c$.

(S_3) 0 is *multiplicatively absorbing*, this is, $\forall\, a \in R,\ a\,0 = 0$.

Every ring (commutative with nonzero identity) is a semiring and there is two distinct ways of consider a bounded distributive lattice as a semiring, namely: taking the sum and multiplication, respectively, as the join and meet in the lattice, or taking the sum and multiplication, respectively, as the meet and join in the lattice. As is usual, we denote a semiring $(R, +, \cdot, 0, 1)$ by R, and the multiplication $a \cdot b$ by ab. The notions of (*proper*) *ideal*, *prime ideal* and (*minimal prime*) *maximal ideal* of a semiring R are defined as in commutative rings ([4]). We denote by $\mathrm{Id}(R)$ and $\mathrm{Spec}(R)$, respectively, the sets of ideals and prime ideals of R, and we set $I \leq R$ for indicate that I is an ideal of R. As in rings, every proper ideal is contained in a maximal ideal, every maximal ideal is prime and every prime ideal contains a minimal prime ([10]). By a *local semiring* we mean a semiring with a unique maximal ideal, and for every ideal I of R, we denote by $\eta(I)$ the *prime radical* of I, this is, the intersection of the prime ideals of R containing I. It is well known that $\eta(I) = \{a \in R : a^n \in I \text{ for some } n \in \mathbb{N}\}$ and if $I = \eta(I)$ then I is *semiprime*. Thus, the improper ideal R is semiprime (the empty intersection of prime ideals),

the ideal $\eta(0)$

is the *prime radical* of R and if $\eta(0) = (0)$ then R is *semiprime*. We denote by $U(R)$ the set of invertible elements of R and if $a \in R$ then $Ra := \{ra : r \in R\}$ is the principal ideal of R generated by a, and $\eta(a) := \eta(Ra)$. Also, for every ideal I of R, the set $(I : a) := \{r \in R : ra \in I\}$ is an ideal of R and in particular, $(0 : a) := \{r \in R : ra = 0\}$ is the *annihilator of* a.

See [10] for more details about the semiring theory and their applications, and for the lattice theory see [6].

3   Prime spectrum of a semiring

For every ideal I of a semiring R, we denote by $(I)_0 := \{P \in \mathrm{Spec}(R) : I \subseteq P\}$ and $D_0(I) := \mathrm{Spec}(R) \backslash (I)_0$. Also, $(x)_0 := (\{x\})_0$ and $D_0(x) := D_0(\{x\})$ for every $x \in R$. Then,

the family $\{(I)_0 : I \leq R\}$ satisfies the axioms of closed sets for a topology t_Z on $\mathrm{Spec}(R)$, called the *Zariski topology*, and the space $(\mathrm{Spec}(R), t_Z)$ is the *prime spectrum* of R. If $P \in \mathrm{Spec}(R)$

then $\overline{P}^{\,t_Z} = (P)_0$ and thus, $\{P\}$ is t_Z-closed if and only if P is a maximal ideal.

Further, in general, the complete lattice $\mathrm{Spec}(R)^\cap$ of semiprime ideals of R is isomorphic to the dual of the lattice t_Z^*, via $I \mapsto (I)_0$, where the meet in t_Z^* is the intersection and the join of a family $\{(I_j)_0\}_{j \in J}$ in t_Z^* is $(\bigcap_{j \in J} I_j)_0$. Hence, $\mathrm{Spec}(R)^\cap$ is a distributive complete lattice.

A space (X, τ) is *spectral* if it is a compact T$_0$-space such that the compact open subsets of X form a base of τ which is closed under finite intersections, and it is a *sober space*, this is, the non-empty irreducible τ-closed subsets of X are of the form $\overline{x}^\tau$, for some (unique) $x \in X$. Also, a τ-closed subset F of X is *irreducible* if whenever $F \subseteq F_1 \cup F_2$ with $F_i \in \tau^*$, we have either $F \subseteq F_1$ or $F \subseteq F_2$. It is well-known that a space is spectral if and only if it is homeomorphic to the prime spectrum of a commutative (semiprime) ring, which we call the *Hochster's Theorem* ([14]). Moreover, in [22] is pointed out that the prime spectrum of a topology is spectral. We see that, in general, the prime spectrum of a semiring is a spectral space. In Proposition 7.20 of [10] it is proved that the prime spectrum of a semiring (non necessarily commutative) is a compact T$_0$-space.

Theorem 3.1 *Let R be a semiring. Then, $\mathrm{Spec}(R)$ is a spectral space.*

Proof. Let $X = \mathrm{Spec}(R)$ and $I = Rx_1 + \cdots + Rx_n$ a finitely generated (f.g.) ideal of R. We see that $D_0(I)$ is compact and thus, $X = D_0(R)$ and $D_0(a)$ with $a \in R$, are too. In fact, if $D_0(I) \subseteq \bigcup_{s \in S} D_0(s)$ for some $S \subseteq R$ and RS is the ideal of R generated by S, then $(RS)_0 = \bigcap_{s \in S} (s)_0 \subseteq (I)_0$ and thus, $I \subseteq \eta(RS)$ and there exists $m \geq 1$ integer such that every $x_i^m \in RS$. Hence, every $x_i^m = \sum_{j=1}^{k_i} a_{ij} s_{ij}$ where $k_i \in \mathbb{N}$, $a_{ij} \in R$ and $s_{ij} \in S$. It follows that $D_0(I) \subseteq \bigcup_{i,j} D_0(s_{ij})$ and the family $\{D_0(I) : I \text{ is a f.g. ideal of } R\}$ is a base of t_Z (since this contains the base $\{D_0(a) : a \in R\}$) formed by compact open sets and it is closed under finite intersections, since if I, J are f.g. ideals of R then by the commutativity of R, the product IJ is also a f.g. ideal of R such that $D_0(I) \cap D_0(J) = D_0(IJ)$. Finally, we see that X is a sober space. Let P be a prime ideal of R and let I, J be ideals of R such that $(P)_0 \subseteq (I)_0 \cup (J)_0$. Then, $(P)_0 \subseteq (IJ)_0$ and $IJ \subseteq P$ and thus, $I \subseteq P$ or $J \subseteq P$. Hence, $(P)_0 \subseteq (I)_0$ or $(P)_0 \subseteq (J)_0$. Conversely, suppose Q is a semiprime ideal of R such that $(Q)_0$ is a non-empty irreducible in t_Z^* and let $a, b \in R$ such that $ab \in Q$. Then, $(Q)_0 \subseteq (ab)_0 = (a)_0 \cup (b)_0$ and by hypothesis, $(Q)_0 \subseteq (a)_0$ or $(Q)_0 \subseteq (b)_0$. If $(Q)_0 \subseteq (a)_0$ then $\eta(a) \subseteq Q$ and $a \in Q$ (otherwise, $b \in Q$). Hence, Q is a prime ideal of R and X is a spectral space. $\square$

