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Geometrical Inequalities Unconventional Demonstrated

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Abstract

In this paper we have synthetized more geometric inequalities at which we can give unconventional solutions. The applications and some solutions are selected from the bibliography attached to this paper. We consider that the solutions that we can give to the presented inequalities are interesting and can be used also in other cases.

The applications are divided into three categories. For the first category will be used Jensen' s inequality. For second class, will transform the inequalities in optimization problems with restrictions. And to the third category, will be used in the proofs, the vectorial calculation.

key words. Jensen inequality; Optimization problems; Vectorial calculation; Geometric inequalities.

AMS subject classifications. 97D10.

I. Applications of Jensen' s Inequality

"The geometry everywhere exists". Leibniz

Theorem 1 *(the Jensen' s inequality)*

If $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is a convex (concave) function, then for all $x_1, \ldots, x_n \in I$ and for all $q_1, \ldots, q_n \in \mathbb{R}_+$ with $q_1 + \ldots + q_n = 1$ the following inequality take place:

$$f(q_1x_1 + \ldots + q_nx_n) \leq (\geq) q_1f(x_1) + \ldots + q_nf(x_n)$$

1. *In a convex polygon $A_1A_2 \ldots A_n$, there is relationship:*

$$\sum_{k=1}^{n} \sin^\alpha A_k \leq n \sin^\alpha \frac{2\pi}{n}$$

for all $0 < \alpha < 1$. ([1])

Proof. We use the function $f:(0,\pi)\to\mathbb{R},\quad f(x)=\sin^{\alpha}x$. From Jensen' s inequality we have

$$\sin^{\alpha}\left(\frac{1}{n}\sum_{k=1}^{n}A_k\right)=\sin^{\alpha}\left(\pi-\frac{2\pi}{n}\right)\geq\frac{1}{n}\sum_{k=1}^{n}\sin^{\alpha}A_k.$$

2. *In any triangle ABC, with $A,\quad B,\quad C\in\left(0,\frac{\pi}{2}\right)$, occurring relations ([2]):*

a) $\sin A+\sin B+\sin C\leq\frac{3\sqrt{3}}{2}$*;*

b) $\cos A\cos B\cos C\leq\frac{1}{8}$

c) $\cos A+\cos B+\cos C\leq\frac{3}{2}$*;*

d) $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\leq\frac{1}{8}$

e) $\sqrt{\cos A}+\sqrt{\cos B}+\sqrt{\cos C}\leq\frac{3\sqrt{3}}{2}$*;*

f) $tgA+tgB+tgC\geq 3\sqrt{3}$*;*

g) $a+b+c\geq\frac{a^2}{b+c-a}+\frac{b^2}{a+c-b}+\frac{c^2}{a+b-c}$;

h) $r_a+r_b+r_c\geq 9r$.

Proof. We use: a) $f(x)=\sin x$; b) $f(x)=\ln\cos x$; c) $f(x)=\cos x$; $f(x)=\ln\sin\frac{x}{2}$; e) $f(x)=\sqrt{\cos x}$; f) $f(x)=\mathrm{tg}x$; g) $f(x)=\frac{x^2}{2p-2x}-x$;
h) $r_a=\frac{S}{p-a},\ r=\frac{S}{p},\ f(x)=\frac{p}{p-x}$.

3. *If a,b,c,d are the sides of an convex polygon, then ([2]):*

$$\frac{a}{b+c+d-a}+\frac{b}{a+c+d-b}+\frac{c}{a+b+d-c}+\frac{d}{a+b+c-d}\geq 2.$$

Proof. We use $f(x)=\frac{x}{2p-2x},\ x\in(0,p)$.

