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A study of a new subclass of multivalent analytic functions

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Abstract

After introducing a new linear differential operator, we introduce and study certain new subclasses of analytic and multivalent functions in the open unit disk. Some inclusion relationships also discussed in particular with reference to an integral operator.

key words. Analytic functions, Differential operator, Inclusion relationship

AMS(MOS) subject classifications.

1 Introduction and preliminaries

Let $H[a,n]$ be the class of analytic functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots .$$

We define $A(p) \subseteq H[a,n]$ be the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in \mathbb{N} = \{1, 2, 3, \cdots\} \tag{1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$.

A function $f \in A(p)$ is said to belong to the class of p-valent starlike functions of order ξ in $\mathbb{U}$, and is denoted by $S_p^*(\xi)$, if it satisfies

$$\text{Re}\left[\frac{zf'(z)}{f(z)}\right] > \xi, \quad 0 \leq \xi < p, p \in \mathbb{N}, z \in \mathbb{U}. \tag{2}$$

The class $S_p^*(0)$ was introduced by Goodman [2] whereas Patil and Thakare [1] generalized this idea to get the class $S_p^*(\xi)$.

Owa [3] introduced a class $C_p(\xi)$, the class of p-valent convex functions of order ξ in $\mathbb{U}$ as

$$\text{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] > \xi, \quad 0 \leq \xi < p, p \in \mathbb{N}, z \in \mathbb{U}. \tag{3}$$

The class $C_p(0)$ was introduced by Goodman [2].

It follows from (2) and (3) that

$$f(z) \in C_p(\xi) \text{ if and only if } \frac{zf'(z)}{p} \in S_p^*(\xi), 0 \leq \xi < p, p \in \mathbb{N}, z \in \mathbb{U}.$$

A function $f \in A(p)$ is said to belong to the class of p-valent close-to-convex functions of order ρ and type ξ in $\mathbb{U}$, and is denoted by $K_p(\rho,\xi)$, if there exists a function $g(z) \in S_p^*(\xi)$ such that

$$\operatorname{Re}\left[\frac{zf'(z)}{g(z)}\right] > \rho, \quad 0 \leq \rho, \xi < p, p \in \mathbb{N}, z \in \mathbb{U}. \tag{4}$$

The class $K_p(\rho,\xi)$ was studied by Aouf [4] and the class $K_1(\rho,\xi)$ was studied by Libera [5].

Noor [6, 7] introduced and studied the classes $K_p^*(\rho,\xi)$ and $K_1^*(\rho,\xi)$.

A function $f \in A(p)$ is said to belong to the class $K_p^*(\rho,\xi)$ of p-valent quasi-convex functions of order ρ and type ξ in $\mathbb{U}$, if there exists a function $g(z) \in C_p(\xi)$ such that

$$\operatorname{Re}\left[\frac{(zf'(z))'}{g'(z)}\right] > \rho, \quad 0 \leq \rho, \xi < p, p \in \mathbb{N}, z \in \mathbb{U}. \tag{5}$$

Similarly from (4) and (5) we have

$$f(z) \in K_p(\xi) \text{ if and only if } \frac{zf'(z)}{p} \in K_p^*(\xi), 0 \leq \xi < p, p \in \mathbb{N}, z \in \mathbb{U}.$$

For $f \epsilon A(p)$ we define the operator as follows:

$$\begin{aligned}
\Theta_p^0(\beta,\gamma) f(z) &= f(z); \\
(p(\gamma+1)+\beta)\,\Theta_p^1(\beta,\gamma) f(z) &= \beta f(z) + p(\gamma+1)\left(\frac{zf'(z)}{p}\right); \\
&\vdots \\
\Theta_p^n(\beta,\gamma) f(z) &= D\left(\Theta_p^{n-1}(\beta,\gamma)\right).
\end{aligned}$$

This gives rise to

$$\Theta_p^n(\beta,\gamma) f(z) = z^p + \sum_{k=p+1}^{\infty}\left(\frac{\beta+k(\gamma+1)}{\beta+p(\gamma+1)}\right)^n a_k z^k, \quad \beta,\gamma \geq 0, p \in \mathbb{N}. \tag{6}$$

This operator generalizes certain differential operators which already exist in literature as under.