Corollary 3.1 *For every semiring R there exists a (semiprime) commutative ring A such that $\mathrm{Spec}(R)$ and $\mathrm{Spec}(A)$ are homeomorphic spaces.*

Proof. It follows from Theorem 3.1 and Hochster's Theorem ([14]).   □

Since every closed subspace of a spectral space is spectral, for every proper ideal I of a semiring R there exists a commutative ring $R(I)$ such that the subspace $(I)_0$ is homeomorphic to $\text{Spec}(R(I))$. If R is a ring then, by the correspondence theorem of prime ideals, we can choose $R(I)$ as the quotient ring R/I. A space (X, τ) is *irreducible* if every pair of nonempty τ-open subsets of X have nonempty intersection. The following result is a consequence from Corollary 3.1 and the corresponding result for commutative rings ([4], Chap. 1, Ex. 19).

Corollary 3.2 *Let R be a semiring. Then, $\text{Spec}(R)$ is an irreducible space if and only if $\eta(0)$ is a prime ideal. Further, in such a case, $\eta(0)$ is the unique minimal prime ideal of R.*

We say that (X, τ) is a *supercompact space* if every τ-open covering of X contains X, this is, X is a supercompact element of the complete lattice $(t_Z, \subseteq)$ in the sense of [7]. This notion of supercompactness for topological spaces is stronger than the usual supercompactness introduced by de Groot in [11] (see also [21]), for example, consider a semiring with only two prime ideals which are maximal ideals.

Theorem 3.2 *A semiring R is local if and only if $\text{Spec}(R)$ is a supercompact space.*

Proof. Let $X = \text{Spec}(R)$. Suppose R is local with maximal ideal Q and that there exists a t_Z-open covering $\{\mathcal{U}_j\}_{j \in S}$ of X such that every $\mathcal{U}_j \neq X$. Then, there exist proper ideals I_j of R such that $Q \in (I_j)_0 = X \backslash \mathcal{U}_j$ for every $j \in S$. But then, $Q \in \bigcap_{j \in S}(I_j)_0 = $ which is a contradiction. Conversely, suppose X is a supercompact space and that $|\text{Max}(R)| > 1$. Since $X = \bigcup_{P \in \text{Max}(R)} D_0(P)$, there exists $P \in \text{Max}(R)$ such that $(P)_0 = $ which is not possible.   □

The atoms of the lattice $(t_Z, \subseteq)$ are of the form $D_0(a)$ where $a \in R$ such that $\eta(a)$ *covers* $\eta(0)$ in $(\text{Spec}(R)^{\cap}, \subseteq)$, this is, if $I \in \text{Spec}(R)^{\cap}$ such that $\eta(0) \subsetneq I \subseteq \eta(a)$ then $I = \eta(a)$. Also, the coatoms of the lattice $(t_Z, \subseteq)$ are the sets $D_0(Q)$ where Q is a maximal ideal.

We see a description of the *clopen* (this is, open and closed) subsets of $\text{Spec}(R)$. If R is a commutative ring then the clopen subsets of $\text{Spec}(R)$ are of the form $D_0(e)$ where $e = e^2 \in R$ (Lemma 2.1 in [16]). We say that $a \in R$ is *η-complemented* if there exists $b \in R$ such that $a + b = 1$ and $ab \in \eta(0)$, and $comp_\eta(R)$ denotes the set of η-complemented elements of R.

Lemma 3.1 *Let R be a semiring and $\mathcal{U}$ a subset of $\text{Spec}(R)$. Then, $\mathcal{U}$ is a t_Z-clopen in $\text{Spec}(R)$ if and only if $\mathcal{U} = D_0(a)$ for some $a \in comp_\eta(R)$.*

Proof. It is straightforward. □

Corollary 3.3 *Let R be a semiring. Then, $\mathrm{Spec}(R)$ is a connected space if and only if $0, 1$ are the unique η-complemented elements of R.*

Recall that (X, τ) is a *nested space* if the lattice $(\tau, \subseteq)$ is linearly ordered.

