Revista Notas de Matemática
Vol.6(2), No. 297, 2010, pp. 58-62
http://www.saber.ula.ve/notasdematematica
Pre-prints
Departamento de Matemáticas
Facultad de Ciencias
Universidad de Los Andes

On some series of functions

Amelia Bucur and José Luis López-Bonilla

Abstract

The study of the Fourier series convergence is a frequent subject in the speciality literature (see [3], [4]). It is known that the Fourier series convergence is very useful in the technical problems: the electromagnetic field and the loses in the case of an elliptic screen; the calculus of the rectangular plates, the simple leant beam actioned by a movable charge, the dynamic system of the metal bars in which the current crosses, the torsion of a rectangular bar, the quasi-stationary electromagnetic field, the motion of an elastic bar (see[1]). In this paper we will present some results concerning the Fourier series.

key words. Fourier series, Convergence.

AMS subject classifications. 47A10

1 Introduction

Let $c_0, c_1, ..., c_n, ...$ be a sequence of real constants, and $f_0, f_1, ..., f_n, ...$ a sequence of functions defined, for example, in the interval (0,1). In this paper we investigate some of the properties of the series

$$\sum_{n=0}^{\infty}[\underline{+}c_n f_n(x)] \tag{1}$$

which may be obtained from the standard series

$$\sum_{n=0}^{\infty} c_n f_n(x) \tag{2}$$

by interchanging the signs of the terms in a quite arbitrary way. The series (1) may be written in a slightly different way. Let us define on [0,1) a sequence of functions $\phi_0, \phi_1, ..., \phi_n, ...$ by the following conditions:

$\phi_n(t) = \phi_0(2^n t)(n = 1, 2, ...)$, where $\phi_0(t) = 1$ for $0 < t < \frac{1}{2}; \phi_0(t) = -1$ for $\frac{1}{2} < t < 1; \phi_0(0) = \phi_0(\frac{1}{2}) = 0; \phi_0(t+1) = \phi_0(t)$. In other words the functions $\phi_n(t)$ may be obtained by dividing the interval (0, 1) into 2^{n+1} equal sub-intervals, and putting $\phi_n(t) = 1, \phi_n(t) = -1$ in the interior of alternate intervals, and $\phi_n(t) = 0$ at the end points of the intervals. The functions ϕ_n are Rademacher functions. They form a normal orthogonal system for the interval (0, 1), i.e.

$$\int_0^1 \phi_m(t)\phi_n(t)dt \begin{cases} = 0 & , m \neq n \\ = 1 & , m = n \end{cases}$$

To every infinite sequence consisting of the signs $\pm$ there corresponds a sequence $\alpha_0, \alpha_1, ..., \alpha_n, ...$ whose terms are either 0 or 1, and which is obtained from the first sequence by replacing $(+)$ by 0 and (-) by 1. Hence to every sequence of $\pm$ there corresponds a real number $0 \leq t \leq 1$, whose development in the scale of notation 2 is

$$0\ \alpha_0\ \alpha_1\ \alpha_2.... \tag{3}$$

Conversely to every t from the interval (0,1) there corresponds an expansion of the form (3) and thus a sequence of signs $\pm$. Thus neglecting a denumerable set of t which possesses two developments (3), we have a one-to-one correspondence between the sequences $\pm$ and the real numbers $0 \leq t \leq 1$, and (neglecting this denumerable set) all the series (1) can be written in the form of a single series

$$\sum_{n=0}^{\infty} c_n f_n(x)\ \phi_n(t) \tag{4}$$

depending on the variable x as well as on the parameter t.

Let us denote, quite formally, the series (4) by $S_t(x)$. If a property (P) is a possessed by all the series $S_t(x)$, except perhaps for values of t enclosed in a set of measure zero, then we shall say that almost all the series (4)(or (1)) enjoy the property (P).

Let us consider a property (P) and lat E be the (measurable) set of t $(0 \leq t \leq 1)$ for which the series $S_t(x)$ possesses the property (P). If (P) is of such a character that, if we change a finite number of terms of a series $S_t(x)$ which has the property (P), the new series must also possess the property (e.g. if (P) denotes the convergence of $S_{t_0}(x)$ for almost all x), then, as is well known, the measure of E must be either 1 or 0.

In this paper we consider some properties of almost all series (4). It will appear that in all the problems a decisive role is played by the convergence or divergence of the series $\sum c_n^2$. We suppose, except where otherwise stated, that c_n, f_n are real.