- $\beta=\lambda, \gamma=0$ we get $\Theta_p^n(n,\lambda,0)$ of Aghalary et al. differential operator [10].
- $\beta=\lambda, \gamma=0$ and $p=1$ we get Cho-Kim [11] and Cho-Srivastava [12] differential operator.

- $\beta = 1, \gamma = 0$ and $p = 1$ we get Uralegaddi and Somanatha differential operator [13].
- $\beta = 0, \gamma = 0$ and $p = 1$ we get Salagean differential operator [14].
- $\beta = l, \gamma = 0$ we get Kumar et al. differential operator [15] and Srivastava et al. differential operator [16].

Note that

$$(\gamma+1)\, z\left(\Theta_p^n(\beta,\gamma) f(z)\right)' = (p(\gamma+1)+\beta)\, \Theta_p^{n+1}(\beta,\gamma) f(z) - \beta\Theta_p^n(\beta,\gamma) f(z).$$

Now for this linear operator $\Theta_p^n(\beta,\gamma)$ we define the following classes:

$$S_n^*(p,\xi,\beta,\gamma) = \left\{f \in A(p) : \Theta_p^n(\beta,\gamma) f \in S_p^*(\xi)\right\};$$
$$C_n(p,\xi,\beta,\gamma) = \left\{f \in A(p) : \Theta_p^n(\beta,\gamma) f \in C_p(\xi)\right\};$$
$$K_n(p,\rho,\xi,\beta,\gamma) = \left\{f \in A(p) : \Theta_p^n(\beta,\gamma) f \in K_p(\rho,\xi)\right\};$$

and

$$K_n^*(p,\rho,\xi,\beta,\gamma) = \left\{f \in A(p) : \Theta_p^n(\beta,\gamma) f \in K_p^*(\rho,\xi)\right\}.$$

Next, we establish the various inclusion relationships for these classes and investigate an integral operator in these classes.

2 Inclusion relationships

In order to prove our main results, we require the following lemma.

Lemma 2.1 *[8, 9] Let $\varphi(\mu,\upsilon)$ be a complex function such that $\varphi : D \to \mathbb{C}$, $D \subset \mathbb{C}\times\mathbb{C}$, and let $\mu = \mu_1 + i\mu_2$, $\upsilon = \upsilon_1 + i\upsilon_2$. Suppose that $\varphi(\mu,\upsilon)$ satisfies the following conditions:*

i. *$\varphi(\mu,\upsilon)$ is continuous in D;*

ii. *$(1,0) \in D$ and $\operatorname{Re}\varphi(1,0) > 0$;*

iii. *$\operatorname{Re}\varphi(i\mu_2,\upsilon_1) \le 0$ for all $(i\mu_2,\upsilon_1) \in D$ such that $\upsilon_1 \le -\frac{1}{2}(1+\mu_2^2)$.*

Let $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ be analytic in $\mathbb{U}$, such that $(h(z), zh'(z)) \in D$ for all $z \in \mathbb{U}$. If $\operatorname{Re}\{\varphi(h(z), zh'(z))\} > 0 (z \in \mathbb{U})$, then $\operatorname{Re}\{h(z)\} > 0$ for $z \in \mathbb{U}$.

Theorem 2.1 *Let $f \in A(p)$, $0 \leq \xi < 1$, $\beta, \gamma \geq 0, n \in \mathbb{N}$ then*

$$S^*_{n+1}(p,\xi,\beta,\gamma) \subseteq S^*_n(p,\xi,\beta,\gamma) \subseteq S^*_{n-1}(p,\xi,\beta,\gamma).$$

Proof 2.1 *Let $f(z) \in S^*_{n+1}(p,\xi,\beta,\gamma)$. Now to prove that $f(z) \in S^*_n(p,\xi,\beta,\gamma)$ it is enough to show that*

$$\mathrm{Re}\left(\frac{z(\Theta_p^n(\beta,\gamma) f(z))'}{\Theta_p^n(\beta,\gamma) f(z)}\right) > \xi,\ 0 \leq \xi < 1, z \in \mathbb{U}.$$