Theorem 3.3 *Let R be a semiring. Then, the following conditions are equivalent:*

(a) $\mathrm{Spec}(R)$ *is a nested space.*

(b) *The complete lattice $(\mathrm{Spec}(R)^{\cap}, \subseteq)$ is linearly ordered.*

(c) *For every $a, b \in R$, there exists $n \in \mathbb{N}$ such that either a/b^n or b/a^n in R.*

(d) *For every $a, b \in R$, the ideals $\eta(a)$ and $\eta(b)$ are comparable by inclusion.*

(e) *For every $a, b \in R$, the t_Z-open sets $D_0(a)$ and $D_0(b)$ are comparable by inclusion.*

Proof. It is clear that $(b) \Rightarrow (d) \Leftrightarrow (c)$ and $(b) \Rightarrow (a) \Rightarrow (e)$. We see $(d) \Rightarrow (b)$. Suppose (d) and let $I, J \in \mathrm{Spec}(R)^{\cap}$ such that $I \not\subseteq J$ and $J \not\subseteq I$. Then, there exist $a \in I \backslash J$ and $b \in J \backslash I$. But then, by hypothesis, we can suppose $\eta(a) \subseteq \eta(b)$ and so, $a \in \eta(b) \subseteq J$ which is a contradiction. Hence, (b) is holds. Finally, for prove $(d) \Leftrightarrow (e)$ note that, if $a, b \in R$ then: $\eta(a) \subseteq \eta(b)$ if and only if $(b)_0 \subseteq (a)_0$ if and only if $D_0(a) \subseteq D_0(b)$. □

The following are the versions for semirings of two fundamental results in commutative algebra (see [1] and [20], and Lemma 1.1 in [13], respectively).

Lemma 3.2 *Let S be a multiplicatively closed subset of a semiring R and let I be an ideal of R such that $I \cap S = $. Then, there exists an ideal P of R maximal respect to the property $P \cap S = $ and $I \subseteq P$. Further, every such an ideal is prime.*

Lemma 3.3 *A prime ideal P of a semiring R is minimal prime if and only if for every $x \in P$, there exists $a \in R \backslash P$ such that ax is nilpotent. Hence, in such a case, $P = \bigcup_{a \in R \backslash P}(\eta(0) : a)$.*

Proof. Suppose P is a minimal prime ideal of R, let $S = R \backslash P$ and $x \in P$. Then, the set $T = \bigcup_{n \in \mathbb{N}} S x^n$ is a multiplicatively closed subset of R which contains S (since $x^0 = 1$). If $T \cap \eta(0) = $ then, by Lemma 3.2, there exists a prime ideal Q of R such that $T \cap Q = $ and thus, $Q \subseteq P$ and $x \notin Q$ contradicting the minimality of P. Hence, $T \cap \eta(0) \neq $ and there exist $a \in S$

and $n \in \mathbb{N}$ such that $ax^n \in \eta(0)$ and it follows $ax \in \eta(0)$. Conversely, suppose the sufficiency condition. If P is not minimal prime then there exists a minimal prime ideal Q of R properly contained in P and there exists $x \in P \backslash Q$. By hypothesis, we will can choose $a \in R \backslash P$ such that $ax \subseteq \eta(0) \subseteq Q$ which is a contradiction. $\square$

A space (X, τ) is an R_0-*space* if for every $U \in \tau$ and $x \in U$, we have $\overline{x}^\tau \subseteq U$, or equivalently, if the specialization pre-order on X induced by τ is an equivalence relation ([17]). Also, a subset Y of X is λ-*closed* if $Y = \overline{Y}^\tau \cap \widehat{Y}^\tau$ ([3]). Also, a point $x \in X$ is λ-*closed* if the singleton $\{x\}$ is λ-closed. Every (closed) kerneled subset is λ-closed and a space is a *weak* R_0-*space* if every λ-closed point of this is kerneled. It is well known that a space is a T_0-space if and only if each of its points is λ-closed, and a space is a T_1-space if and only if it is both T_0 and a (weak) R_0-space (Theorem 2.9 in [3]). Further, (X, τ) is an R_1-*space* if for every pair $x, y \in X$ with $\overline{x}^\tau \neq \overline{y}^\tau$, there exist $U, V \in \tau$ such that $\overline{x}^\tau \subseteq U$, $\overline{y}^\tau \subseteq V$ and $U \cap V = $ ([17]).

Theorem 3.4 *Equivalent conditions for a semiring* R:

(a) *Every prime ideal of R is maximal.*

(b) $\mathrm{Spec}(R)$ *is a* T_1-*space.*

(c) $\mathrm{Spec}(R)$ *is a* T_2-*space.*

(d) $\mathrm{Spec}(R)$ *is an* R_0-*space.*

(e) $\mathrm{Spec}(R)$ *is a weak* R_0-*space.*

(f) $\mathrm{Spec}(R)$ *is an* R_1-*space.*

(g) *Every point of* $\mathrm{Spec}(R)$ *is kerneled.*

Proof. It is clear that $(c) \Rightarrow (b) \Leftrightarrow (a)$, and since $\mathrm{Spec}(R)$ is a T_0-space and by the identities $T_1 = T_0 + R_0 = T_0 + \mathrm{weak}\, R_0$, we have $(b) \Leftrightarrow (d) \Leftrightarrow (e)$. Also, since $T_2 = T_1 + R_1$ and $R_1 \Rightarrow R_0$, we have $(c) \Leftrightarrow (f)$. We see $(a) \Rightarrow (c)$. Suppose that every prime ideal of R is maximal and we see that $\mathrm{Spec}(R)$ is a T_2-space. Let P and Q be distinct points of $\mathrm{Spec}(R)$. It is sufficient to verify that there exist $a \in R \backslash Q$ and $b \in R \backslash P$ such that $ab \in \eta(R)$, since, in such a case, $D_0(a)$ and $D_0(b)$ are disjoint t_Z-open neighborhoods of the points Q and P, respectively. By hypothesis, the ideals P and Q are not comparable by inclusion and thus, there exists $a \in P \backslash Q$. Now, by the minimality of P and Lemma 3.3, there exists $b \in R \backslash P$ such that ab is nilpotent and thus, a and

b satisfy the required conditions. Finally, since $\mathrm{Spec}(R)$ is a T_0-space, every point of $\mathrm{Spec}(R)$ is λ-closed and we have $(e) \Leftrightarrow (g)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

The equivalence of $(a) - (c)$ in Theorem 3.4 are well-known for commutative rings and we present here the version for semirings.

