# Dual spacelike elastic biharmonic curves with spacelike principal normal according to dual Bishop frames $\mathbb{D}_1^3$

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### Abstract

In this paper, we study dual spacelike elastic biharmonic curves with spacelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$.

**key words.** Dual space curve, dual Bishop frame, biharmonic curve.

**AMS subject classifications.** 58E20.

## 1 Introduction

Dual numbers were introduced by W. K. Clifford (1849-1879) as a tool for his geometrical investigations [2]. After him E. Study used dual numbers and dual vectors in his research on the geometry of lines and kinematics [8]. He devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the points of dual unit sphere $\mathbb{S}^2$ and the directed lines in $\mathbb{R}^3$.

E. Study devoted special attention to the representation of directed lines by dual unit vectors and defined the mapping that is known by his name. There exists one-to-one correspondence between the vectors of dual unit sphere $\mathbb{S}^2$ and the directed lines of space of lines $\mathbb{R}^3$ [8]. The existence of the dual numbers has been noticed in some papers concerning supermathematics e.g. [14]. The most interesting use of dual numbers in field theory can be found in a series of papers by Wald [15].

In this paper, we study dual spacelike elastic biharmonic curves with spacelike principal normal in dual Lorentzian space $\mathbb{D}_1^3$. We characterize curvature and torsion of dual spacelike biharmonic curves with spacelike principal normal in terms of dual Bishop frame in dual Lorentzian space $\mathbb{D}_1^3$.

## 2   Preliminaries

If $\varphi$ and $\varphi^*$ are real numbers and $\varepsilon^2 = 0$ the combination $\hat{\varphi} = \varphi + \varphi^*$ is called a dual number. The symbol $\varepsilon$ designates the dual unit with the property $\varepsilon^2 = 0$. In analogy with the complex numbers W.K. Clifford defined the dual numbers and showed that they form an algebra, not a field. Later, E.Study introduced the dual angle subtended by two nonparallel lines $\mathbb{E}^3$, and defined it as $\hat{\varphi} = \varphi + \varphi^*$ in which $\varphi$ and $\varphi^*$ are, respectively, the projected angle and the shortest distance between the two lines.

In the Euclidean 3-Space $\mathbb{E}^3$, lines combined with one of their two directions can be represented by unit dual vectors over the the ring of dual numbers. The important properties of real vector analysis are valid for the dual vectors. The oriented lines $\mathbb{E}^3$ are in one to one correspondence with the points of the dual unit sphere $\mathbb{D}^3$.

A dual point on $\mathbb{D}^3$ corresponds to a line in $\mathbb{E}^3$, two different points of $\mathbb{D}^3$ represents two skew lines in $\mathbb{E}^3$. A differentiable curve on $\mathbb{D}^3$ represents a ruled surface $\mathbb{E}^3$.

The set

$$\mathbb{D}^3 = \left\{ \hat{\varphi} : \hat{\varphi} = \varphi + \varepsilon \varphi^*, \ \varphi, \varphi^* \in \mathbb{E}^3 \right\}$$

is a module over the ring $\mathbb{D}$.

The elements of $\mathbb{D}^3$ are called dual vectors. Thus a dual vector $\hat{\varphi}$ can be written

$$\hat{\Omega} = \Omega + \varepsilon \Omega^*,$$

where $\varphi$ and $\varphi^*$ are real vectors in $\mathbb{R}^3$.

The Lorentzian inner product of dual vectors $\hat{\varphi}$ and $\hat{\psi}$ in $\mathbb{D}^3$ is defined by

$$\left\langle \hat{\Omega}, \hat{\psi} \right\rangle = \langle \Omega, \psi \rangle + \varepsilon \left( \langle \Omega, \psi^* \rangle + \langle \Omega^*, \psi \rangle \right),$$

with the Lorentzian inner product $\varphi$ and $\psi$

$$\langle \Omega, \psi \rangle = -\Omega_1 \psi_1 + \Omega_2 \psi_2 + \Omega_3 \psi_3,$$

where $\Omega = (\Omega_1, \Omega_2, \Omega_3)$ and $\psi = (\psi_1, \psi_2, \psi_3)$.

Therefore, $\mathbb{D}^3$ with the Lorentzian inner product $\left\langle \hat{\Omega}, \hat{\psi} \right\rangle$ is called 3-dimensional dual Lorentzian space and denoted by of $\mathbb{D}_1^3$.