We assume that

$$\frac{z(\Theta_p^n(\beta,\gamma) f(z))'}{\Theta_p^n(\beta,\gamma) f(z)} = \xi + (p-\xi)h(z),\ 0 \leq \xi < 1, z \in \mathbb{U}. \tag{7}$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Simultaneously applying (6) and (7) we conclude that

$$\left(p + \frac{\beta}{\gamma+1}\right)\frac{\Theta_p^{n+1}(\beta,\gamma) f(z)}{\Theta_p^n(\beta,\gamma) f(z)} = \xi + (p-\xi)h(z) + \frac{\beta}{\gamma+1},\ 0 \leq \xi < 1, z \in \mathbb{U}. \tag{8}$$

Differentiating (8) with respect to z logarithmically, we obtain

$$\frac{z(\Theta_p^{n+1}(\beta,\gamma) f(z))'}{\Theta_p^{n+1}(\beta,\gamma) f(z)} - \xi = (p-\xi)h(z) + \frac{(\gamma+1)(p-\xi)zh'(z)}{\beta + (\gamma+1)\xi + (p-\xi)h(z)},$$

$$L = \frac{\alpha+\beta}{\mu+\upsilon}, 0 \leq \xi < 1, z \in \mathbb{U}.$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ and $zh'(z) = \upsilon = \upsilon_1 + i\upsilon_2$, we define the function $\varphi(\mu,\upsilon)$ by:

$$\varphi(\mu,\upsilon) = (p-\xi)\mu + \frac{(\gamma+1)(p-\xi)\upsilon}{\beta + (\gamma+1)\xi + (p-\xi)\mu}. \tag{9}$$

Then from (9) we have

i. $\varphi(\mu,\upsilon)$ *is continuous in* $D = (\mathbb{C} - \frac{\beta+(\gamma+1)\xi}{\xi-p}) \times \mathbb{C}$;

ii. $(1,0) \in D$ *and* $\mathrm{Re}\{\varphi(1,0)\} > p - \xi$;

For the third condition we proceed as follows:

$$\mathrm{Re}\{\varphi(i\mu_2,\upsilon_1)\} = \frac{[p + (\gamma+1)\xi](\gamma+1)(p-\xi)\upsilon_1}{(p+(\gamma+1)\xi)^2 + (p-\xi)^2\mu_2^2},$$

which implies

$$\mathrm{Re}\{\varphi(i\mu_2,\upsilon_1)\} \leq -\frac{[p + (\gamma+1)\xi](\gamma+1)(p-\xi)(1+\mu_2^2)}{(p+(\gamma+1)\xi)^2 + (p-\xi)^2\mu_2^2} < 0.$$

Hence, the function $\varphi(\mu,\upsilon)$ satisfies the conditions of Lemma 2.1. Therefore $\mathrm{Re}\{h(z)\} > 0 (z \in \mathbb{U})$, *that is, $f(z) \in S^*_n(p,\xi,\beta,\gamma)$. This completes the proof of Theorem 2.*

Theorem 2.2 *Let $f \in A(p)$, $0 \le \xi < 1$, $\beta, \gamma \ge 0, n \in \mathbb{N}_0$ then*

$$C_{n+1}(p,\xi,\beta,\gamma) \subseteq C_n(p,\xi,\beta,\gamma) \subseteq C_{n-1}(p,\xi,\beta,\gamma).$$

Proof 2.2 *Let* $f \in C_{n+1}(p,\xi,\beta,\gamma) \Rightarrow \Theta_p^{n+1}(\beta,\gamma) f \in C_p(\xi) \Leftrightarrow \frac{z}{p}(\Theta_p^{n+1}(\beta,\gamma) f)' \in S_p^*(\xi) \Rightarrow \Theta_p^{n+1}(\beta,\gamma)(\frac{zf'}{p}) \in S_p^*(\xi) \Rightarrow \frac{zf'}{p} \in S_{n+1}^*(p,\xi,\beta,\gamma) \subseteq S_n^*(p,\xi,\beta,\gamma) \Rightarrow \frac{zf'}{p} \in S_n^*(p,\xi,\beta,\gamma) \Rightarrow \Theta_p^{n}(\beta,\gamma)(\frac{zf'}{p}) \in S_p^*(\xi) \Rightarrow \frac{z}{p}(\Theta_p^{n}(\beta,\gamma) f)' \in S_p^*(\xi) \Leftrightarrow \Theta_p^{n}(\beta,\gamma) f \in C_p(\xi)$.