4.*Let $P_1P_2\dots P_n$ a polygon. If $a_i=P_iP_{i+1},i=\overline{1,n},n\geq 3,P_{n+1}=P_1$,*

$p \in [1, \infty),\ m = \dfrac{\sum_{i=1}^{n} a_i}{n}$, *then:*

$$\frac{a_1^P}{-a_1^p + a_2^p + \ldots + a_n^p} + \frac{a_2^p}{a_1^p - a_2^p + \ldots + a_n^p} + \ldots + \frac{a_n^p}{a_1^p + a_2^p + \ldots - a_n^p} \geq \frac{n}{2} \frac{m^p}{s - m^p}$$

Proof. We use the convex function $f : (0, s^{\frac{1}{p}}) \to \mathbb{R},\ f(x) = \dfrac{x^p}{s - x^p}$.

Let $a_1, a_2, \ldots, a_n \in (0, s^{\frac{1}{p}})$. Then:

$$2 \sum_{i=1}^{n} \frac{a_i^p}{s - a_i^p} \geq n \frac{m^p}{s - m^p},$$

which is equivalent with the inequality from the application.

5. *Let $A_1A_2 \ldots A_n$ a convex polygon with area S and sides $a_1 = A_1A_2, \ldots, a_n = A_nA_1$ and M an arbitrary point inside the polygon. If $d_1, d_2, \ldots, d_n$ are the distances from this point at the sides $A_1A_2, A_2A_3, \ldots, A_nA_1$ then*

$$\sum_{i=1}^{n} \frac{1}{a_i^\alpha d_i^\alpha} \geq \frac{n^{\alpha+1}}{2^\alpha S^\alpha},$$

for all $\alpha \in \mathbb{R}_+^*$; the minimum is reached for the center of weight of the polygon.

Proof. We note $S_1, S_2, \ldots, S_n$ the areas of triangles $A_1MA_2, A_2MA_3, \ldots, A_nMA_1$. The inequality becomes

$$\sum_{i=1}^{n} \frac{1}{S_i^\alpha} \geq \frac{n^{\alpha+1}}{S^\alpha}.$$

We consider $f : \mathbb{R}_+^* \to \mathbb{R}_+^*,\ f(x) = \dfrac{1}{x^\alpha}$(a strictly convex function) and $q_1 = q_2 = \ldots = q_n$. From Jensen's inequality, we have: $f\left(\sum_{i=1}^{n} q_i S_i\right) \leq \sum_{i=1}^{n} q_i f(S_i)$, or $\dfrac{n^{\alpha+1}}{S^\alpha} \leq \sum_{i=1}^{n} \dfrac{1}{S_i^\alpha}$, because $\sum_{i=1}^{n} S_i = S$.

6. ([2]) *To show that in any ABC triangle, with the angles $a < b < c$, we have the double inequality*

$$-\frac{1}{2} < \frac{\sin a}{(a-b)(a-c)} + \frac{\sin b}{(b-c)(b-a)} + \frac{\sin c}{(c-a)(c-b)} < 0.$$

Proof. Relationship required is equivalent to

$$-\frac{1}{2} < (b-a)(c-b) < \frac{c-b}{c-a} \sin a - \sin b + \frac{b-a}{c-a} \sin c < 0.$$

The function $f:[0,\pi]\to[0,1],\ f(x)=\sin x$ is strictly convex.
Thus, we note $\lambda=\frac{c-b}{c-a}\in(0,1),\ 1-\lambda=\frac{b-a}{c-a}$ and $\lambda a+(1-\lambda)c=b$.

Thus, implies: $f[\lambda a+(1-\lambda)c]>f(a)+(1-\lambda)f(b)$ or

$$0>-\sin b+\frac{c-b}{c-a}\sin a+\frac{b-a}{c-a}\sin c.$$

Let $E=-\sin b+\lambda\sin a+\sin c-\lambda\sin c=2\sin\frac{c-b}{2}\cos\frac{b+c}{2}-2\lambda\sin\frac{c-a}{2}\cos\frac{a+c}{2}$
$=(c-b)\left[g\left(\frac{c-b}{2}\right)\sin\frac{a}{2}-g\left(\frac{c-a}{2}\right)\sin\frac{b}{2}\right]$,
where $g:(0,\pi)\to\mathbb{R}_+,\ g(x)=\frac{\sin x}{x}$ is strictly descending. We have $g\left(\frac{c-b}{2}\right)>g\left(\frac{c-a}{2}\right)$, which implies:

$$E>(c-b)g\left(\frac{c-b}{2}\right)\left(\sin\frac{a}{2}-\sin\frac{b}{2}\right)=-4\sin\frac{c-b}{2}\sin\frac{b-a}{4}\cos\frac{a+b}{4}.$$