For $\hat{\Omega} \neq 0$, the norm $\left\|\hat{\Omega}\right\|$ of $\hat{\Omega}$ is defined by

$$\left\|\hat{\Omega}\right\| = \sqrt{\left\langle \hat{\Omega}, \hat{\Omega} \right\rangle}.$$

A dual vector $\hat{\Omega} = \varphi + \varepsilon \varphi^*$ is called dual spacelike vector if $\left\langle \hat{\Omega}, \hat{\Omega} \right\rangle > 0$ or $\hat{\Omega} = \mathbf{0}$, dual timelike vector if $\left\langle \hat{\Omega}, \hat{\Omega} \right\rangle < 0$ and dual null (lightlike) vector if $\left\langle \hat{\Omega}, \hat{\Omega} \right\rangle = 0$ for $\hat{\Omega} \neq 0$.

Therefore, an arbitrary dual curve, which is a differentiable mapping onto $\mathbb{D}_1^3$, can locally be dual space-like, dual time-like or dual null, if its velocity vector is respectively, dual spacelike, dual timelike or dual null.

# 3   Spacelike Dual Biharmonic Curves with Spacelike Principal Normal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Let $\hat{\gamma} = \gamma + \varepsilon \gamma^* : I \subset R \to \mathbb{D}_1^3$ be a $C^4$ dual spacelike curve with spacelike principal normal by the arc length parameter $s$. Then the unit tangent vector $\hat{\gamma}' = \hat{\mathbf{t}}$ is defined, and the principal normal is $\hat{\mathbf{n}} = \frac{1}{\hat{\kappa}} \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}}$, where $\hat{\kappa}$ is never a pure-dual. The function $\hat{\kappa} = \left\| \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}} \right\| = \kappa + \varepsilon \kappa^*$ is called the dual curvature of the dual curve $\hat{\gamma}$. Then the binormal of $\hat{\gamma}$ is given by the dual vector $\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$. Hence, the triple $\left\{ \hat{\mathbf{t}}, \hat{\mathbf{n}}, \hat{\mathbf{b}} \right\}$ is called the Frenet frame fields and the Frenet formulas may be expressed

$$\begin{aligned} \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{t}} &= \hat{\kappa} \hat{\mathbf{n}}, \\ \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{n}} &= -\hat{\kappa} \hat{\mathbf{t}} + \hat{\tau} \hat{\mathbf{b}}, \\ \nabla_{\hat{\mathbf{t}}} \hat{\mathbf{b}} &= \hat{\tau} \hat{\mathbf{n}}, \end{aligned} \tag{3.1}$$

where $\hat{\tau} = \tau + \varepsilon \tau^*$ is the dual torsion of the timelike dual curve $\hat{\gamma}$. Here, we suppose that the dual torsion $\hat{\tau}$ is never pure-dual. In addition,

$$\begin{aligned} g\left(\hat{\mathbf{t}}, \hat{\mathbf{t}}\right) = 1, \ g\left(\hat{\mathbf{n}}, \hat{\mathbf{n}}\right) = 1, \ g\left(\hat{\mathbf{b}}, \hat{\mathbf{b}}\right) = -1, \\ g\left(\hat{\mathbf{t}}, \hat{\mathbf{n}}\right) = g\left(\hat{\mathbf{t}}, \hat{\mathbf{b}}\right) = g\left(\hat{\mathbf{n}}, \hat{\mathbf{b}}\right) = 0. \end{aligned} \tag{3.2}$$

In the rest of the paper, we suppose everywhere $\hat{\kappa} \neq 0$ and $\hat{\tau} \neq 0$.

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{t}} = \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2,$$
$$\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_1 = -\hat{k}_1\hat{\mathbf{t}},$$
$$\nabla_{\hat{\mathbf{t}}}\hat{\mathbf{m}}_2 = -\hat{k}_2\hat{\mathbf{t}},$$
(3.3)

where

$$g\left(\hat{\mathbf{t}},\hat{\mathbf{t}}\right) = 1,\ \ g\left(\hat{\mathbf{m}}_1,\hat{\mathbf{m}}_1\right) = 1,\ \ g\left(\hat{\mathbf{m}}_2,\hat{\mathbf{m}}_2\right) = -1,$$
$$g\left(\hat{\mathbf{t}},\hat{\mathbf{m}}_1\right) = g\left(\hat{\mathbf{t}},\hat{\mathbf{m}}_2\right) = g\left(\hat{\mathbf{m}}_1,\hat{\mathbf{m}}_2\right) = 0.$$
(3.4)