Theorem 2.3 *Let $f \in A(p)$, $0 \le \xi < 1$, $\beta, \gamma \ge 0, n \in \mathbb{N}_0$ then*

$$K_{n+1}(p,\xi,\beta,\gamma) \subseteq K_n(p,\xi,\beta,\gamma) \subseteq K_{n-1}(p,\xi,\beta,\gamma).$$

Proof 2.3 *Let $f(z) \in K_{n+1}(p,\xi,\beta,\gamma)$ then by definition there exist $g(z) \in S_{n+1}^*(p,\xi,\beta,\gamma)$ such that*

$$\mathrm{Re}\left(\frac{z(\Theta_p^{n+1}(\beta,\gamma) f(z))'}{\Theta_p^{n+1}(\beta,\gamma) g(z)}\right) > \rho,\ 0 \le \rho < 1, z \in \mathbb{U}.$$

We assume that

$$\left(\frac{z(\Theta_p^{n}(\beta,\gamma) f(z))'}{\Theta_p^{n}(\beta,\gamma) g(z)}\right) = \rho + (p-\rho)h(z),\ 0 \le \rho < 1, z \in \mathbb{U}, \tag{10}$$

where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$. Using (6), we have

$$\frac{z(\Theta_p^{n+1}(\beta,\gamma) f(z))'}{\Theta_p^{n+1}(\beta,\gamma) g(z)} = \frac{(\gamma+1)\, z(\Theta_p^{n}(\beta,\gamma)\, zf'(z))' + \beta(\Theta_p^{n}(\beta,\gamma)\, zf'(z))}{(\gamma+1)\, z(\Theta_p^{n}(\beta,\gamma)\, g(z))' + \beta(\Theta_p^{n}(\beta,\gamma)\, g(z))}. \tag{11}$$

Since $g(z) \in S_{n+1}^*(p,\xi,\beta,\gamma) \subseteq S_n^*(p,\xi,\beta,\gamma)$, so let

$$\frac{z(\Theta_p^{n}(\beta,\gamma) g(z))'}{\Theta_p^{n}(\beta,\gamma) g(z)} = \xi + (p-\xi)H(z).$$

So from (11) we have

$$\frac{z(\Theta_p^{n+1}(\beta,\gamma) f(z))'}{\Theta_p^{n+1}(\beta,\gamma) g(z)} = \frac{\frac{(\gamma+1)z(\Theta_p^n(\beta,\gamma)f'(z))'}{\Theta_p^n(\beta,\gamma)g(z)} + \frac{\beta z(\Theta_p^n(\beta,\gamma)zf'(z))}{\Theta_p^n(\beta,\gamma)g(z)}}{\frac{(\gamma+1)z(\Theta_p^n(\beta,\gamma)g(z))'}{\Theta_p^n(\beta,\gamma)g(z)} + \beta}. \tag{12}$$

As

$$\frac{z(\Theta_p^{n}(\beta,\gamma) f(z))'}{\Theta_p^{n}(\beta,\gamma) g(z)} = \rho + (p-\rho)h(z)$$

therefore we have

$$\frac{(\Theta_p^{n}(\beta,\gamma)\, zf'(z))}{\Theta_p^{n}(\beta,\gamma) g(z)} = \rho + (p-\rho)h(z)$$

which gives

$$\frac{\left(\Theta_p^n\left(\beta,\gamma\right)zf'(z)\right)'}{\left(\Theta_p^n\left(\beta,\gamma\right)zf'(z)\right)}-\frac{\left(\Theta_p^n\left(\beta,\gamma\right)g(z)\right)'}{\Theta_p^n\left(\beta,\gamma\right)g(z)}=\frac{\rho+(p-\rho)h'(z)}{\rho+(p-\rho)h(z)} \tag{13}$$

After substituting the result of (13) into (12) we have

$$\frac{z(\Theta_p^{n+1}\left(\beta,\gamma\right)f(z))'}{\Theta_p^{n+1}\left(\beta,\gamma\right)g(z)}-\rho=(p-\rho)h(z)+\frac{(\gamma+1)\,(p-\rho)zh'(z)}{(\gamma+1)\,(\xi+(p-\xi)H(z))+\beta}.$$