But, $\sin\frac{c-b}{2}<\frac{c-b}{2},\ \sin\frac{b-a}{4}<\frac{b-a}{4}$ and $\cos\frac{a+b}{4}<1$, so,

$$E>-4\cdot\frac{c-b}{2}\cdot\frac{b-a}{4}=-\frac{1}{2}(c-b)(c-a).$$

II. In this section, we will transform the inequalities in optimization problems with restrictions.

"Every problem contains within itself the seeds of its own solution" Edward Somers

We use in demonstrations, the following theorem:

Theorem 2 *(Karush-Kuhn-Tucker [3])*

Let the problem

$$(8)\begin{cases}\min\limits_{x\in M} f(x)\\ g_i(x)\le 0,\ i=\overline{1,p}\\ h_k(x)=0\ k=\overline{1,q}\end{cases},$$

where M is a nonempty subset form $\mathbb{R}^n$ space and f,g_i,h_k are the domain M for $i=\overline{1,p},\ k=\overline{1,q}$

Let $S=\{x\in M|g_i(x)\le 0\ i=\overline{1,p},\ h_k(x)=0,\ k=\overline{1,q}\}$ the admissible solutions set for problem (8).

If $x^0\in S$ and the following conditions take place:

(i) $x^0\in$ int M;

(ii) the function f is differentiable in x^0;

(iii) the functions $g_1, \dots, g_p, h_1, \dots, h_q$ are partial differentiable in relation to each variable in x^0;

(iv) for each $y \in \mathbb{R}^n$ $\{0\}$ which is a solution for the following system:

$$\begin{cases} < y, \triangledown g_i(x^0) > \leq 0, & i \in I(x^0) = \{i | g_i(x^0) = 0\} \\ < y, \triangledown h_k(x^0) > = 0, & k = \overline{1, q} \end{cases}$$

there is a number $t_0 > 0$ and a function $\gamma : [0, t_0] \to \mathbb{R}^n$ derivable in origin, so: $\gamma(t) \in S$ for each $t \in [0, t_0]$;

$\gamma(0) = x^0$; $\dfrac{d\gamma}{dt}(0) = y$.

If x^0 is a local minimum point of function f relative to S, then there is a point $(v^0, w^0) \in \mathbb{R}^p_+ \times \mathbb{R}^q$ for which will be true the condition of Kuhn - Tucker:

a) $g_i(x^0) v_i^0 = 0, \quad i = \overline{1, p}$;

b) $\triangledown f(x^0) + \sum\limits_{i=1}^{p} v_i^0 \triangledown g_i(x^0) + \sum\limits_{k=1}^{q} w_k^0 \triangledown h_k(x^0) = 0.$

7. [4] *To show that in any ABC rectangular triangle with the sides a, b, c the following inequality take place:*

$$(a + b + c)(ab + bc + ca) > kabc, \qquad k = 5 + 3\sqrt{2}. \tag{1}$$

Proof. We consider the following problem of optimization:

$$\begin{cases} \min \left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3} \right) \\ x_1^2 \geq x_2^2 + x_3^2 \\ x_1 + x_2 + x_3 = 1 \\ x_1 > 0, \ x_2 > 0, \ x_3 > 0 \end{cases}$$

and let: $M = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : \ x_1 > 0, \ x_2 > 0, \ x_3 > 0\}$ and $f, g, h : M \to \mathbb{R}$,

$f(x) = \dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3}$;

$g(x) = -x_1^2 + x_2^2 + x_3^2$;

$h(x) = x_1 + x_2 + x_3 - 1$.