Here, we shall call the set $\{\hat{\mathbf{t}},\hat{\mathbf{m}}_1,\hat{\mathbf{m}}_1\}$ as Bishop trihedra, $\hat{k}_1$ and $\hat{k}_2$ as Bishop curvatures. Here $\tau(s) = \hat{\theta}'(s)$ and $\hat{\kappa}(s) = \sqrt{\left|\hat{k}_1^2 - \hat{k}_2^2\right|}$. Thus, Bishop curvatures are defined by

$$\hat{k}_1 = \hat{\kappa}(s)\cosh\hat{\theta}(s),$$
$$\hat{k}_2 = \hat{\kappa}(s)\sinh\hat{\theta}(s).$$
(3.5)

**Theorem 3.1.** *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$\hat{k}_2^2 - \hat{k}_1^2 = \hat{\Omega},$$
$$\hat{k}_1'' + \hat{k}_1^3 - \hat{k}_2^2\hat{k}_1 = 0,$$
$$-\hat{k}_2'' + \hat{k}_2^3 - \hat{k}_1^2\hat{k}_2 = 0,$$
(3.6)

*where $\hat{\Omega}$ is dual constant of integration, [5].*

**Lemma 3.2.** *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length. $\hat{\gamma}$ is a non-geodesic spacelike dual biharmonic curve if and only if*

$$-\hat{k}_1^2 + \hat{k}_2^2 = \hat{\Omega},$$
$$\hat{k}_1'' + \hat{k}_1\hat{\Omega} = 0,$$
$$\hat{k}_2'' + \hat{k}_2\hat{\Omega} = 0,$$
(3.7)

where $\hat{\Omega} = \Omega + \varepsilon\Omega^*$ is constant of integration, [5].

**Corollary 3.3.** *Let* $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ *be a non-geodesic spacelike dual curve with spacelike principal normal parametrized by arc length.* $\hat{\gamma}$ *is a non-geodesic spacelike dual biharmonic curve if and only if*

$$k_1^2 - k_2^2 = -\Omega, \tag{3.8}$$

$$k_1 k_1^* - k_2 k_2^* = -\Omega^*. \tag{3.9}$$

# 4   Dual Spacelike Elastic Biharmonic Curves with Timelike Binormal in the Dual Lorentzian Space $\mathbb{D}_1^3$

Consider regular curve (curves with nonvanishing velocity vector) in dual Lorentzian space $\mathbb{D}_1^3$ defined on a fixed interval $I = [a_1, a_2]$:

$$\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3.$$

We will assume (for technical reasons) that the curvature $\hat{\kappa}$ of $\hat{\gamma}$ is nonvanishing.

The elastica minimizes the bending energy

$$\Pi(\hat{\gamma}) = \int_{\hat{\gamma}} \hat{\kappa}(s)^2 \, ds$$

with fixed length and boundary conditions. Accordingly, let $\alpha_1$ and $\alpha_2$ be points in $\mathbb{D}_1^3$ and $\alpha_1', \alpha_2'$ nonzero vectors. We will consider the space of smooth curves

$$\Xi = \left\{ \hat{\gamma} : \hat{\gamma}(a_i) = \hat{\alpha}_i, \ \hat{\gamma}'(a_i) = \hat{\alpha}_i' \right\},$$

and the subspace of unit-speed curves

$$\Xi_u = \left\{ \hat{\gamma} \in \Omega : \left\| \hat{\gamma}' \right\| = 1 \right\}.$$

Later on we need to pay more attention to the precise level of differentiability of curves, but we will ignore that for now.

$\Pi^\lambda : \Omega \longrightarrow \mathbb{D}$ is defined by

$$\Pi^\lambda(\hat{\gamma}) = \frac{1}{2} \int_{\hat{\gamma}} \left[ \left\| \hat{\gamma}'' \right\| + \hat{\Lambda}(t) \left( \left\| \hat{\gamma}' \right\| - 1 \right) \right] dt,$$

where $\hat{\Lambda}(t) = \Lambda(t) + \varepsilon\Lambda^*(t)$ is a pointwise dual multiplier, constraining speed.