Taking $h(z)=\mu=\mu_1+i\mu_2$ *and* $zh'(z)=\upsilon=\upsilon_1+i\upsilon_2$, *we define the function* $\varphi(\mu,\upsilon)$ *by:*

$$\varphi(\mu,\upsilon)=(p-\rho)\mu+\frac{(\gamma+1)\,(p-\rho)\upsilon}{(\gamma+1)\,(\xi+(p-\xi)H(z))+\beta}.$$

It is easy to see that the function $\varphi(\mu,\upsilon)$ *satisfies the conditions (i) and (ii) of Lemma 2.1 in* $D=\mathbb{C}\times\mathbb{C}$. *To verify the condition (iii), we proceed as follows;*

$$\mathrm{Re}\left[\varphi(i\mu_2,\upsilon_1)\right]=\frac{(\gamma+1)\,\upsilon_1(p-\rho)[\beta+\xi\,(\gamma+1)+(\gamma+1)\,(p-\xi)h_1(x_1,y_1)]}{[\beta+\xi\,(\gamma+1)+(\gamma+1)\,(p-\xi)h_1(x_1,y_1)]^2+[(\gamma+1)\,(p-\xi)h_2(x_1,y_1)]^2}.$$

Where $H(z)=h_1(x_1,y_1)+ih_2(x_2,y_2)$, $h_1(x_1,y_1)$ *and* $h_2(x_2,y_2)$ *being functions of* x *and* y *and* $\mathrm{Re}(h_1(x_1,y_1))>0$. *By putting* $\upsilon_1\leq-\frac{1}{2}(1+\mu_2^2)$, *we obtain*

$$\mathrm{Re}\left[\varphi(i\mu_2,\upsilon_1)\right]=-\frac{(\gamma+1)\,(1+\mu_2^2)(p-\rho)[\beta+\xi\,(\gamma+1)+(\gamma+1)\,(p-\xi)h_1(x_1,y_1)]}{[\beta+\xi\,(\gamma+1)+(\gamma+1)\,(p-\xi)h_1(x_1,y_1)]^2+[(\gamma+1)\,(p-\xi)h_2(x_1,y_1)]^2}<0.$$

Hence, the function $\varphi(\mu,\upsilon)$ *satisfies the conditions of Lemma 2.1. Implies* $\mathrm{Re}\{h(z)\}>0 (z\in\mathbb{U})$, *that is,* $f(z)\in K_n(p,\xi,\beta,\gamma)$.

Similarly we can prove the following theorem.

Theorem 2.4 *Let* $f\in A\left(p\right)$, $0\leq\xi<1$, $\beta,\gamma\geq0, n\in\mathbb{N}_0$ *then*

$$K_{n+1}^*(p,\xi,\beta,\gamma)\subseteq K_n^*(p,\xi,\beta,\gamma)\subseteq K_{n-1}^*(p,\xi,\beta,\gamma).$$

3 Integral operator

For $c>-p$ and $f(z)\in A\left(p\right)$, the integral operator $L_{c,p}(f):A\left(p\right)\rightarrow A\left(p\right)$ is defined by

$$L_{c,p}(f)=\frac{c+p}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{14}$$

The operator $L_{c,p}(f)$ was introduced by Bernardi [17]. For more detailed about the operator $L_{c,p}(f)$, we refer the reader to study to [5, 18, 19].

Theorem 3.1 *Let $c > -p$, $0 \leq \xi < p$. If $f(z) \in S_n^*(p, \xi, \beta, \gamma)$, then $L_c(f) \in S_n^*(p, \xi, \beta, \gamma)$.*

Proof 3.1 *From (14) we have*

$$z(\Theta_p^n(\beta, \gamma) L_{c,p} f(z))' = (c+p) \Theta_p^n(\beta, \gamma) f(z) - c(\Theta_p^n(\beta, \gamma) L_{c,p} f(z)).$$

To prove that $L_{c,p}(f) \in S_n^*(p, \xi, \beta, \gamma)$. We suppose that

$$\frac{z(\Theta_p^n(\beta, \gamma) L_c f(z))'}{\Theta_p^n(\beta, \gamma) L_c f(z)} = \xi + (p - \xi) h(z). \tag{15}$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$.