Now, let $\mathcal{F}_P := \{I \in \mathrm{Spec}(R)^{\cap} : I \neq R, I \nsubseteq P\}$ for every prime ideal P of R.

Then, $\widehat{P}^{\,t_Z} = \bigcap_{I \in \mathcal{F}_P} D_0(I)$ and

if $\mathcal{F}_P \neq$ then $\mathcal{F}_P$ is closed under nonempty finite intersections.

The following result is straightforward.

Theorem 3.5 *Let R be a semiring. Then, the following conditions are equivalent:*

   (a) *R is a local semiring.*

   (b) *There exists a prime ideal P of R such that $\mathcal{F}_P = $.*

   (c) *There exists a prime ideal P of R such that $\widehat{P}^{\,t_Z} = \mathrm{Spec}(R)$.*

Further, in such a case, P is the unique maximal ideal of R.

In the literature there is other notion of weak R_0-space due to Di Maio ([17]), which is independent of the given in [3]. Thus, we say (X, τ) is an R_0^*-*space* (a weak R_0-space in the Di Maio's sense) if $\bigcap_{x \in X} \overline{x}^{\,\tau} = $, or equivalently, if $\tau(x) \neq \{X\}$ for every $x \in X$. By Theorems 3.2 and 3.5 and Hochter's Theorem, every spectral space is either supercompact or an R_0^*-space. In general, every space is either supercompact (in our sense) or an R_0^*-space.

4   Regularity and normality

We say that a point x of a space (X, τ) is a *regular point* if for every $F \in \tau^*$ with $x \in X \backslash F$, there exist $U, V \in \tau$ such that $x \in U$, $F \subseteq V$ and $U \cap V = $. Thus, a space is *regular* if each of its points is regular. We see a description of the regular points of the space $\mathrm{Spec}(R)$.

Let P be a prime ideal of a semiring R. If $\mathcal{F}_P = $ then P is the unique maximal ideal of R (Theorem 3.5) and the empty set is the unique t_Z-closed subset of $\mathrm{Spec}(R)$ which does not contain P and so, P is a regular point of $\mathrm{Spec}(R)$. Hence, in our study about the regularity of $\mathrm{Spec}(R)$ we can suppose the nonlocal case.

Theorem 4.1 *Let R be a semiring and P a prime ideal of R such that $\mathcal{F}_P \neq$. Then, the following conditions are equivalent:*

(a) *P is a regular point of $\mathrm{Spec}(R)$.*

(b) *For every $I \in \mathcal{F}_P$, there exist $a \in R\backslash P$ and an ideal J of R such that $I + J = R$ and $Ja \subseteq \eta(0)$.*

(c) *For every $x \in R\backslash P$, there exist $a \in R\backslash P$ and an ideal J of R such that $Rx + J = R$ and $Ja \subseteq \eta(0)$.*

Further, in such a case, P is a maximal ideal of R.

Proof. Let $X = \mathrm{Spec}(R)$. We see $(a) \Leftrightarrow (b)$. Let $I \in \mathcal{F}_P$. Then, are equivalent:

(1) There exist $\mathcal{U}, \mathcal{V} \in t_Z$ such that $P \in \mathcal{U}$, $(I)_0 \subseteq \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = $.

(2) There exist $a \in R$ and $J \leq R$ with $P \in D_0(a)$, $(I)_0 \subseteq D_0(J)$ and $D_0(a) \cap D_0(J) = $.

(3) There exist $a \in R\backslash P$ and $J \leq R$ such that $(I)_0 \cap (J)_0 = $ and $(a)_0 \cup (J)_0 = X$.

(4) There exist $a \in R\backslash P$ and $J \leq R$ such that $(I + J)_0 = $ and $(aJ)_0 = X$.

(5) There exist $a \in R\backslash P$ and $J \leq R$ such that $I + J = R$ and $aJ \subseteq \eta(0)$.

It is clear that $(c) \Rightarrow (b)$. We see $(b) \Rightarrow (c)$. Suppose (b) and let $x \in R\backslash P$. If $x \in U(R)$ then we set $a = x$ and $J = \eta(0)$. Suppose that $x \notin U(R)$. Then, $I = \eta(x) \in \mathcal{F}_P$ and by hypothesis, there exist $a \in R\backslash P$ and an ideal J of R such that $I + J = R$ and $Ja \subseteq \eta(0)$. Let $i \in I$ and $j \in J$ such that $1 = i + j$. Then, there exists $n \in \mathbb{N}$ such that $i^n \in Rx$ and by the Newton's formula, $1 = (i+j)^n \in Rx + J$. The last part is a consequence of (c) and the inclusions $J \subseteq (\eta(0) : a) \subseteq P$ for every $a \in R\backslash P$.   □

For every prime ideal P of a semiring R, we set $I(P) := \bigcap\{Q \in \mathrm{Spec}(R) : Q \neq P\}$ and we say P is *weak η-redundant* if $I(P) = \eta(0)$.

Corollary 4.1 *Let P be a non weak η-redundant maximal ideal of a semiring R. Then, P is a regular point of $\mathrm{Spec}(R)$.*

Proof. Since $\eta(0) \subsetneq I(P)$, we have $I(P) \nsubseteq P$ and there exists $x \in I(P)\backslash P$. Thus, if $a = x$ and $J = P$ then $a \in R\backslash P$ and $Ja \subseteq \eta(0)$ and by Theorem 4.1, P is a regular point of $\mathrm{Spec}(R)$.   □

The converse of Corollary 4.1 need not to be holds (for example, consider a local domain which is not a field).