**Theorem 4.1.** *(Noether's Theorem) If $\hat{\gamma}$ is a solution curve and $W$ is an infinitesimal symmetry, then*

$$\hat{\gamma}''.W' + \left(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''\right).W$$

*is constant. In particular, for a translational symmetry, $W$ is constant; so*

$$\left(\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}'''\right).W = \text{constant}.$$

Letting $W$ range over all translations, we get

$$\hat{\Lambda}\hat{\gamma}' - \hat{\gamma}''' = \hat{J}, \tag{4.1}$$

for $\hat{J}$ some constant field and

$$\hat{J} = J + \varepsilon J^*$$

**Theorem 4.2.** *Let $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ be a dual spacelike elastic biharmonic curves with timelike binormal according to Bishop frame. Then,*

$$\Lambda(s) = 0 \text{ and } \Lambda^*(s) = 0. \tag{4.2}$$

**Proof.** Now it is helpful to assume dual biharmonic curve $\hat{\gamma}$ is parametrized by arclength $s$. If we use dual Bishop frame (3.3), yields

$$\begin{aligned}
\hat{\gamma}' &= \hat{\mathbf{t}} \\
\hat{\gamma}'' &= \hat{k}_1\hat{\mathbf{m}}_1 - \hat{k}_2\hat{\mathbf{m}}_2, \\
\hat{\gamma}''' &= \left(\hat{k}_2^2 - \hat{k}_1^2\right)\hat{\mathbf{t}} + \hat{k}_1'\hat{\mathbf{m}}_1 - \hat{k}_2'\hat{\mathbf{m}}_2.
\end{aligned} \tag{4.3}$$

By means of dual function, $\varepsilon^2 = 0$ reduces to

$$\begin{aligned}
\hat{\gamma}' &= \mathbf{t} + \varepsilon\mathbf{t}^*, \\
\hat{\gamma}'' &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2 + \varepsilon(k_1^*\mathbf{m}_1 + k_1\mathbf{m}_1^* - k_2^*\mathbf{m}_2 - k_2\mathbf{m}_2^*), \\
\hat{\gamma}''' &= \left(k_2^2 - k_1^2\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2 + \varepsilon(\left(k_2^2 - k_1^2\right)\mathbf{t}^* \\
&\quad + \left(2k_2k_2^* - 2k_1k_1^*\right)\mathbf{t} + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*.
\end{aligned} \tag{4.4}$$

If we calculate the real and dual parts of this equation, we get the following relations

$$\begin{aligned}
\gamma' &= \mathbf{t}, \\
\gamma'' &= k_1\mathbf{m}_1 - k_2\mathbf{m}_2, \\
\gamma''' &= \left(k_2^2 - k_1^2\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2,
\end{aligned}$$

and

$$\begin{aligned}
\gamma^{*\prime} &= \mathbf{t}^*, \\
\gamma^{*\prime\prime} &= k_1^*\mathbf{m}_1 + k_1\mathbf{m}_1^* - k_2^*\mathbf{m}_2 - k_2\mathbf{m}_2^*, \\
\gamma^{*\prime\prime\prime} &= \left(k_2^2 - k_1^2\right)\mathbf{t}^* + \left(2k_2k_2^* - 2k_1k_1^*\right)\mathbf{t} \\
&\quad + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*.
\end{aligned}$$

Using (4.1), we get

$$\begin{aligned}
J &= \left(k_2^2 - k_1^2 - \Lambda\right)\mathbf{t} + k_1'\mathbf{m}_1 - k_2'\mathbf{m}_2, \\
J^* &= \left(k_2^2 - k_1^2 - \Lambda\right)\mathbf{t}^* + \left(2k_2k_2^* - 2k_1k_1^* - \Lambda^*\right)\mathbf{t} \\
&\quad + k_1^{*\prime}\mathbf{m}_1 + k_1'\mathbf{m}_1^* - k_2^{*\prime}\mathbf{m}_2 - k_2'\mathbf{m}_2^*.
\end{aligned} \tag{4.5}$$