Using (14) and (15) we conclude that

$$\frac{z(\Theta_p^n(\beta, \gamma) L_c f(z))'}{\Theta_p^n(\beta, \gamma) L_c f(z)} = (c+p) \frac{\Theta_p^n(\beta, \gamma) f(z)}{\Theta_p^n(\beta, \gamma) L_c f(z)} - c.$$

Differentiating w.r.t. z we have

$$\frac{z(\Theta_p^n(\beta, \gamma) L_c f(z))'}{\Theta_p^n(\beta, \gamma) L_c f(z)} - \xi = (p - \xi) h(z) + \frac{(p-\xi) z h'(z)}{\xi + (p-\xi) h(z) + c}$$

Taking $h(z) = \mu = \mu_1 + i\mu_1$ and $zh'(z) = \upsilon = \upsilon_1 + i\upsilon_1$, we define the function $\varphi(\mu, \upsilon)$ by:

$$\varphi(\mu, \upsilon) = (p - \xi)\mu + \frac{(p-\xi)\upsilon}{\xi + c + (p-\xi)\mu}.$$

It is easy to see that the function $\varphi(\mu, \upsilon)$ satisfies the conditions (i) and (ii) of Lemma 2.1 in $D = (\mathbb{C} - \{\frac{\xi+c}{\xi-p}\}) \times \mathbb{C}$. To verify the condition (iii), we proceed as follows;

$$\mathrm{Re}\,[\varphi(i\mu_2, \upsilon_1)] = \frac{(\xi+c)(p-\xi)\upsilon_1}{[\xi+c]^2 + [(p-\xi)\mu_2]^2} \leq \frac{-(\xi+c)(p-\xi)(1+\mu_2^2)}{2[\xi+c]^2 + 2[(p-\xi)\mu_2]^2} < 0.$$

Hence, the function $\varphi(\mu, \upsilon)$ satisfies the conditions of Lemma 2.1. Implies $\mathrm{Re}\{h(z)\} > 0 (z \in \mathbb{U})$, *that is, $L_{c,p}(f) \in S_n^*(p, \xi, \lambda, \alpha, \beta, \mu)\cdot$ This completes the proof of Theorem 3.1.*

Theorem 3.2 *Let $c > -p$, $0 \leq \xi < p$. If $f(z) \in C_n(p, \xi, \beta, \gamma)$, then $L_c(f) \in C_n(p, \xi, \beta, \gamma)$.*

Proof 3.2 *Since $f(z) \in C_n(p, \xi, \beta, \gamma) \Rightarrow \frac{zf'(z)}{p} \in S_n^*(p, \xi, \beta, \gamma) \Rightarrow L_{c,p}(\frac{zf'(z)}{p}) \in S_n^*(p, \xi, \beta, \gamma) \Rightarrow \frac{z}{p}(L_{c,p} f)' \in S_n^*(p, \xi, \beta, \gamma) \Leftrightarrow L_{c,p} f \in C_n(p, \xi, \beta, \gamma)$.*

Theorem 3.3 *Let $c > -p$, $0 \leq \xi < p$. If $f(z) \in K_n(p, \xi, \beta, \gamma)$, then $L_c(f) \in K_n(p, \xi, \beta, \gamma)$.*

Proof 3.3 *Since* $f(z) \in K_n(p,\xi,\beta,\gamma)$ *implies* $\Theta_p^n(\beta,\gamma) f \in K_p(\rho,\xi)$ *or*

$$\mathrm{Re}\left(\frac{z(\Theta_p^n(\beta,\gamma) f(z))'}{\Theta_p^n(\beta,\gamma) g(z)}\right) > \rho.$$

We suppose that

$$\left(\frac{z(\Theta_p^n(\beta,\gamma) L_c f(z))'}{\Theta_p^n(\beta,\gamma) L_c g(z)}\right) = \rho + (p-\rho)h(z), \; z \in \mathbb{U}. \tag{16}$$

Where $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$.