Corollary 4.2 *Let R be a semiring. Then, the following conditions are equivalent:*

(a) $\mathrm{Spec}(R)$ *is a regular space.*

(b) *For every prime ideal P of R and every $I \in \mathcal{F}_P$, there exist $a \in R \backslash P$ and an ideal J of R such that $I + J = R$ and $Ja \subseteq \eta(0)$.*

(c) *For every prime ideal P of R and every $x \in R \backslash P$, there exist $a \in R \backslash P$ and an ideal J of R such that $Rx + J = R$ and $Ja \subseteq \eta(0)$.*

(d) *Every prime ideal of R is maximal.*

Proof. By Theorem 4.1, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$ and $(a) \Rightarrow (d)$.

Also, $(d) \Rightarrow (a)$ by Theorem 3.4 and since every compact T_2-space is regular. $\qquad \square$

Recall that a space (X, τ) is *normal* if for every pair F, G of disjoint τ-closed subsets of X, there exist $U, V \in \tau$ such that $F \subseteq U$, $G \subseteq V$ and $U \cap V = $. In such a case, we say that F, G are *normal-separated.* Also, we say that two ideals I, J of a semiring R are *comaximals* if $I + J = R$. The following result is straightforward.

Lemma 4.1 *Let R be a semiring, $X = \mathrm{Spec}(R)$ and I, J comaximal ideals of R. Then, $(I)_0$ and $(J)_0$ are normal-separated if and only if there exist $I', J' \in X^{\cap}$ such that $I + I' = J + J' = R$ and $I' \cap J' = \eta(0)$.*

Corollary 4.3 *Let R be a semiring. Then, the following conditions are equivalent:*

(a) $\mathrm{Spec}(R)$ *is a normal space.*

(b) *For every pair I, J of comaximal (semiprime) ideals of R, there exist (semiprime) ideals I', J' of R such that $I + I' = J + J' = R$ and $I' \cap J' = \eta(0)$.*

(c) *For every pair $a, b \in R$ such that $a + b = 1$, there exist $a', b' \in R$ such that $Ra + Ra' = Rb + Rb' = R$ and $a'b' \in \eta(0)$.*

Proof. It follows from Lemma 4.1. $\qquad \square$

We say that R is a *Gelfand semiring* (or a *pm-semiring*) if every prime ideal P of R is contained in a unique maximal ideal $\mu_R(P)$ ([19]). Note that this is not equivalent to Golan's definition ([10]), since any local ring with finite characteristic is a good witness. If $\mathrm{Max}(R)$ is the subspace of $\mathrm{Spec}(R)$ formed by the maximal ideals of R then the function $\mu_R : \mathrm{Spec}(R) \to \mathrm{Max}(R)$ is continuous (and thus, a retraction), closed and send closed disjoint subsets of $\mathrm{Spec}(R)$ into closed disjoint subsets of $\mathrm{Max}(R)$. The proof of these properties of μ_R are similar to the case of commutative rings ([19]). For every $P \in \mathrm{Spec}(R)$, we set $\mathcal{O}_P$ the intersection of the prime ideals of R contained in P. Thus, $\mathcal{O}_P = \bigcup_{a \in R \setminus P}(\eta(0) : a)$ and as for commutative rings ([19], Theorem 2.1) we have the following result.

Theorem 4.2 *Equivalent conditions for a semiring R:*

(a) R *is a Gelfand semiring.*

(b) $\mathrm{Max}(R)$ *is a retract of* $\mathrm{Spec}(R)$.

(c) *For every maximal ideal M of R, M is the unique maximal ideal containing $\mathcal{O}_M$.*

(d) $\mathrm{Spec}(R)$ *is a normal space.*

Further, in such a case, the function μ_R is the unique retraction of $\mathrm{Spec}(R)$ onto $\mathrm{Max}(R)$ and $\mathrm{Max}(R)$ is a T_2-subspace of $\mathrm{Spec}(R)$.

If $\mathrm{Max}(R)$ is a T_2-subspace of $\mathrm{Spec}(R)$ and $J(R) := \bigcap \mathrm{Max}(R)$ then every prime ideal in $(J(R))_0$ is contained in a unique maximal ideal of R. Thus, if $J(R) = \eta(0)$ then $\mathrm{Max}(R)$ is a T_2-space if and only if R is a Gelfand semiring. This remark is pointed out in [19].

5   kerneled, isolated and open-regular points

In this section we characterize the kerneled, isolated and open-regular points of the prime spectrum of a semiring and we use these results to characterize when it is a $\mathrm{T}_{1/4}$, $\mathrm{T}_{1/2}$ and $\mathrm{T}_{3/4}$-space, respectively. Recall that a point x of a space (X, τ) is *isolated* if $\{x\}$ is τ-open, and a subset A of (X, τ) is a *generalized closed set* if $\overline{A} \subseteq U$ whenever $U \in \tau(A)$ ([15]). Also, a space is called a $\mathrm{T}_{1/4}$-*space* if each of its points is either kerneled or closed, and it is a $\mathrm{T}_{1/2}$-*space* if every generalized closed set is closed ([15]), or equivalently, if each of its points is either isolated or closed ([8]). Further, a space is a $\mathrm{T}_{3/4}$-*space* if each of its points is either closed or open-regular, where an *open-regular set* is an open set which is the interior of its closure. In general, $\mathrm{T}_{3/4} \Rightarrow \mathrm{T}_{1/2} \Rightarrow \mathrm{T}_{1/4}$

and since every isolated point is kerneled, to describe when the prime spectrum of a semiring is either a $T_{1/4}$-space or a $T_{1/2}$-space we need first to know the kerneled points of this.