If we take the derivative of $\hat{J}$ with respect to $s$, we get

$$\begin{aligned}
\hat{J}_s &= \left(-\Lambda_s - k_1'k_1 + k_2'k_2\right)\mathbf{t} + \varepsilon[\left(-\Lambda_s - k_1'k_1 + k_2'k_2\right)\mathbf{t}^* \\
&\quad + \left(-\Lambda_s^* + k_2'k_2^* + k_2^{*\prime}k_2 - k_1'k_1^* - k_1^{*\prime}k_1\right)\mathbf{t}] \\[6pt]
&\quad + [k_1'' - k_1\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_1 \\
&\quad + \varepsilon[k_1^{*\prime\prime} - \left(k_2^2 - k_1^2 - \Lambda\right)k_1^* + \left(2k_2k_2^* - 2k_1k_1^* - \Lambda^*\right)k_1]\mathbf{m}_1 \\
&\quad + \varepsilon[k_1'' - k_1\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_1^* \\[6pt]
&\quad - [k_2'' - k_2\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_2 \\
&\quad - \varepsilon[k_2^{*\prime\prime} - \left(k_2^2 - k_1^2 - \Lambda\right)k_2^* + \left(2k_2k_2^* - 2k_1k_1^* - \Lambda^*\right)k_2]\mathbf{m}_2 \\
&\quad - \varepsilon[k_2'' - k_2\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_2^*.
\end{aligned} \tag{4.6}$$

Then we calculate the real and dual parts of this equation, we get the following relations

$$
\begin{aligned}
J_s =\ & \left(-\Lambda_s - k_1' k_1 + k_2' k_2\right) \mathbf{t} + [k_1'' - k_1\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_1 \\
& -[k_2'' - k_2\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_2,
\end{aligned}
$$

$$
\begin{aligned}
J_s^* =\ & \left(-\Lambda_s - k_1' k_1 + k_2' k_2\right) \mathbf{t}^* + \left(-\Lambda_s^* + k_2' k_2^* + k_2^{*\prime} k_2 - k_1' k_1^* - k_1^{*\prime} k_1\right)\mathbf{t} \\
& +[k_1^{*\prime\prime} - \left(k_2^2 - k_1^2 - \Lambda\right) k_1^* + \left(2 k_2 k_2^* - 2 k_1 k_1^* - \Lambda^*\right) k_1]\mathbf{m}_1 \\
& +[k_1'' - k_1\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_1^* \\
& -[k_2^{*\prime\prime} - \left(k_2^2 - k_1^2 - \Lambda\right) k_2^* + \left(2 k_2 k_2^* - 2 k_1 k_1^* - \Lambda^*\right) k_2]\mathbf{m}_2 \\
& -[k_2'' - k_2\left(k_2^2 - k_1^2 - \Lambda\right)]\mathbf{m}_2^*.
\end{aligned}
$$

Thus, by taking into consideration that (3.3) and (3.4), complete the proof of the theorem.

**Corollary 4.3.** *Let* $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ *be a dual spacelike elastic biharmonic curves with timelike binormal according to Bishop frame. Then,*

$$
\begin{aligned}
J_s = \left(-k_1' k_1 + k_2' k_2\right) \mathbf{t} + [k_1'' - k_1\left(k_2^2 - k_1^2\right)]\mathbf{m}_1 \\
-[k_2'' - k_2\left(k_2^2 - k_1^2\right)]\mathbf{m}_2
\end{aligned}
\tag{4.7}
$$

$$
\begin{aligned}
J_s^* = \left(-k_1' k_1 + k_2' k_2\right) \mathbf{t}^* + \left(k_2' k_2^* + k_2^{*\prime} k_2 - k_1' k_1^* - k_1^{*\prime} k_1\right)\mathbf{t} \\
+[k_1^{*\prime\prime} - \left(k_2^2 - k_1^2\right) k_1^* + \left(2 k_2 k_2^* - 2 k_1 k_1^*\right) k_1]\mathbf{m}_1 \\
+[k_1'' - k_1\left(k_2^2 - k_1^2\right)]\mathbf{m}_1^* \\
-[k_2^{*\prime\prime} - \left(k_2^2 - k_1^2\right) k_2^* + \left(2 k_2 k_2^* - 2 k_1 k_1^*\right) k_2]\mathbf{m}_2 \\
-[k_2'' - k_2\left(k_2^2 - k_1^2\right)]\mathbf{m}_2^*
\end{aligned}
$$

**Proof.** Using (4.2) and (4.6), we have (4.7). This completes the proof.

**Corollary 4.4.** *Let* $\hat{\gamma} : I \longrightarrow \mathbb{D}_1^3$ *be a dual spacelike elastic biharmonic curves with timelike binormal according to Bishop frame. Then* $\hat{J}$ *is a Killing vector field.*

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