The inequality (1) is equivalent with the following problem:

$$\min\{f(x) : x \in M, \ g(x) \leq 0, h(x) = 0\}.$$

We demonstrate that the problem has at least one solution. Let S be the set of possible solutions.

$$S\left(\frac{47}{5}\right)=\left\{x\in S: f(x)\leq \frac{47}{5}\right\}\neq\varnothing \quad \left(\ni\left(\frac{5}{12},\frac{4}{12},\frac{3}{12}\right)\right)$$

is bounded and closed.

$\Rightarrow$ f has at least one minimum point relating to S $\Rightarrow$ the problem has at least one solution.

Following we demonstrate that the problem has actually just one solution. Let $x^0=(x_1^0,x_2^0,x_3^0)$ a problem′s solution.

$$\Rightarrow\ni (v^0,w^0)\in\mathbb{R}_+\times\mathbb{R} \text{ so that } \begin{cases} v^0[-(x_1^0)^2+(x_2^0)^2+(x_3^0)^2]=0\\ -\dfrac{1}{(x_1^0)^2}-2v^0x_1^0+w^0=0\\ -\dfrac{1}{(x_2^0)^2}+2v^0x_2^0+w^0=0\\ -\dfrac{1}{(x_3^0)^2}+2v^0x_3^0+w^0=0. \end{cases}$$

In other words $(x_1^0,x_2^0,x_3^0,v^0,w^0)$ is a solution of the system

$$\begin{cases} v(-x_1^2+x_2^2+x_3^2)=0\\ -\dfrac{1}{(x_1^2)^2}-2vx_1+w=0\\ -\dfrac{1}{(x_2^2)^2}-2vx_2+w=0\\ -\dfrac{1}{(x_3^2)^2}-2vx_3+w=0\\ (x_1,x_2,x_3)\in S, \quad v\geq 0. \end{cases}$$

Case I: $v=0$

$\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)\notin S$ $\Rightarrow$ the system has no solution with $v=0$.

$$\text{Case II: } v>0 \begin{cases} -(x_1)^2+(x_2)^2+(x_3)^2=0\\ -\dfrac{1}{(x_1^2)^2}-2vx_1+w=0\\ -\dfrac{1}{(x_2^2)^2}-2vx_2+w=0\\ -\dfrac{1}{(x_3^2)^2}-2vx_3+w=0\\ x_1+x_2+x_3=1. \end{cases}$$

$x^0=(-1+\sqrt{2},1-\frac{\sqrt{2}}{2},1-\frac{\sqrt{2}}{2})$.

8. [4] $ab+bc+ca\leq k(a+b+c)^2, \quad k=-\dfrac{5}{2}+2\sqrt{2}$

Proof. $a>b,\ a>c,\ \ a^2>b^2+c^2$.

$t = \left(\frac{a}{a+b+c}, \frac{b}{a+b+c}, \frac{c}{a+b+c}, \right) = (x,y,z) =$ admissible solution for the problem.

$$\begin{cases} \max(xy+yz+zx) \\ x^2 \geq y^2+z^2 \\ x+y+z=1 \\ x \geq 0,\ y \geq 0,\ z \geq 0. \end{cases}$$

In the maximum point the function′s value is k.

$\left.\begin{array}{l} S - \text{ nonempty} \\ S - \text{ compact} \\ f - \text{ continuous} \end{array}\right\} \Rightarrow$ the problem has at least a solution. We propose to determinate the solution.

$M = \mathbb{R}^3_+$.

$f(x) = -(xy+yz+zx)$,

$g(x) = -x^2+y^2+z^2$,

$g(x) = x+y+z-1$.

$\begin{cases} \bigtriangledown f(t_0) + v^0 \bigtriangledown g(t^0) + w^0 \bigtriangledown (t^0) = 0 \\ v^0 g(t^0) = 0. \end{cases} \Rightarrow (x_0, y_0, z_0, v^0, w^0)$ is solution for the following system:

$$\begin{cases} v(-x^2+y^2+z^2) = 0 \\ y+z+2vx-w=0 \\ x+y-2vz-w=0 \\ (x,y,z) \in S,\ \geq 0. \end{cases}$$

Case I: $v = 0$ $\quad \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right) \notin S$

Case II: $v > 0$ $\quad x_0 = -1+\sqrt{2},\ y_0 = z_0 = \frac{1-\sqrt{2}}{2},\ v^0 = \frac{3}{2},\ w^0 = 4\sqrt{2}-5.$

$\Rightarrow\ t^0 = (x_0, y_0, z_0) = \left(-1+\sqrt{2}, \frac{1-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2} \right)$ is the only solution of the problem.