Lemma 5.1 *Let R be a semiring and P a prime ideal of R. Then, P is a kerneled point of* $\mathrm{Spec}(R)$ *if and only if P is minimal prime.*

Proof. It follows of the identity $\widehat{P}^{t_Z} = \{Q \in \mathrm{Spec}(R) : Q \subseteq P\}$. $\qquad\qquad\square$

Corollary 5.1 *Let R be a semiring. Then,* $\mathrm{Spec}(R)$ *is a $T_{1/4}$-space if and only if every prime ideal of R is either maximal or minimal prime.*

Proof. The closed points of $\mathrm{Spec}(R)$ are the maximal ideals of R. Now, use Lemma 5.1. $\qquad\square$

By Corollary 5.1, a semiring have Krull-dimension at most 1 if and only if its prime spectrum is a $T_{1/4}$-space. This result is a generalization of Proposition 3.1.3 in [5].

To characterize the isolated points of $\mathrm{Spec}(R)$ we need the following notion which generalize the ideals considered by Picavet ([23]) and Gilmer ([9]). Let $\mathcal{F}$ be a nonempty family of ideals of a semiring R. We say that an ideal I of R is *absolutely $\mathcal{F}$-irreducible* if for every subset $\{I_j\}_{j \in S}$ of $\mathcal{F}$ such that $\bigcap_{j \in S} I_j \subseteq I$, there exists $j \in S$ such that $I_j \subseteq I$. If this condition holds for $\mathcal{F} = \mathrm{Spec}(R)$ we say I is *absolutely prime-irreducible* and if it holds for $\mathcal{F} = \mathrm{Id}(R)$ then I is *absolutely irreducible*. Every absolutely irreducible ideal is *completely irreducible*, this is, is not the intersection of a family of overideals ([12]). Also, the absolutely irreducible prime ideals are the prime ideals satisfying the property (#) studied by Gilmer in [9], as well as the absolutely prime-irreducible prime ideals are the prime ideals with the property (##). Moreover, the maximal ideal of a local semiring is absolutely irreducible and the ring of integers contain no absolutely prime-irreducible proper ideals.

Note that if $\mathcal{F}$ is a nonempty family of ideals of a semiring R then an ideal of R is absolutely $\mathcal{F}$-irreducible if and only if it is absolutely $\mathcal{F}^{\cap}$-irreducible.

In particular, an ideal of R is absolutely prime-irreducible if and only if it is absolutely $\mathrm{Spec}(R)^{\cap}$-irreducible.

Theorem 5.1 *Let R be a semiring and P a prime ideal of R. Then, P is an isolated point of* $\mathrm{Spec}(R)$ *if and only if P is an absolutely prime-irreducible minimal prime ideal.*

Proof. Let $X = \mathrm{Spec}(R)$. If $X = \{P\}$ the result is immediate. Suppose $X \neq \{P\}$ and that P is an isolated point of X and let I be an ideal of R such that $\{P\} = D_0(I)$. Since every isolated

point is kerneled, P is minimal prime (Lemma 5.1). Now, let $\{Q_j\}_{j \in S}$ be a subset of X such that $\bigcap_{j \in S} Q_j \subseteq P$. If every $Q_j \nsubseteq P$ then $I \subseteq Q_j$ for every $j \in S$ which is a contradiction. Thus, there exists $j \in S$ such that $Q_j \subseteq P$. Conversely, suppose P is an absolutely prime-irreducible minimal prime ideal, and let $I = I(P) = \bigcap\{Q \in \mathrm{Spec}(R) : Q \neq P\}$. Then, $I \nsubseteq P$ and $P \in D_0(I)$. Also, if $Q \in D_0(I)$ then $Q = P$ (otherwise, $I \subseteq Q$). Hence, $\{P\} = D_0(I)$ and P is an isolated point of X. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 5.2 *Let R be a semiring. Then, $\mathrm{Spec}(R)$ is a $\mathrm{T}_{1/2}$-space if and only if every prime ideal of R is either maximal or absolutely prime-irreducible minimal prime.*

Corollary 5.3 *Let R be a semiring. Then, $\mathrm{Spec}(R)$ is a discrete space if and only if every prime ideal of R is absolutely prime-irreducible minimal prime.*

The conditions "absolutely prime-irreducible" and "minimal prime" are independent for prime ideals, since the zero ideal is the unique minimal prime ideal of the ring of integers which is not absolutely prime-irreducible, and the unique maximal ideal of any local domain which is not a field is absolutely irreducible and it is not minimal prime. On the other hand, by the last part of the proof of Theorem 5.1 we have the following result.

Lemma 5.2 (Prime dichotomy) *Let P be a prime ideal of a semiring R. Then, we have $D_0(I(P)) \subseteq \{P\}$. Hence, only one of the following conditions is holds: $I(P) = \eta(0)$ or P is an absolutely prime-irreducible minimal prime ideal.*

Corollary 5.4 *A minimal prime ideal Q of a semiring R is absolutely prime-irreducible if and only if $I(Q) \neq \eta(0)$. Also, a prime ideal P of R is an isolated point of $\mathrm{Spec}(R)$ if and only if $I(P) \neq \eta(0)$.*

Corollary 5.5 *Every prime ideal of a semiring is either weak η-redundant or an isolated point of its prime spectrum.*

We say that a point x of a space (X, τ) is an *open-regular point* if the singleton $\{x\}$ is an open-regular in (X, τ), or equivalently, if x is the unique interior point of $\overline{x}^\tau$. For every semiring R, we describe the open-regular points of the space $X = \mathrm{Spec}(R)$. If $|X| = 1$ then X is a $\mathrm{T}_{3/4}$-space. Thus, we suppose $|X| \geq 2$. Also, if $X = (P)_0$ then P is the smallest prime ideal of R and in such a case, P is an open-regular point of X if and only if $X = \{P\}$. Hence, in that follows, we suppose $|X| \geq 2$ and $X \neq (P)_0$ for every $P \in X$.