2 Main result

Lemma 1 *Let $\phi_0, \phi_1, ..., \phi_n, ...$ denote Rademacher's functions, and let the series*

$$\sum_{n=0}^{\infty} c_n^2$$

be convergent. Then the series

$$\sum_{n=0}^{\infty} c_n \phi_n(t) \tag{5}$$

converges almost everywhere.

Proof. See[2]. This result was proved by Rademacher. A very simple proof has been given by Kolmogoroff.

From Lemma it is easy to deduce the following theorem.

Theorem 1 *Let $f_0, f_1, ..., f_n, ...$ be a system of real functions defined in the interval (0, 1), and satisfying the inequality*

$$\int_0^1 f_n^2(x)dx \le A \quad n \in \mathbb{N}. \tag{6}$$

We suppose that

$$\sum_{n=0}^{\infty} c_n^2 < \infty$$

Then almost all the series (1) converge for almost all x, to a sum of function which belongs to the class L^2.

Proof. From the convergence of $\sum c_n^2$ and from (6) it follows that the series

$$\sum_{n=0}^{\infty} c_n^2 f_n^2(x)$$

converges in a set E of measure 1. For every x contained in E, we see by Lemma that the series

$$\sum_{n=0}^{\infty} c_n f_n(x)\ \phi_n(t) \tag{7}$$

converges for $t \in F_x$, where F_x is a set contained in the interval (0, 1) and $m(F_x) = 1$. Let us denote by P the plane set of points contained in the square $0 \leq x \leq 1, 0 \leq t \leq 1$, for which the series (7) is convergent. Then we have

$$m(P) = \int_0^1 m(F_x)dx = 1.$$

We denote by s(x; t) $(0 \leq x, t \leq 1)$ the sum of the series (7), where this sum exists. If we prove that

$$\int_0^1 \int_0^1 [s(x,t)]^2 dx\ dt < \infty, \tag{8}$$

then it will follow by a well-known theorem that $\int_0^1 [s(x,t)]^2 dx$ exists for almost all values of t. In order to prove (8), it is sufficient, by Faton's well-known lemma, to prove that

$$\int_0^1 \int_0^1 [s_N(x;t)]^2 dx\ dt = 0(1), \tag{9}$$

where $s_N(x;t)$ denotes the sum

$$\sum_{n=0}^{N} c_n f_n(x) \phi_n(t).$$

But the integral on the left-hand side of (9) is

$$\int_0^1 \{\sum_0^N a_n^2 f_n^2(x)\} dx \leq A \sum_0^\infty a_n^2,$$

and this proves the required result.

Remark 1 *The results remain true even if the numbers c_n and the functions f_n are allowed to assume complex values, provided that we replace the condition $\sum c_n^2 < \infty$ by $\sum |c_n|^2 < \infty$. In the particular, if we write $f_n(x) = e^{inx}$, and replace the interval $(0 \leq x \leq 1)$ by $(0 \leq x \leq 2\pi)$ we obtain the following result.*

Corollary 1 *We suppose that $\sum |c_n|^2 \leq 1$ and write*

$$S_t(z) = \sum_{n=0}^{\infty} c_n z^n \phi_n(t) \quad (|z| \leq 1).$$

Then, for almost all t, we have

$$\int_{0}^{2\pi} exp\{\lambda|S_t(e^{i\phi})|^2\}d\phi < \infty,$$

and thus

$$\int_{0}^{2\pi} exp\{\lambda|S_t(re^{i\phi})|^2\}d\phi = 0(1) \ as \ r \to 1,$$

λ denoting an arbitrary positive number.

References

[1] M. Boncuţ, A. Bucur, *Matematici speciale*, Editura Alma Mater, Sibiu, 2002.

[2] E.V. Dobrescu, L. Dobrescu Purice, *Matematici speciale*, Editura Didactică şi Pedagogică, Bucureşti, 1967.

[3] M. Roşculeţ, *Serii trigonometrice*, Editura Academiei Române, Bucureşti, 1991.

[4] A. Vescan, *Sumabilitatea seriilor*, Editura Tehnică, Bucureşti, 1965.

Amelia Bucur
University "Lucian Blaga" of Sibiu,
Department of Mathematics
Str. Dr.I. Raţiu, nr.7, 550012 - Sibiu, Romania
e-mail: ameliabucur@go.ro

José Luis López-Bonilla
Sección de Estudios de Posgrado e Investigación,
Escuela Superior de Ingeniería Mecánica y Eléctrica
Instituto Politécnico Nacional
Edif. Z-4, 3er piso, Col. Lindavista, CP 07738, México, D.F.
e-mail: lopezbjl@mexico.com