$\Rightarrow$ the maximum for the propose function is $(xy+yz+zx)_{t^0} = k \quad \Rightarrow xy+yz+zx \leq k$.

9. [4] $\begin{cases} \max(x_1, x_2, x_3) \\ (x_1)^2 \geq (x_2)^2 + (x_3)^2 \\ x_1+x_2+x_3 = 1 \\ x_1 > 0,\ x_2 > 0,\ x_3 > 0. \end{cases}$

$(a+b+c)^3 > kabc, \quad \frac{a}{a+b+c} \cdot \frac{b}{a+b+c} \cdot \frac{c}{a+b+c} < \frac{1}{k} = \frac{5\sqrt{2}-7}{2}.$

$x^0 = \left(-1+\sqrt{2}, \frac{1-\sqrt{2}}{2}, \frac{1-\sqrt{2}}{2} \right)$ is the only solution for the problem.

III. The method of vectorial calculation

"Mathematics don't posses only the true, but the supreme beauty, as the sculpture does. "

Bertrand Russel

10. [5] *Demonstrate the inequality*

$$\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{ab}$$

where a, b, c are the sides for a triangle, and $a > c,\ b > c$.

Proof. We choose the vectors $\overrightarrow{v_1} = (\sqrt{c}, \sqrt{b-c})$ and $\overrightarrow{v_2} = (\sqrt{a-c}, \sqrt{c})$.

$$|\overrightarrow{v_1} \cdot \overrightarrow{v_2}| \leq ||\overrightarrow{v_1}|| \cdot ||\overrightarrow{v_2}|| \Leftrightarrow \sqrt{c} \cdot \sqrt{a-c} + \sqrt{b-c} \cdot \sqrt{c}$$

$$\leq \sqrt{(\sqrt{c})^2 + (\sqrt{b-c})^2} \cdot \sqrt{(\sqrt{a-c})^2 + (\sqrt{c})^2}$$

which is equivalent with $\sqrt{c(a-c)} + \sqrt{c(b-c)} \leq \sqrt{ab}$.

11. [5] *Demonstrate that, if a, b, c are the sides of a triangle with perimeter 1, the following inequalities take place:*

$$2 < \sqrt{a^2 + (b+c)^2} + \sqrt{b^2 + (c+a)^2} + \sqrt{c^2 + (a+b)^2} < 3.$$

Proof. We choose the vectors

$$\overrightarrow{v_1} = (1,1),\ \overrightarrow{v_2} = (1,1),\ \overrightarrow{v_3} = (1,1),$$

$$\overrightarrow{w_1} = (a, b+c), \overrightarrow{w_2} = (b, a+c), \overrightarrow{w_3} = (c, a+b). \tag{2}$$

From relations (2) results:

$$\sqrt{|\overrightarrow{v_1} \cdot \overrightarrow{w_1}|} \leq \sqrt{||\overrightarrow{v_1}|| \cdot ||\overrightarrow{w_1}||};$$

$$\sqrt{|\overrightarrow{v_2} \cdot \overrightarrow{w_2}|} \leq \sqrt{||\overrightarrow{v_2}|| \cdot ||\overrightarrow{w_2}||};$$

$$\sqrt{|\overrightarrow{v_3} \cdot \overrightarrow{w_3}|} \leq \sqrt{||\overrightarrow{v_3}|| \cdot ||\overrightarrow{w_3}||}. \tag{3}$$