Lemma 5.3 *Let I be an ideal of a semiring R and P a prime ideal of R such that $D_0(I)$ is a nonempty subset of $(P)_0$. Then, $P \in D_0(I)$ and P is minimal prime.*

Proof. Let $Q \in D_0(I)$. Then, $I \nsubseteq Q$ and $P \subseteq Q$. Thus, $I \nsubseteq P$ and $P \in D_0(I)$. Also, let H be a minimal prime ideal of R such that $H \subseteq Q$. Then, $I \nsubseteq H$ and $H \in D_0(I) \subseteq (P)_0$. Hence, $P \subseteq H$ and $P = H$. $\square$

For every prime ideal P of R, we set $S_P := R \backslash [U(R) \mathbin{\dot\cup} P]$, $S_P^* := \{a \in S_P : D_0(a) \subseteq (P)_0\}$ and $I_0(P) := \bigcap \{Q \in \mathrm{Min}(R) : Q \neq P\}$. Thus, $S_P^* = \{a \in S_P : aP \subseteq \eta(0)\} \subseteq I_0(P) \backslash P$ and $I(P) \backslash P = \{a \in S_P : D_0(a) = \{P\}\}$. Note that is possible that $S_P^* = $ (Lemma 5.5 below).

Lemma 5.4 *Let P be a prime ideal of a semiring R. Then, $(P)_0$ have nonempty t_Z-interior if and only if there exists $a \in S_P$ such that $\neq D_0(a) \subseteq (P)_0$. Further, in such a case, P is a minimal prime ideal.*

Proof. Suppose I is a semiprime ideal of R such that $\neq D_0(I) \subseteq (P)_0$. By Lemma 5.3, $P \in D_0(I) = \bigcup_{a \in I} D_0(a)$ and there exists $a \in I$ such that $P \in D_0(a)$. But then, $a \in S_P$ (since $I \neq R$ and $a \notin P$) and $\neq D_0(a) \subseteq (P)_0$. Conversely, suppose that $a \in S_P$ such that $\neq D_0(a) \subseteq (P)_0$. Then, $I = \eta(a)$ is a semiprime ideal of R such that $D_0(I) = D_0(a)$. The last part it follows from Lemma 5.3. $\square$

Lemma 5.5 *Let R be a semiring and P an isolated point of $\mathrm{Spec}(R)$. Then, $S_P^* \neq $.*

Proof. If $a \in R$ such that $\{P\} = D_0(a)$ then $a \in S_P^*$. $\square$

Note that if $\mathcal{U} \subseteq \mathrm{Spec}(R)$ and $P \in \mathrm{Spec}(R)$ then $\overline{\mathcal{U}}^{t_Z} = (\bigcap \mathcal{U})_0$ and $I(P) \subseteq \bigcap D_0(P)$.

Theorem 5.2 *Equivalent conditions for a prime ideal P of a semiring R:*

(a) *P is an open-regular point of $\mathrm{Spec}(R)$.*

(b) *P is the unique interior point of $(P)_0$.*

(c) *$S_P^* \neq $ and $D_0(a) = \{P\}$ for every $a \in S_P^*$.*

(d) *$\mathrm{Spec}(R) \backslash \{P\} = \overline{D_0(P)}^{t_Z}$.*

(e) *P is an isolated point of $\mathrm{Spec}(R)$ such that $I(P) = \bigcap D_0(P)$.*

Proof. It is clear that $(a) \Leftrightarrow (b)$. We see $(b) \Leftrightarrow (c)$. Suppose (b). By Lemma 5.5, $S_P^* \neq$ and if $a \in S_P^*$ then $P \in D_0(a)$ and by hypothesis, (c) is holds. Conversely, suppose (c). Then, $S_P^* \neq$ and by Lemma 5.3, P is an interior point of $(P)_0$. Now, let Q be an interior point of $(P)_0$ and $a \in R$ such that $Q \in D_0(a) \subseteq (P)_0$. Then, $a \in S_P^*$ (Lemma 5.3) and by hypothesis, $Q = P$. We see $(b) \Leftrightarrow (d)$. Let $X = \mathrm{Spec}(R)$ and suppose (b). Then $X \backslash \{P\}$ is a closed in X containing $D_0(P)$ and thus, $\overline{D_0(P)}^{t_Z} \subseteq X \backslash \{P\}$. Now, if $Q \in X \backslash \{P\}$ then by hypothesis, every open neighborhood of Q in X intersects $D_0(P)$ and thus, $Q \in \overline{D_0(P)}^{t_Z}$. Conversely, suppose (d). Then $\{P\}$ is open in X and P is an interior point of $(P)_0$. Now, if Q is an interior point of $(P)_0$ and $a \in R$ such that $Q \in D_0(a) \subseteq (P)_0$ then $P = Q$, since, otherwise, $Q \in X \backslash \{P\} = \overline{D_0(P)}^{t_Z}$ and there exists $H \in D_0(P)$ such that $H \in D_0(a) \subseteq (P)_0$ which is a contradiction. Finally, we see $(d) \Leftrightarrow (e)$. Suppose (d). Then, P is an isolated point of X and $I(P) \subseteq \bigcap D_0(P)$. Now, if $a \in \bigcap D_0(P)$ and $Q \neq P$ in X then $Q \in \overline{D_0(P)}^{t_Z}$ and if $a \notin Q$ then $D_0(a)$ intersects $D_0(P)$ which is a contradiction. Conversely, suppose (e). Then, $X \backslash \{P\}$ is a t_Z-closed set containing $D_0(P)$ and thus, $\overline{D_0(P)}^{t_Z} \subseteq X \backslash \{P\}$. Note that, by hypothesis, $\overline{D_0(P)}^{t_Z} = (\bigcap D_0(P))_0 = (I(P))_0$ and hence, $X \backslash \{P\} \subseteq \overline{D_0(P)}^{t_Z}$. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Corollary 5.6 *Let R be a semiring and $X = \mathrm{Spec}(R)$. Then, are equivalent:*