Since we have

$$(c+p)(\Theta_p^n(\beta,\gamma) f(z)) = z(\Theta_p^n(\beta,\gamma) L_c f(z))' + c\Theta_p^n(\beta,\gamma) L_c f(z)).$$

Therefore

$$\left(\frac{z(\Theta_p^n(\beta,\gamma) f(z))'}{\Theta_p^n(\beta,\gamma) g(z)}\right) = \frac{\frac{z(\Theta_p^n(\beta,\gamma)L_c(zf'(z))'}{\Theta_p^n(\beta,\gamma)L_c g(z)} + c\frac{(\Theta_p^n(\beta,\gamma)L_c(zf'(z))}{\Theta_p^n(\beta,\gamma)L_c g(z)}}{\frac{z(\Theta_p^n(\beta,\gamma)L_c(g(z))'}{\Theta_p^n(\beta,\gamma)L_c g(z)} + c}. \tag{17}$$

Since $g(z) \in S_n^*(p,\xi,\beta,\gamma)$ *implies* $L_c(g(z)) \in S_n^*(p,\xi,\beta,\gamma)$. *Let*

$$\frac{z(\Theta_p^n(\beta,\gamma) L_c(g(z))'}{\Theta_p^n(\beta,\gamma) L_c g(z)} = \xi + (p-\xi)H(z), \; \mathrm{Re}(H(z)) > 0, \; z \in \mathbb{U}. \tag{18}$$

$$\Rightarrow \frac{z(\Theta_p^n(\beta,\gamma) L_c(zf'(z))'}{\Theta_p^n(\beta,\gamma) L_c g(z)} = [\rho + (p-\rho)h(z)][\xi + (p-\xi)H(z)] + [(p-\rho)zh'(z)]. \tag{19}$$

Simultaneously solving (17) and (19) we get

$$\left(\frac{z(\Theta_p^n(\beta,\gamma) f(z))'}{\Theta_p^n(\beta,\gamma) g(z)}\right) - \rho = (p-\rho)h(z) + \frac{(p-\rho)zh'(z)}{\xi + (p-\xi)H(z) + c}$$

Taking $h(z) = \mu = \mu_1 + i\mu_2$ *and* $zh'(z) = \upsilon = \upsilon_1 + i\upsilon_2$, *we define the function* $\varphi(\mu,\upsilon)$ *by:*

$$\varphi(\mu,\upsilon) = (p-\rho)\mu + \frac{(p-\rho)\upsilon}{\xi + (p-\xi)H(z) + c}.$$

It is easy to see that the function $\varphi(\mu,\upsilon)$ *satisfies the conditions (i) and (ii) of Lemma 2.1 in* $D = \mathbb{C} \times \mathbb{C}$. *To verify the condition (iii), we proceed as follows;*

$$\mathrm{Re}\left[\varphi(i\mu_2,\upsilon_1)\right] = \frac{\upsilon_1(p-\rho)[(\xi+c) + (p-\xi)h_1(x_1,y_1)]}{[(\xi+c) + (p-\xi)h_1(x_1,y_1)]^2 + [(p-\xi)h_2(x_2,y_2)]^2}.$$

Where $H(z) = h_1(x_1, y_1) + ih_2(x_2, y_2)$, $h_1(x_1, y_1)$ *and* $h_2(x_2, y_2)$ *being functions of* x *and* y *and* $\mathrm{Re}(h_1(x_1, y_1)) > 0$. *By putting* $\upsilon_1 \le -\frac{1}{2}(1+\mu_2^2)$, *we obtain*

$$\mathrm{Re}\left[\varphi(i\mu_2, \upsilon_1)\right] = -\frac{1}{2}\frac{(1+\mu_2^2)(p-\rho)[(\xi+c)+(p-\xi)h_1(x_1,y_1)]}{[(\xi+c)+(p-\xi)h_1(x_1,y_1)]^2+[(p-\xi)h_2(x_2,y_2)]^2} < 0.$$

Hence, the function $\varphi(\mu, \upsilon)$ *satisfies the conditions of Lemma 2.1. Implies* $\mathrm{Re}\{h(z)\} > 0 (z \in \mathbb{U})$, *that is,* $L_c(f) \in K_n(p, \xi, \beta, \gamma)$. *This completes the proof of Theorem 3.3.*

Similarly we can prove the following theorem.

Theorem 3.4 *Let* $c > -p$, $0 \le \xi < p$. *If* $f(z) \in K_n^*(p, \xi, \beta, \gamma)$, *then* $L_c(f) \in K_n^*(p, \xi, \beta, \gamma)$.

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