Summarizing member by member, form relations (2) we obtain:

$$\sqrt{a+(b+c)} + \sqrt{b+(c+a)} + \sqrt{c+(a+b)}$$

$$\leq \sqrt{2}\sqrt{a^2+(b+c)^2} + \sqrt{2}\sqrt{b^2+(c+a)^2} + \sqrt{2}\sqrt{c^2+(a+b)^2}.$$

Replacing $a+b+c$ with 1, we have:

$$\frac{3}{\sqrt{2}} \leq \sqrt{a^2+(b+c)^2} + \sqrt{b^2+(c+a)^2} + \sqrt{c^2+(a+b)^2}. \tag{4}$$

But $\dfrac{3}{\sqrt{2}} \geq 2$, so

$$2 < \sqrt{a^2+(b+c)^2} + \sqrt{b^2+(c+a)^2} + \sqrt{c^2+(a+b)^2} \tag{5}$$

On the other sides, we choose the vectors:

$$\overrightarrow{x_1} = (a,b), \overrightarrow{x_2} = (b,c), \overrightarrow{x_3} = (c,a)$$

$$\overrightarrow{y_1} = (0,c), \overrightarrow{y_2} = (0,a), \overrightarrow{y_3} = (0,b) \tag{6}$$

Summarizing member by member and putting the coordinates from the relations (5), we obtain:

$$\sqrt{a^2+(b+c)^2} + \sqrt{b^2+(c+a)^2} + \sqrt{c^2+(a+b)^2}$$

$$\leq \sqrt{a^2+b^2} + \sqrt{c^2} + \sqrt{b^2+c^2} + \sqrt{a^2} + \sqrt{c^2+a^2} + \sqrt{b^2} =$$

$$\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{c^2+a^2} + a + b + c. \tag{7}$$

Because a, b, c are positive and at least one of them is strictly bigger than 0, we obtain:

$$\sqrt{a^2+b^2} \leq a+b, \quad \sqrt{b^2+c^2} \leq b+c, \quad \sqrt{c^2+a^2} \leq c+a$$

$$\sqrt{a^2+b^2} + \sqrt{b^2+c^2} + \sqrt{c^2+a^2} \leq 2(a+b+c). \tag{8}$$

From relations (7) and (8) it results:

$$\sqrt{a^2+(b+c)^2} + \sqrt{b^2+(c+a)^2} + \sqrt{c^2+(a+b)^2} < 3(a+b+c) = 3. \tag{9}$$

From relations (5) and (9) we obtain the double inequality from the problem.

12. [4] *Let $a, b, c, d \geq 0$. Show that:*

$$\sqrt{a+b+c+d}+\sqrt{b+c+d}+\sqrt{c+d}+\sqrt{d} \geq \sqrt{a+4b+9c+16d}.$$

Proof. Let the vectors

$$\overrightarrow{v_1}=(\sqrt{a},\sqrt{b},\sqrt{c},\sqrt{d}), \overrightarrow{v_2}=(0,\sqrt{b},\sqrt{c},\sqrt{d}), \overrightarrow{v_3}=(0,0,\sqrt{c},\sqrt{d}), \overrightarrow{v_4}=(0,0,0,\sqrt{d}).$$

We have

$$||\overrightarrow{v_1}||+||\overrightarrow{v_2}||+||\overrightarrow{v_3}||+||\overrightarrow{v_4}|| \geq ||\overrightarrow{v_1}+\overrightarrow{v_2}+\overrightarrow{v_3}+\overrightarrow{v_4}|| \tag{10}$$

so,

$$\sqrt{(\sqrt{a})^2+(\sqrt{b})^2+(\sqrt{c})^2+(\sqrt{d})^2}+\sqrt{(\sqrt{b})^2(\sqrt{c})^2(\sqrt{d})^2}+\sqrt{(\sqrt{c})^2(\sqrt{d})^2}+\sqrt{(\sqrt{d})^2}$$

$$\geq \sqrt{(\sqrt{a})^2+(2\sqrt{b})^2+(3\sqrt{c})^2+(4\sqrt{d})^2} \tag{11}$$

relation that is equivalent with the one from the statement.

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