(a) *X is a $\mathrm{T}_{3/4}$-space.*

(b) *Every point P of X is either closed or an isolated point such that $D_0(a) = \{P\}$ for every $a \in S_P^*$.*

(c) *Every point P of X is either closed or satisfies $X \backslash \{P\} = \overline{D_0(P)}^{t_Z}$.*

(d) *$dim(R) \leq 1$ and $X \backslash \{P\} = \overline{D_0(P)}^{t_Z}$ for every nonmaximal prime ideal P of R.*

(e) *Every point P of X is either closed or an isolated point such that $I(P) = \bigcap D_0(P)$.*

(f) *Every prime ideal P of R is either maximal or an absolutely prime-irreducible ideal such that $I(P) = \bigcap D_0(P)$.*

Further, in such a case, the set $\mathrm{Min}(R) \backslash \mathrm{Max}(R)$ is a discrete space.

References

[1] Allen, P.J., Neggers, J. and Kim, H.S., *Ideal theory in commutative A-semirings*, Kyungpook Math. J. 46 (2006), 261-271.

[2] Arenas, F.G., Dontchev, J., and Ganster, M., *On some weaker forms of Alexandroff spaces*, Arabian J. Science and Engineering 23 (1998), 79-89.

[3] Arenas, F.G., Dontchev, J., and Ganster, M., *On λ-sets and the dual of generalized continuity*, Questions and Answers Gen. Topology 15 (1) (1997), 3-13.

[4] Atiyah, M.F., and Macdonald, I.G., *Introduction to commutative algebra*, Addison-Wesley, Reading, Mass., 1969.

[5] Ávila, J.A., Spec(R) *y axiomas de separación entre* T_0 *y* T_1, Divulg. Mat. 13 (2) (2005), 90-98.

[6] Baldes, R., and Dwinger, Ph., *Distributive lattices*, University of Missouri Press, Columbia, Missouri, 1974.

[7] Banaschewski, B., and Niefield, S.B., *Projective and supercoherent frames*, J. Pure Appl. Algebra 70 (1991), 45-51.

[8] Dunham, W., $T_{1/2}$-*spaces*, Kyungpook Math. J. 17 (1977), 161-169.

[9] Gilmer, R., *An intersection condition for prime ideals*, in *Factorization in integral domains* (Iowa City, 1996), Lecture Notes in Pure and Appl. Math. 189, Marcel Dekker, New York, 1997.

[10] Golan, J.S., *Semirings and their applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[11] Groot, J. de, *Supercompactness and superextensions*, in: *Contributions to extension theory of topological structures*, Symp. Berlin, 1967, Deutscher Verlag Wiss., Berlin, 1969, 89-90.

[12] Heinzer, W., and Olberding, B., *Unique irredundant intersections of completely irreducible ideals*, J. of Algebra 287 (2005), 432-448.

[13] Henricksen, M., and Jerison, M., *The space of minimal prime ideals of a commutative ring*, General topology and its relations to modern algebra and analysis, Vol. I, Academic Press and Publishing House of the Czechoslovak Academy of Sciences, Prague, 1962, pp. 199-203.

[14] Hochster, M., *Prime ideal structure in commutative rings*, Trans. Amer. Math. Soc. 142 (1969), 43-60.

[15] Levine, N., *Generalized closed sets in topology*, Rend. Circ. Mat. Palermo 19 (2) (1970), 89-96.

[16] Lu, D., and Yu, W., *On prime spectrums of commutative rings*, Comm. Algebra 34 (2006), 2667-2672.

[17] Di Maio, J., *A separation axiom weaker than* R_0, Indian J. Pure Appl. Math. 2 (12) (1985), 373-385.

[18] Maki, H., *Generalized Λ-sets and the associated closure operator*, The Special Issue in Commemoration of Prof. Kazusada IKEDA's Retirement, 1. Oct. 1986, 139-146.

[19] De Marco, G., and Orsatti, A., *Commutative rings in which every prime ideal is contained in a unique maximal ideal*, Proc. Amer. Math. Soc. 30 (3) (1971), 459-466.

[20] Matsumura, H., *Commutative algebra*, second edition, Benjamin, Reading, Mass., 1980.

[21] Mill, J., van, *Supercompactness and Wallman extensions*, Math. Centre Tracts 85, Amsterdam, 1977.

[22] Pérez E., D., *Semi-rings and spectral spaces*, Lecture Notes in Math. 1328 (1988), 219-226.

[23] Picavet, G., *Autor des idéaux premiers de Goldman d'un anneau commutatif*, Ann. Univ. Clermont-Ferrand 44 (1975), 73-90.

Alirio Peña
Departamento de Matemáticas,
Facultad Experimental de Ciencias, Universidad del Zulia,
Maracaibo, Venezuela
e-mail: apena@luz.edu.ve

Luz M. Ruza
Departamento de Matemáticas,
Núcleo Rafael Rangel, Universidad de Los Andes,
Trujillo, Venezuela
e-mail: ruza@ula.ve

Jorge Vielma
Departamento de Matemáticas,
Facultad de Ciencias, Universidad de Los Andes
Mérida 5101, Venezuela
e-mail: vielma@ula.ve